Belief Functions Contextual Discounting and Canonical Decompositions

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Abstract

In this article, the contextual discounting of a belief function, a classical discounting generalization, is extended and its particular link with the canonical disjunctive decomposition is highlighted. A general family of correction mechanisms allowing one to weaken the information provided by a source is then introduced, as well as the dual of this family allowing one to strengthen a belief function.

Keywords: Belief Functions, Discounting, Reinforcement, Canonical Decompositions.

1. Introduction

In the Dempster-Shafer theory of belief functions [3, 21], the reliability of a source of information is classically taken into account by the discounting operation [21, page 252], which transforms a belief function into a weaker, less informative one. This operation is usually important in uncertain information management [1, 2, 7, 8, 9, 11, 14, 18, 17, 19, 27].

Introduced in [13], the contextual discounting is a refinement of the discounting operation. It takes into account the fact that the reliability of a source of information can be expected to depend on the true answer of the question of interest.

For instance, in medical diagnosis, depending on his/her speciality, experience or training, a physician may be more or less competent to diagnose some types of diseases. Likewise, in target recognition, a sensor may be

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more capable of recognizing some types of targets while being less effective for other types.

In this contextual model, the agent in charge of the fusion process or the decision making can hold knowledge regarding the reliability of a source of information in different contexts, which forms a partition of the universe of discourse. For example, a sensor in charge of recognizing targets can be more or less reliable depending on the fact that the target is a helicopter (h), an airplane (a) or a rocket(r), subsets $\{h\}$, $\{r\}$ and $\{a\}$ forming the finest partition of the universe $\Omega = \{a, b, c\}$). However, this previous model can not handle a reliability knowing that the target is a helicopter or a rocket ($\{h, r\}$) as well as a reliability knowing that the target is an airplane or a helicopter ($\{a, h\}$). Sets $\{a, h\}$ and $\{h, r\}$ do not form a partition of Ω.

This last step is reached in this article. It is then shown that the contextual discounting exposed previously in [13] is a particular case of a more general correction process [15, 12] allowing the discounting of a belief function in a finer way. In particular, a simple expression of this mechanism is given in the form of disjunctive combinations. At last, the dual version of the contextual discounting, allowing one to reinforce a belief function, is also introduced. This article extends deeply a first version of this work presented in [16].

To develop the justifications of these mechanisms, belief functions are interpreted as expressing weighted opinions, irrespective of any underlying probability distributions, and the Transferable Belief Model [23, 24, 26] is adopted.

This article is organized as follows. Background material needed on belief functions is recalled in Section 2. Contextual discounting is extended in Section 3. A discussion is next launched in Section 4. A dual reinforcement process is introduced in Section 5, and finally, Section 6 concludes this article.

2. Belief functions: basic concepts

2.1. Representing information

Let us consider an agent Ag in charge of making a decision regarding the answer to a given question Q of interest.

Let $\Omega = \{\omega_1, \ldots, \omega_K\}$, called the *frame of discernment*, be the finite set containing the possible answers to question Q.

The information held by agent Ag regarding the answer to question Q can be quantified by a *basic belief assignment (BBA)* or a *mass function*

m^{Ω}_{Ag}, defined as a function from 2^{Ω} to $[0,1]$, and verifying:

$$\sum_{A \subseteq \Omega} m^{\Omega}_{Ag}(A) = 1 \ . \tag{1}$$

Function m^{Ω}_{Ag} describes the state of knowledge of agent Ag regarding the answer to question Q belonging to Ω. By extension, it also represents an item of evidence that induces such a state of knowledge. The quantity $m^{\Omega}_{Ag}(A)$ is interpreted as the part of the unit mass allocated to the hypothesis: "the answer to question Q is in the subset A of Ω".

When there is no ambiguity, the full notation m^{Ω}_{Ag} will be simplified to m^{Ω}, or even m.

The following definitions and notations are considered.

Definition 1 (Focal element). *A subset A of Ω such that $m(A) > 0$ is called a* focal element *of m.*

Definition 2 (Categorical BBA). *A BBA m with only one focal element A is said to be* categorical *and is denoted m_A; we thus have $m_A(A) = 1$.*

Definition 3 (Vacuous mass function). *Total ignorance is represented by the BBA m_{Ω}, called the* vacuous mass function*.*

Definition 4. *A BBA m is said to be:*

- dogmatic *if $m(\Omega) = 0$;*
- non-dogmatic *if $m(\Omega) > 0$;*
- normal *if $m(\emptyset) = 0$;*
- subnormal *if $m(\emptyset) > 0$;*
- simple *if m has no more than two focal sets, Ω being included.*

Definition 5 (Negation of a BBA). *Function $\overline{m}$ denotes the* negation of *m [6], defined by $\overline{m}(A) = m(\overline{A})$, for all $A \subseteq \Omega$ such that $\overline{A}$ is the complement of A in Ω.*

Definition 6. *The belief, plausibility, implicability and commonality functions associated with a mass function m are defined, respectively, as:*

$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B), \tag{2}$$

$$pl(A) = \sum_{B \cap A \neq \emptyset} m(B), \quad (3)$$

$$b(A) = bel(A) + m(\emptyset) = 1 - pl(\overline{A}), \quad (4)$$

and

$$q(A) = \sum_{B \supseteq A} m(B), \quad (5)$$

for all $A \subseteq \Omega$.

Functions bel, pl, b and q are all in one-to-one correspondence [28], and represent then the same information.

2.2. Combining pieces of information

Two BBAs m_1 and m_2 induced by distinct and reliable sources of information can be combined using the *conjunctive rule of combination (CRC)*, also referred to as the *unnormalized Dempster's rule of combination*, defined for all $A \subseteq \Omega$ by:

$$m_1 \textcircled{\cap} m_2(A) = \sum_{B \cap C = A} m_1(B) m_2(C) \ . \quad (6)$$

Alternatively, if we only know that at least one of the sources is reliable, BBAs m_1 and m_2 can be combined using the *disjunctive rule of combination (DRC)*, defined for all $A \subseteq \Omega$ by:

$$m_1 \textcircled{\cup} m_2(A) = \sum_{B \cup C = A} m_1(B) m_2(C) \ . \quad (7)$$

2.3. Marginalization and vacuous extension on a product space

A mass function defined on a product space $\Omega \times \Theta$ may be *marginalized* on Ω by transferring each mass $m^{\Omega \times \Theta}(B)$ for $B \subseteq \Omega \times \Theta$ to its projection on Ω:

$$m^{\Omega \times \Theta \downarrow \Omega}(A) = \sum_{\substack{B \subseteq \Omega \times \Theta, \\ \text{Proj}(B \downarrow \Omega) = A}} m^{\Omega \times \Theta}(B), \quad (8)$$

for all $A \subseteq \Omega$ where $\text{Proj}(B \downarrow \Omega)$ denotes the projection of B onto Ω.

Conversely, it is usually not possible to retrieve the original BBA $m^{\Omega \times \Theta}$ from its marginal $m^{\Omega \times \Theta \downarrow \Omega}$ on Ω. However, the *least committed*, or *least informative BBA* [22] such that its projection on Ω is $m^{\Omega \times \Theta \downarrow \Omega}$ may be computed. This defines the *vacuous extension* of m^{Ω} in the product space $\Omega \times \Theta$, noted $m^{\Omega \uparrow \Omega \times \Theta}$, and given by:

$$m^{\Omega \uparrow \Omega \times \Theta}(B) = \begin{cases} m^{\Omega}(A) & \text{if } B = A \times \Theta, A \subseteq \Omega, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

2.4. Conditioning and ballooning extension on a product space

Conditional beliefs represent knowledge that is valid provided that an hypothesis is satisfied. Let m be a mass function and $B \subseteq \Omega$ an hypothesis; the *conditional belief function* $m[B]$ is given by:

$$m[B] = m \circledcap m_B. \tag{10}$$

If $m^{\Omega\times\Theta}$ is defined on the product space $\Omega \times \Theta$, and θ is a subset of Θ, the conditional BBA $m^\Omega[\theta]$ is defined by combining $m^{\Omega\times\Theta}$ with $m_\theta^{\Theta\uparrow\Omega\times\Theta}$, and marginalizing the result on Ω:

$$m^\Omega[\theta] = \left(m^{\Omega\times\Theta} \circledcap m_\theta^{\Theta\uparrow\Omega\times\Theta}\right)^{\downarrow\Omega}. \tag{11}$$

Assume now that $m^\Omega[\theta]$ represents the agent's beliefs on Ω conditionally on θ, i.e., in a context where θ holds. There are usually many BBAs on $\Omega\times\Theta$, whose conditioning on θ yields $m^\Omega[\theta]$. Among these, the least committed one is defined for all $A \subseteq \Omega$ by:

$$m^\Omega[\theta]^{\Uparrow\Omega\times\Theta}(A \times \theta \cup \Omega \times \overline{\theta}) = m^\Omega[\theta](A). \tag{12}$$

This operation is referred to as the *deconditioning* or *ballooning extension* [22] of $m^\Omega[\theta]$ on $\Omega \times \Theta$.

2.5. Discounting

When receiving a piece of information represented by a mass function m, agent Ag may have some doubts regarding the reliability of the source that provided this information. Such metaknowledge can be taken into account using the discounting operation introduced by Shafer [21, page 252], and defined by:

$$^\alpha m = (1-\alpha)m + \alpha\, m_\Omega\ , \tag{13}$$

where $\alpha \in [0,1]$.

A discount rate α equal to 1, means that the source is not reliable and the piece of information it provides cannot be taken into account, so Ag's knowledge remains vacuous: $m_{Ag}^\Omega = {}^1m = m_\Omega$. On the contrary, a null discount rate indicates that the source is fully reliable and the piece of information is entirely accepted: $m_{Ag}^\Omega = {}^0m = m$. In practice, however, agent Ag usually does not know for sure whether the source is reliable or not, but has some degree of belief expressed by:

$$\begin{cases} m_{Ag}^{\mathcal{R}}(\{R\}) &= 1-\alpha \\ m_{Ag}^{\mathcal{R}}(\mathcal{R}) &= \alpha, \end{cases} \tag{14}$$

where $\mathcal{R} = \{R, NR\}$, R and NR standing, respectively, for *"the source is reliable"* and *"the source is not reliable"*. This formalization yields expression (13), as demonstrated by Smets in [22, Section 5.7].

The discounting operation (13) of a BBA m is also equivalent to the disjunctive combination (7) of m with the mass function m_0^Ω defined by:

$$m_0^\Omega(A) = \begin{cases} \beta & \text{if } A = \emptyset \\ \alpha & \text{if } A = \Omega \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

with $\alpha \in [0, 1]$ and $\beta = 1 - \alpha$.

Indeed:

$$m \ovee m_0^\Omega(A) = m(A) m_0^\Omega(\emptyset) = \beta m(A) = {}^\alpha m(A), \ \forall A \subset \Omega \ , \tag{16}$$

and

$$\begin{aligned} m \ovee m_0^\Omega(\Omega) &= m(\Omega) m_0^\Omega(\emptyset) + m_0^\Omega(\Omega) \textstyle\sum_{A \subseteq \Omega} m(A) \\ &= \beta m(\Omega) + \alpha = {}^\alpha m(\Omega) \ . \end{aligned} \tag{17}$$

2.6. Contextual Discounting based on a coarsening

Let $\Theta = \{\theta_1, \ldots, \theta_L\}$ be a coarsening of Ω, which means that $\theta_1, \ldots, \theta_L$ form a partition of Ω [21, chapter 6].

Unlike (14), in the contextual model, agent Ag is assumed to hold beliefs on the reliability of the source of information conditionally on each θ_ℓ, $\ell \in \{1, \ldots, L\}$:

$$\begin{cases} m_{Ag}^{\mathcal{R}}[\theta_\ell](\{R\}) &= 1 - \alpha_\ell = \beta_\ell \\ m_{Ag}^{\mathcal{R}}[\theta_\ell](\mathcal{R}) &= \alpha_\ell \ . \end{cases} \tag{18}$$

For all $\ell \in \{1, \ldots, L\}$, $\beta_\ell + \alpha_\ell = 1$, and β_ℓ represents the degree of belief that the source is reliable knowing that the true answer of the question of interest belongs to θ_ℓ.

In the same way as in the discounting operation (13), agent Ag considers that the source can be in two states: reliable or not reliable [22, 13]:

- If the source is reliable (state R), the information m_S^Ω it provides becomes Ag's knowledge. Formally, $m_{Ag}^\Omega[\{R\}] = m_S^\Omega$.
- If the source is not reliable (state NR), the information m_S^Ω it provides is discarded, and Ag remains in a state of ignorance: $m_{Ag}^\Omega[\{NR\}] = m_\Omega$.

The knowledge held by agent Ag, based on the information m_S^Ω from a source S as well as metaknowledge $m_{Ag}^{\mathcal{R}}$ concerning the reliability of the source can then be computed by:

- Deconditioning the L BBAs $m_{Ag}^{\mathcal{R}}[\theta_\ell]$ on the product space $\Omega \times \mathcal{R}$ using (12);
- Deconditioning $m_{Ag}^{\Omega}[\{R\}]$ on the same product space $\Omega \times \mathcal{R}$ using (12) as well;
- Combining them using the CRC (6);
- Marginalizing the result on Ω using (8).

Formally:

$$m_{Ag}^{\Omega}[m_S^\Omega, m_{Ag}^{\mathcal{R}}] = \left(\textcircled{\cap}_{\ell=1}^{L} m_{Ag}^{\mathcal{R}}[\theta_\ell]^{\Uparrow \Omega \times \mathcal{R}} \textcircled{\cap} m_{Ag}^{\Omega}[\{R\}]^{\Uparrow \Omega \times \mathcal{R}} \right)^{\downarrow \Omega} . \tag{19}$$

As shown in [13], the resulting BBA m_{Ag}^{Ω} , only depends on m_S and on the vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_L)$ of discount rates. It is then denoted by ${}^{\boldsymbol{\alpha}}_{\Theta}m$.

Proposition 1 ([13, Proposition 8]). *Contextual discounting* ${}^{\boldsymbol{\alpha}}_{\Theta}m$ *of a BBA* m *is equal to the disjunctive combination of* m *with a BBA* m_0^Ω *such that:*

$$m_0^\Omega = m_1^\Omega \textcircled{\cup} m_2^\Omega \textcircled{\cup} \ldots \textcircled{\cup} m_L^\Omega \ , \tag{20}$$

where each m_ℓ^Ω, $\ell \in \{1, \ldots, L\}$, *is defined by:*

$$m_\ell^\Omega(A) = \begin{cases} \beta_\ell & \text{if } A = \emptyset \\ \alpha_\ell & \text{if } A = \theta_\ell \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Remark 1. *Two special cases of this discounting operation can be considered.*

- *If* $\Theta = \{\Omega\}$ *denotes the trivial partition of* Ω *in one class, combining* m *with* m_0 *defined by (15) is equivalent to combining* m *with* m_0 *defined by (20), so this contextual discounting operation is identical to the* classical discounting operation.
- *If* $\Theta = \Omega$, *the finest partition of* Ω, *this discounting is simply called* contextual discounting *and denoted* ${}^{\boldsymbol{\alpha}}m$. *It is defined by the disjunctive combination of* m *with the BBA* $m_1^\Omega \textcircled{\cup} m_2^\Omega \textcircled{\cup} \ldots \textcircled{\cup} m_K^\Omega$, *where each* m_k^Ω, $k \in \{1, \ldots, K\}$ *is defined by* $m_k^\Omega(\emptyset) = \beta_k$ *and* $m_k^\Omega(\{\omega_k\}) = \alpha_k$.

2.7. Canonical conjunctive and disjunctive decompositions

In [25], extending the notion of separable BBA introduced by Shafer [21, chapter 4], Smets shows that each non-dogmatic BBA m can be uniquely decomposed into a conjunctive combination of *generalized simple BBAs (GSBBAs)*, denoted $A^{w(A)}$ with $A \subset \Omega$, and defined from 2^Ω to $\mathbb{R}$ by:

$$A^{w(A)}: \begin{array}{lll} \Omega & \mapsto & w(A) \\ A & \mapsto & 1 - w(A) \\ B & \mapsto & 0 \ , \ \forall B \in 2^\Omega \setminus \{A, \Omega\} \ , \end{array} \tag{22}$$

with $w(A) \in (0, \infty)$.

The function w: $2^\Omega \setminus \{\Omega\} \to (0, \infty)$ is yet another representation of a non-dogmatic mass function and is called the *conjunctive weight function.*

Let us note that the higher is the weight $w(A)$, the higher is the incertitude on A.

The canonical conjunctive decomposition of a non-dogmatic BBA m is then given by:

$$m = \textcircled{\cap}_{A \subset \Omega} A^{w(A)} \ . \tag{23}$$

In [4], Denœux introduces another decomposition: the canonical disjunctive decomposition of a subnormal BBA into *negative GSBBAs (NGSBBAs)*, denoted $A_{v(A)}$ with $A \supset \emptyset$, and defined from 2^Ω to $\mathbb{R}$ by:

$$A_{v(A)}: \begin{array}{lll} \emptyset & \mapsto & v(A) \\ A & \mapsto & 1 - v(A) \\ B & \mapsto & 0 \ , \ \forall B \in 2^\Omega \setminus \{\emptyset, A\} \ , \end{array} \tag{24}$$

with $v(A) \in (0, \infty)$.

Every subnormal BBA m can be canonically decomposed into a disjunctive combination of NGSBBAs:

$$m = \textcircled{\cup}_{A \supset \emptyset} A_{v(A)} \ . \tag{25}$$

Indeed, as remarked in [4], the negation of a BBA m can also be conjunctively decomposed as soon as m is subnormal:

$$\begin{array}{rcl} \overline{m} & = & \textcircled{\cap}_{A \subset \Omega} A^{\overline{w}(A)} \text{ (as } \overline{m} \text{ is non-dogmatic)} \\ \Rightarrow \ m & = & \overline{\textcircled{\cap}_{A \subset \Omega} A^{\overline{w}(A)}} = \textcircled{\cup}_{A \subset \Omega} \overline{A^{\overline{w}(A)}} = \textcircled{\cup}_{A \supset \emptyset} A_{\overline{w}(\overline{A})} \ . \end{array} \tag{26}$$

The relation between functions v and w is then $v(A) = \overline{w}(\overline{A})$ for all $A \neq \emptyset$, and function v: $2^\Omega \setminus \{\emptyset\} \to (0, \infty)$, called the *disjunctive weight function*, is another representation of a subnormal mass function.

Practically, functions w and v have the following properties [4]:

- for all w:

$$\prod_{A\subset\Omega} w(A) = m(\Omega)\ , \tag{27}$$

- for all v:

$$\prod_{A\supset\emptyset} v(A) = m(\emptyset)\ , \tag{28}$$

- for all subset $A \subset \Omega$: $A^{w_1(A)} \circledcap A^{w_2(A)} = A^{w_1(A)w_2(A)}$,
- for all subset $A \supset \emptyset$: $A_{v_1(A)} \circledcup A_{v_2(A)} = A_{v_1(A)v_2(A)}$,
- function w can be conveniently obtained from the commonality function q as follows:

$$\forall A \subset \Omega, \quad w(A) = \frac{\prod_{B\supseteq A, Parity(|A|)\neq Parity(|B|)} q(B)}{\prod_{B\supseteq A, Parity(|A|)=Parity(|B|)} q(B)}\ , \tag{29}$$

  where $Parity(n)$ means the parity of an integer n ($Parity(n) = 0$ if n is even, 1 otherwise),

- likewise, function v can be computed from the implicability function b as follows:

$$\forall A \supset \emptyset, \quad v(A) = \frac{\prod_{B\subseteq A, Parity(|A|)\neq Parity(|B|)} b(B)}{\prod_{B\subseteq A, Parity(|A|)=Parity(|B|)} b(B)}\ , \tag{30}$$

For "quasi-Bayesian" BBAs, another convenient way to compute w is given by the following property.

Proposition 2 ([4, Proposition 1]). *Let m be a BBA which focal sets are Ω, A_1, A_2, ..., A_n, and possibly $\emptyset$, such that the n subsets A_k verifies $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$. The conjunctive weight function w associated with m is then defined by:*

$$w(A) = \begin{cases} \frac{m(\Omega)}{m(A_k)+m(\Omega)} & \text{if } A = A_k\ , \\ m(\Omega)\prod_{k=1}^{n}(1+\frac{m(A_k)}{m(\Omega)}) & \text{if } A = \emptyset\ , \\ 1 & \text{otherwise.} \end{cases} \tag{31}$$

Remark 2. *If m has only one focal element in addition to $\emptyset$ and Ω, Proposition 2 holds as well.*

The dual version of this property to compute disjunctive weights can be obtained as follows.

Proposition 3. *Let m be a BBA which focal sets are $\emptyset$, A_1, A_2, $\ldots$, A_n, and possibly Ω, such that the n subsets A_k verifies $A_i \cup A_j = \Omega$ for all $i, j \in \{1, \ldots, n\}$. The disjunctive weight function v associated with m is then defined by:*

$$v(A) = \begin{cases} \frac{m(\emptyset)}{m(A_k)+m(\emptyset)} & \text{if } A = A_k \ , \\ m(\emptyset) \prod_{k=1}^{n} (1 + \frac{m(A_k)}{m(\emptyset)}) & \text{if } A = \Omega \ , \\ 1 & \text{otherwise.} \end{cases} \tag{32}$$

Proof 1. *Focal sets of $\overline{m}$ are Ω, $\overline{A_1}$, $\overline{A_2}$, $\ldots$, $\overline{A_n}$, and possibly $\emptyset$, such that $\overline{A_i} \cap \overline{A_j} = \emptyset$ for all $i, j \in \{1, \ldots, n\}$. Consequently, from Proposition 2, the conjunctive weight function $\overline{w}$ associated with $\overline{m}$ is given by:*

$$\overline{w}(A) = \begin{cases} \frac{\overline{m}(\Omega)}{\overline{m}(\overline{A_k})+\overline{m}(\Omega)} & \text{if } A = \overline{A_k} \ , \\ \overline{m}(\Omega) \prod_{k=1}^{n} (1 + \frac{\overline{m}(\overline{A_k})}{\overline{m}(\Omega)}) & \text{if } A = \emptyset \ , \\ 1 & \text{otherwise.} \end{cases} \tag{33}$$

Then:

$$v(A) = \overline{w}(\overline{A}) = \begin{cases} \frac{\overline{m}(\Omega)}{\overline{m}(\overline{A_k})+\overline{m}(\Omega)} & \text{if } \overline{A} = \overline{A_k} \ , \\ \overline{m}(\Omega) \prod_{k=1}^{n} (1 + \frac{\overline{m}(\overline{A_k})}{\overline{m}(\Omega)}) & \text{if } \overline{A} = \emptyset \ , \\ 1 & \text{otherwise,} \end{cases} \tag{34}$$

and, as $\overline{m}(A) = m(\overline{A})\ \forall A$, Equation (32) is obtained.

□

Remark 3. *Proof 1 also implies, from Remark 2, that if m has only one focal element in addition to $\emptyset$ and Ω, Equation (32) is still valid.*

3. Extending the contextual discounting

In this section, the contextual discounting operation on a coarsening is extended to any subsets of the frame of discernment, and a general formulation linked with the canonical disjunctive decomposition of a BBA is developed.

According to the previous definitions (22) and (24), BBAs m_ℓ, $\ell \in \{1,\ldots,L\}$, defined in (21) by $m_\ell(\emptyset)=\beta_\ell$ and $m_\ell(\theta_\ell)=\alpha_\ell$, can be denoted $\theta_{\ell_{\beta_\ell}}$ or θ_{β_ℓ} in a simple way.

From (20) and (25), the contextual discounting on a coarsening $\Theta = \{\theta_1,\ldots,\theta_L\}$ of Ω of a subnormal BBA m is thus defined by:

$$\begin{aligned} {}^{\boldsymbol{\alpha}}_{\Theta}m &= m \,\circledcup\, \theta_{\beta_1} \circledcup \ldots \circledcup \theta_{\beta_L} \\ &= \circledcup_{A\supset\emptyset} A_{v(A)} \circledcup \theta_{\beta_1} \circledcup \ldots \circledcup \theta_{\beta_L} \ . \end{aligned}$$

In particular:

- The classical discounting (15) of a subnormal BBA $m = \circledcup_{A\supset\emptyset} A_{v(A)}$ is defined by:
$$^{\alpha}m = \Omega_{\beta v(\Omega)} \circledcup_{\Omega\supset A\supset\emptyset} A_{v(A)} \ ; \tag{35}$$
- The contextual discounting (Remark 1) of a subnormal BBA $m = \circledcup_{A\supset\emptyset} A_{v(A)}$ is defined by:
$$^{\boldsymbol{\alpha}}m = \circledcup_{\omega_k\in\Omega} \{\omega_k\}_{\beta_k v(\{\omega_k\})} \circledcup_{A\subset\Omega, |A|>1} A_{v(A)} \ . \tag{36}$$

These discounting operations can then be viewed as particular cases of a more general correction mechanism defined by:

$$^{\boldsymbol{\alpha}\cup}m = \circledcup_{A\supset\emptyset} A_{\beta_A v(A)}, \tag{37}$$

where $\beta_A \in [0,1]$ for all $A \neq \emptyset$ and $\boldsymbol{\alpha}$ is the vector $\{\alpha_A\}_{A\neq\emptyset}$.

In [13], the interpretation of each β_A has been given only in the case where the union of the subsets A forms a partition of Ω, β_A being interpreted as the degree of belief held by the agent regarding the fact that the source is reliable, knowing that the value searched belongs to A.

Instead of considering (18), let us now suppose that agent Ag holds beliefs regarding the reliability of the source, conditionally on each subset A of Ω:

$$\begin{cases} m^{\mathcal{R}}_{Ag}[A](\{R\}) &= 1-\alpha_A = \beta_A \\ m^{\mathcal{R}}_{Ag}[A](\mathcal{R}) &= \alpha_A \ , \end{cases} \tag{38}$$

where $\alpha_A \in [0,1]$.

In the same way as in Section 2.6, the knowledge held by agent Ag, based on the information m^{Ω}_{S} from a source and on metaknowledge $m^{\mathcal{R}}_{Ag}$ (38) regarding the reliability of this source, can be computed as follows:

$$m^{\Omega}_{Ag}[m^{\Omega}_S, m^{\mathcal{R}}_{Ag}] = \left(\circledcap_{A\subseteq\Omega} m^{\mathcal{R}}_{Ag}[A]^{\Uparrow\Omega\times\mathcal{R}} \circledcap m^{\Omega}_{Ag}[\{R\}]^{\Uparrow\Omega\times\mathcal{R}} \right)^{\downarrow\Omega} . \tag{39}$$

Proposition 4. *The BBA m^{Ω}_{Ag} resulting from (39) only depends on m^{Ω}_{S} and the vector $\boldsymbol{\alpha} = \{\alpha_A\}_{A\subseteq\Omega}$. It is equal to the disjunctive combination of m^{Ω}_{S} with a BBA m^{Ω}_{0} defined by:*

$$m^{\Omega}_{0}(C) = \prod_{\cup A = C} \alpha_A \prod_{\cup B = \overline{C}} \beta_B, \quad \forall C \subseteq \Omega. \tag{40}$$

Proof 2. *See Appendix A.1.* □

Like in the case of contextual discounting operations considered in Section 2.6, BBA m^{Ω}_{0} defined in Proposition 4 admits a simple decomposition described in the following proposition.

Proposition 5. *The BBA m^{Ω}_{0} defined in Proposition 4 can be rewritten as:*

$$m^{\Omega}_{0} = \textcircled{\cup}_{A\supset\emptyset} A_{\beta_A} \ . \tag{41}$$

Proof 3. *Directly from (40) and the definition (7) of the DRC.* □

From (41), the contextual discounting resulting from (39) of a subnormal BBA $m = \textcircled{\cup}_{A\supset\emptyset} A_{v(A)}$ is then defined by:

$$\textcircled{\cup}_{A\supset\emptyset} A_{v(A)} \textcircled{\cup}_{A\supset\emptyset} A_{\beta_A} = \textcircled{\cup}_{A\supset\emptyset} A_{\beta_A v(A)} = {}^{\boldsymbol{\alpha}\cup}m \ . \tag{42}$$

Correction mechanism ${}^{\boldsymbol{\alpha}\cup}m$ is then the general formulation for a contextual discounting on any subsets of Ω.

The following theorem sums up the contextual discounting operation in its general formulation.

Theorem 1. *On the one hand, agent Ag receives an information m from a source S.*

On the other hand, agent Ag knows that this source is reliable with a degree β_A in different contexts A of Ω, which means that the source is reliable with a degree β_A knowing that the true answer to the question Q of interest belongs to A. Let us note $\mathcal{A}$ the set containing these contexts.

Then, agent Ag's mass function is given by the contextual discounting ${}^{\boldsymbol{\alpha}}m$ of m defined by:

$${}^{\boldsymbol{\alpha}}m = m \ \textcircled{\cup}_{A\in\mathcal{A}} A_{\beta_A} \ . \tag{43}$$

Moreover, if m is subnormal then:

$$\begin{aligned} {}^{\boldsymbol{\alpha}}m &= \textcircled{\cup}_{A\supset\emptyset} A_{v(A)} \ \textcircled{\cup}_{A\in\mathcal{A}} A_{\beta_A} \ , \\ &= \textcircled{\cup}_{A\notin\mathcal{A}} A_{v(A)} \ \textcircled{\cup}_{A\in\mathcal{A}} A_{\beta_A v(A)} \ . \end{aligned} \tag{44}$$

Example 1. *Let us consider a two-lane road section, the question of interest concerning the number of lanes where the traffic is flowing freely. Frame of discernment Ω is equal to $\{\omega_0, \omega_1, \omega_2\}$ with:*

- ω_0 *standing for "0 lane is free": both are blocked,*
- ω_1 *meaning "1 lane is free": one is blocked,*
- ω_2 *signifying "2 lanes are free": traffic is flowing freely on both lanes.*

A source S provides information regarding the traffic on this road section. For instance:

$$\begin{cases} m(\emptyset) & = & .1 \ , \\ m(\{\omega_0\}) & = & .8 \ , \\ m(\Omega) & = & .1 \ . \end{cases} \tag{45}$$

On the other side, you know that the source is very reliable in case of heavy traffic (situation $\{\omega_0, \omega_1\}$), and less reliable when the traffic is rather light (situation $\{\omega_1, \omega_2\}$). Formally, let us suppose that $\beta_{\{\omega_0,\omega_1\}} = .8$ and $\beta_{\{\omega_1,\omega_2\}} = .6$.

Contextual discounting of m is then given by:

$$^{\boldsymbol{\alpha}}m = m \circledcirc \{\omega_0, \omega_1\}_{.8} \circledcirc \{\omega_1, \omega_2\}_{.6} \ . \tag{46}$$

From Remark 3, the disjunctive weight function v associated with m can be computed in the following manner:

$$\begin{cases} v(\{\omega_0\}) & = & \frac{m(\emptyset)}{m(\emptyset)+m(\{\omega_0\})} = \frac{1}{9} \ , \\ v(\Omega) & = & m(\emptyset)(1 + \frac{m(\{\omega_0\})}{m(\emptyset)}) = .1(1 + \frac{.8}{.1}) = .9 \ . \end{cases} \tag{47}$$

Contextual discounting of m knowing $\beta_{\{\omega_0,\omega_1\}} = .8$ and $\beta_{\{\omega_1,\omega_2\}} = .6$ is then given by:

$$\begin{aligned} ^{\boldsymbol{\alpha}}m &= m \circledcirc \{\omega_0, \omega_1\}_{.8} \circledcirc \{\omega_1, \omega_2\}_{.6} \\ &= \{\omega_0\}_{\frac{1}{9}} \circledcirc \Omega_{.9} \circledcirc \{\omega_0, \omega_1\}_{.8} \circledcirc \{\omega_1, \omega_2\}_{.6} \ , \end{aligned} \tag{48}$$

which can be also written by definition from (24) in the following manner:

$$\begin{aligned} ^{\boldsymbol{\alpha}}m &= \begin{cases} \emptyset & \mapsto 1/9 \\ \{\omega_0\} & \mapsto 8/9 \end{cases} \circledcirc \begin{cases} \emptyset & \mapsto .9 \\ \Omega & \mapsto .1 \end{cases} \circledcirc \begin{cases} \emptyset & \mapsto .8 \\ \{\omega_0, \omega_1\} & \mapsto .2 \end{cases} \circledcirc \begin{cases} \emptyset & \mapsto .6 \\ \{\omega_1, \omega_2\} & \mapsto .4 \end{cases} \\ &= \begin{cases} \emptyset & \mapsto .048 \\ \{\omega_0\} & \mapsto .384 \\ \{\omega_0, \omega_1\} & \mapsto .108 \\ \{\omega_1, \omega_2\} & \mapsto .032 \\ \Omega & \mapsto .428 \end{cases} \end{aligned} \tag{49}$$

4. Discussion

4.1. About the mass transfer during a contextual discounting

As it can be observed in the previous example, the contextual discounting allows the transfer of masses on intermediate subsets between focal sets and the frame of discernment Ω. More precisely (cf Equation (43)), contextual discounting consists in transferring each mass to its union with subsets not precisely known by the source.

The following example illustrates a case where a sensor is totally reliable if an object is of a certain type, and not reliable for another type.

Example 2. *Let us consider a sensor in charge of the recognition of two types of objects:* $\Omega = \{a, b\}$*.*

The sensor knows very well how to recognize objects of type a*, whereas it is not the case for objects of type* b*. Which means that:*

- *if an object is of type* a*, the sensor will recognize it;*
- *if an object is of type* b*, the sensor will hesitate and make mistakes.*

For instance, a possible confusion matrix for such a sensor is represented in Table 1.

Table 1: Confusion matrix associated with source S.

truth \ decision	a	b
a	10	0
b	5	5

A contextual discounting knowing that $\beta_{\{a\}} = 1$ *and* $\beta_{\{b\}} = 0$ *(S totally reliable in context* $\{a\}$*, not reliable in context* $\{b\}$*) applied on a mass function* m *provided by* S *is given by:*

$$^{\boldsymbol{\alpha}}m = m \circledcirc \{a\}_1 \circledcirc \{b\}_0 = m \circledcirc \{b\}_0 \ . \tag{50}$$

In particular:

- *if* $m(\{a\}) = 1$ *then* $^{\boldsymbol{\alpha}}m(\{a, b\}) = 1$*,*
- *if* $m(\{b\}) = 1$ *then* $^{\boldsymbol{\alpha}}m(\{b\}) = 1$*.*

In other words, if the source says it is an object of type a: it is an object of type a or an object the source does not recognize. And, in the particular case where the source says it is an object of type b, and the source knows very well the other types of object: it remains that the object is of type b. If it had been an object of type a, the source would have said it, because objects of type a are very well recognized by the source.

4.2. On the notion of reliability

In the simple Example 2, we have a situation where when the source decides b: it is indeed b, and when the source decides a: the truth is a or b. However, the source is totally reliable for a and not for b. The notion of reliability introduced in this article has then to be clearly distinguished from a different notion of reliability which would be linked with the reliability of the decision made by a source.

In the contextual discounting, the definition of the reliability (38) is given conditionally on Ω by:

$$m^{\mathcal{R}}_{Ag}[A](\{R\}) = \beta_A \ , \tag{51}$$

and not conditionally on results of a decision-making process:

$$m^{\mathcal{R}}_{Ag}[\text{“The source decides A”}](\{R\}) = \beta_A \ . \tag{52}$$

Let us remark that this second definition will imply to define the notion of conditioning on processes (even belief functions?) which is not known at present by the authors.

4.3. To be subnormal and non-dogmatic

In order to exploit simple expressions obtained with both conjunctive and disjunctive canonical decompositions, should each mass function be subnormal and non-dogmatic?

As already mentioned in [4], it may be argued that most (if not all) pieces of information provided in real-life applications are imperfect, and then the mass on the frame of discernment should be always strictly positive.

For instance, let us consider a coin tossing and a universe equal to $\{head, tail\}$. As remarked by Denœux, it is absolutely not certain in practice that the coin is perfectly balanced. An appropriate BBA may then be $m(Heads) = 0.5(1-\epsilon)$, $m(Tails) = 0.5(1-\epsilon)$ and $m(\Omega) = \epsilon$ for some small $\epsilon > 0$.

However, we can also add that when we are trying to model a complex real life problem, there are always some approximations, and then some doubts on the model are always possible, so the mass on the conflict should also be strictly positive: in real life, the coin can fall against a book and lands on edge, which may not have been considered.

Consequently, a more appropriate BBA may be $m(Heads) = 0.5(1-\epsilon_1)(1-\epsilon_2)$, $m(Tails) = 0.5(1-\epsilon_1)(1-\epsilon_2)$, $m(\Omega) = \epsilon_1(1-\epsilon_2)$ and $m(\emptyset) = \epsilon_2$ for some small $\epsilon_1, \epsilon_2 > 0$.

4.4. An other approach: combining discountings

Let us consider a separable BBA m, which means that $m = \bigcircledcap_{A\subset\Omega} A^{w(A)}$ with $w(A) \in [0,1]$ for all $A \subset \Omega$.

As recently exposed in [10], a classical discounting on each simple BBA $A^{w(A)}$ can be undertaken with a discount rate $\alpha_A \in [0,1]$, the result being $A^{\beta_A w(A)+\alpha_A}$.

The discounted simple BBAs can then be conjunctively combined which yields to the following discounting operation $\bigcircledcap_{A\subset\Omega} A^{\beta_A w(A)+\alpha_A}$. This operation, restricted to separable BBAs, is different from a contextual operation. The conjunctive combination of discounted BBAs is not a discounting of the BBAs combination in general.

However, as developed in the next section, we can remark that the dual of the contextual discounting operation, which is nevertheless a reinforcement, has a close formulation.

5. A new reinforcement process

In a similar way, a correction mechanism for a non-dogmatic BBA m can be defined from the conjunctive decomposition of m as follows:

$$^{\boldsymbol{\alpha}\cap}m = \bigcircledcap_{A\subset\Omega} A^{\beta_A w(A)} \; ; \tag{53}$$

where $\forall A \subset \Omega, \beta_A \in [0,1]$, and $\boldsymbol{\alpha}$ is the vector $\{\alpha_A\}_{A\subset\Omega}$.

The smaller is the uncertain weight, the higher is the mass on A. This process allows then the reinforcement of a BBA m.

Correction mechanisms $^{\boldsymbol{\alpha}\cap}m$ (37) and $^{\boldsymbol{\alpha}\cup}m$ (53) are related in the following way.

Let us consider a subnormal BBA m, $\overline{m}$ is then non-dogmatic:

$$^{\boldsymbol{\alpha}\cap}\overline{m} = \bigcircledcap_{A\subset\Omega} A^{\beta_A \overline{w}(A)} \; . \tag{54}$$

Then:

$$\begin{aligned}
\overline{^{\boldsymbol{\alpha}\cap}\overline{m}} &= \overline{\bigcircledcap_{A\subset\Omega} A^{\beta_A \overline{w}(A)}} \\
&= \bigcircledcup_{A\subset\Omega} \overline{A^{\beta_A \overline{w}(A)}} \\
&= \bigcircledcup_{A\supset\emptyset} A_{\beta_A \overline{w}(\overline{A})} \\
&= \bigcircledcup_{A\supset\emptyset} A_{\beta_A v(A)} \\
&= {}^{\boldsymbol{\alpha}\cup}m
\end{aligned} \tag{55}$$

These two correction mechanisms can thus be seen as belonging to a general family of correction mechanisms.

In a nutshell, if m is subnormal, a contextual discounting of m is given by:

$$^{\boldsymbol{\alpha}}m = \textcircled{\cup}_{A \supset \emptyset} A_{\beta_A v(A)} \ , \tag{56}$$

and, the negation of a contextual discounting of $\overline{m}$ defines a *dual reinforcement process*:

$$\overline{^{\boldsymbol{\alpha}}\overline{m}} = \overline{\textcircled{\cup}_{A \supset \emptyset} A_{\beta_A \overline{v}(A)}} = \textcircled{\cap}_{A \subset \Omega} A^{\beta_A w(A)} \ . \tag{57}$$

The application of this reinforcement process as well as its comparison with other correction mechanisms [5, 15] has been left for future researches.

6. Conclusion and future work

In this article, the contextual discounting operation of a belief function has been extended to any subsets, and a simple and practical expression, based on disjunctive combinations, to compute it has been given. This expression has highlighted the close relationship between contextual discounting and canonical disjunctive decomposition. The dual expression of this discounting, allowing one to strengthen a belief function, has also been exposed.

Future work will aim at testing it on real data. Likewise, it would also be interesting to automatically learn the coefficients of these correction mechanisms from data, as done for the classical and the contextual discounting operations [7, 13].

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Appendix A. Proofs

Appendix A.1. Proofs of Proposition 4

For each $A \subseteq \Omega$, the deconditioning of $m^{\mathcal{R}}_{Ag}[A]$ on $\Omega \times \mathcal{R}$ is given by:

$$m^{\mathcal{R}}_{Ag}[A]^{\Uparrow\Omega\times\mathcal{R}}(A \times \{R\} \cup \overline{A} \times \mathcal{R}) = \beta_A, \quad \text{(A.1)}$$
$$m^{\mathcal{R}}_{Ag}[A]^{\Uparrow\Omega\times\mathcal{R}}(\Omega \times \mathcal{R}) = \alpha_A. \quad \text{(A.2)}$$

With $A \neq B$:

$$(A \times \{R\} \cup \overline{A} \times \mathcal{R}) \cap (B \times \{R\} \cup \overline{B} \times \mathcal{R}) = (A \cup B) \times \{R\} \cup \overline{(A \cup B)} \times \mathcal{R} \ .$$

Then:

$$\circledcap_{A\subseteq\Omega} m^{\mathcal{R}}_{Ag}[A]^{\Uparrow\Omega\times\mathcal{R}}(C \times \{R\} \cup \overline{C} \times \mathcal{R}) = \prod_{\cup D=\overline{C}} \alpha_D \prod_{\cup E=C} \beta_E, \quad \forall C \subseteq \Omega \ ,$$

or, by exchanging the roles of C and $\overline{C}$:

$$\circledcap_{A\subseteq\Omega} m^{\mathcal{R}}_{Ag}[A]^{\Uparrow\Omega\times\mathcal{R}}(\overline{C} \times \{R\} \cup C \times \mathcal{R}) = \prod_{\cup D=C} \alpha_D \prod_{\cup E=\overline{C}} \beta_E, \quad \forall C \subseteq \Omega \ .$$

It remains to combine conjunctively $m^{\Omega}_{Ag}[\{R\}]^{\Uparrow\Omega\times\mathcal{R}}$ and $\circledcap_{A\subseteq\Omega} m^{\mathcal{R}}_{Ag}[A]^{\Uparrow\Omega\times\mathcal{R}}$ which have focal sets of the form $B \times \{R\} \cup \Omega \times \{NR\}$ and $\overline{C} \times \{R\} \cup C \times \mathcal{R}$, respectively, with $B, C \subseteq \Omega$. The intersection of two such focal sets is:

$$(\overline{C} \times \{R\} \cup C \times \mathcal{R}) \cap (B \times \{R\} \cup \Omega \times \{NR\}) = B \times \{R\} \cup C \times \{NR\} \ ,$$

and it can be obtained only for a particular choice of B and C. Then:

$$\circledcap_{A\subseteq\Omega} m^{\mathcal{R}}_{Ag}[A]^{\Uparrow\Omega\times\mathcal{R}} \circledcap m^{\Omega}_{Ag}[\{R\}]^{\Uparrow\Omega\times\mathcal{R}}(B \times \{R\} \cup C \times \{NR\}) = \left[\prod_{\cup D=C} \alpha_D \prod_{\cup E=\overline{C}} \beta_E\right] m^{\Omega}_{S}(B) \ . \quad \text{(A.3)}$$

Finally, the marginalization of this BBA on Ω is given for all subsets A of Ω, by:

$$^{\boldsymbol{\alpha}}m(A) = \sum_{B\cup C=A} \left[\prod_{\cup D=C} \alpha_D \prod_{\cup E=\overline{C}} \beta_E \right] m_S^{\Omega}(B) \ . \qquad \text{(A.4)}$$

Let us note that the above proof has many similarities with proofs presented in [13, Sections A.1 and A.3].

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