0-1 Combinatorial optimization problems with qualitative and uncertain profits

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Abstract. Recent works have studied 0-1 combinatorial optimization problems where profits of items are measured on a qualitative scale such as "low", "medium" and "high". In this study, we extend this body of work by allowing these profits to be both qualitative and uncertain. In the first step, we use probability theory to handle uncertainty. In the second step, we use evidence theory to handle uncertainty. We combine their approaches with approaches in decision making under uncertainty that utilize the Maximum Expected Utility principle and generalized Hurwicz criterion, to compare solutions. We show that under probabilistic uncertainty and a special case of evidential uncertainty where the focal sets are rectangles, the task of identifying the non-dominated solutions can be framed as solving a multi-objective version of the considered problem. This result mirrors that of the case of qualitative profits with no uncertainty.

Keywords: Combinatorial optimization · Multiple objective optimization · Belief function · Decision making under uncertainty.

1 Introduction

A 0-1 combinatorial optimization problem (01COP) can be seen as the selection of a subset of items from a given collection of subsets, with the objective of maximizing the total profits of the chosen items. Usually, these values are represented quantitatively using a vector in $\mathbb{R}^n_{\geq 0}$.

In many real-life situations, accurately assessing the exact numerical values of items can be challenging due to limited information availability. It is often much easier to make qualitative comparisons between these values. As an example, although most people will find it hard to determine the exact weights of a laptop and a smartphone, they can certainly say that the laptop is heavier.

Given an order between items, a mapping from the items to real values is called a representation of this order if it maintains the empirical relations among the items. The matroid optimization problem [6] is a special case of 01COPs in which the optimality of solutions is independent of the choice of representation. However, in measurement theory [7], it is known that in most cases, the optimality of solutions does depend crucially on the choice of representation, *i.e.,* a solution is optimal for one representation but is not optimal for other representation.

Recently, in [9], the authors studied the Knapsack problem (KP) where profits of items are measured in a qualitative scale such as "low", "medium", "high". To deal with the above-mentioned issue, they provided a new way to compare solutions *i.e.,* a solution x is preferred to a solution y if x has higher profit than y for any representation of the qualitative scale. They called a solution x non-dominated if there is no other solution which is strictly preferred to x and proceeded to enumerate all non-dominated solutions. In [9], they also observed a strong connection between KP with qualitative profits and multi-objective KP and this link is studied in greater details for 01COPs in a very recent paper [5].

In this paper, we further extend the works [5,9] by allowing profits of items to be both qualitative and uncertain. First, we utilize the traditional probabilistic framework to model uncertainty. Subsequently, following recent work encompassing a wide class of optimization problems [11], of which the 01COP is a subclass, we employ evidence theory [10], which is more general than probability theory, to represent uncertainty. It is worth noting that such evidential uncertainty, *i.e.,* belief functions on ordinal variables, *e.g.* on the profit of some items, can be obtained from statistical data using, for instance, the approach described in [1].

In both cases, we adopt approaches in decision-making under uncertainty that utilize, respectively, the Maximum Expected Utility principle and the generalized Hurwicz criterion, to compare solutions, which still results in the concept of non-dominated solutions. Lastly, we show that under probabilistic uncertainty and a special case of evidential uncertainty where the so-called focal sets are rectangles, finding non-dominated solutions can be framed as solving a multi-objective version of the considered problem, which is similar to that of the case with no uncertainty.

The rest of this paper is organized as follows. Section 2 presents necessary background material. Section 3 quickly summarizes the works [5,9]. Section 4 presents the main results of the paper, where uncertainty is added and treated. The paper ends with a conclusion.

2 Preliminaries

In this section, we present necessary background for the rest of the paper. Throughout the paper, we denote by $[m]$ the set $\{1, \dots, m\}$.

2.1 Evidence theory

Let $\Omega = \{\omega_1, \dots, \omega_q\}$ be the set, called frame of discernment, of all possible values of a variable ω. In evidence theory [10], partial knowledge about the true (unknown) value of ω is represented by a mapping $m : 2^{\Omega} \mapsto [0, 1]$ called mass function and such that $\sum_{A \subseteq \Omega} m(A) = 1$ and $m(\emptyset) = 0$, where mass $m(A)$ quantifies the amount of belief allocated to the fact of knowing only that $\omega \in A$. A subset $A \subseteq \Omega$ is called a focal set of m if $m(A) > 0$. If all focal sets of m are singletons, then m is equivalent to a probability distribution. The mass function m gives rise to *belief* and *plausibility* measures defined as follows, respectively:

$$Bel(A) = \sum_{B \subseteq A} m(B) \text{ and } Pl(A) = \sum_{B \cap A \neq \emptyset} m(B), \ \forall A \subseteq \Omega. \tag{1}$$

We can also consider the set $\mathcal{P}(m)$ of all probability measures on Ω which are compatible with m, defined as $\mathcal{P}(m) = \{P : P(A) \geq Bel(A) \ \forall A \subseteq \Omega\}$.

2.2 Multi-objective optimization problem

A multi-objective optimization problem can be written as

$$\max \ \{f_1(x), \ldots, f_m(x)\} \tag{2}$$
$$x \in \mathcal{X}. \tag{3}$$

The notion of Pareto dominance is usually used for multi-objective optimization problems. The feasible solution x is said to Pareto dominate the feasible solution y, denoted by $x \succ_{Pareto} y$ if

$$f_i(x) \geq f_i(y) \ \forall i \in [m] \text{ and } \exists j \in [m] \text{ such that } f_j(x) > f_j(y). \tag{4}$$

As the objectives (2-3) are typically conflicting, there is usually no solution x that simultaneously maximizes all $f_i(x)$. Instead, we seek to find all so-called efficient feasible solutions of (2-3), defined as:

$$x \in \mathcal{X} \text{ such that } \nexists y \in \mathcal{X}, y \succ_{Pareto} x. \tag{5}$$

We refer to the book [4] for a comprehensive discussion on this subject.

2.3 0-1 Combinatorial optimization problem

A general 0-1 Combinatorial Optimization Problem (01COP) can be expressed as follows. Let $\mathcal{S}$ be a set of n items. Each item i has a profit r_i, represented as a vector $r \in \mathbb{R}^n_+$. The profit of a subset of $\mathcal{S}$ is obtained by summing the profits of the items within it. The goal of the decision-maker is to find a subset having maximum profit among a predefined collection $\mathcal{X} \subseteq 2^{\mathcal{S}}$ of subsets of $\mathcal{S}$. This problem can be modeled using a binary vector $x \in \{0,1\}^n$, where each element x_i indicates whether item i is included in the subset (1) or not (0). The 01COP can then be written as:

$$\begin{aligned} \max \ & r^T x \\ & x \in \mathcal{X} \subseteq \{0,1\}^n. \end{aligned} \tag{01COP}$$

The Knapsack problem (KP) is one of the most important problems in the class 01COP, which will serve as a running example throughout the paper. It is defined as follows.

Example 1 (The 0-1 knapsack problem (01KP)). Suppose a company has a budget of W and needs to choose which items to manufacture from a set of n possible items, each with a production cost of w_i and fixed profit of r_i. The 01KP involves selecting a subset of items to manufacture that maximizes the total profit while keeping the total production costs below W. The 01KP can be formulated as

$$\begin{aligned} \max \sum_{i=1}^{n} r_i x_i & \\ \sum_{i=1}^{n} w_i x_i \leq W & \\ x_i \in \{0,1\} \quad i \in [n]. & \end{aligned} \tag{01KP}$$

3 01COP with qualitative levels

In this section, we quickly summarize the works in [5,9]. In many applications, we can only express profits of items on a finite scale of qualitative levels. More precisely, let $(\mathcal{L}, \prec) = \{l_1, \ldots, l_k\}$ be a fixed scale with k levels $l_1 \prec \ldots \prec l_k$. Profits of items are then represented by a fixed vector $r \in \mathcal{L}^n$.

Example 2. Consider 5 items whose profits are measured in the qualitative scale $\mathcal{L} = \{\text{"}low\text{"}, \text{"}medium\text{"}, \text{"}high\text{"}\}$. Their profits are recorded by a vector $r = \{\text{"}low\text{"}, \text{"}medium\text{"}, \text{"}high\text{"}, \text{"}high\text{"}, \text{"}medium\text{"}\}$ in $\mathcal{L}^5$.

In general, the set of items can include absolutely unprofitable items, resulting in the qualitative levels set $\mathcal{L}$ having a level that signifies "no profit at all". However, the decision-maker can always remove all such items from the outset. Due to this, we exclude the case involving the "no profit at all" level. A mapping $v : \mathcal{L} \to \mathbb{R}_{>0}$ is called a representation of $\mathcal{L}$ if

$$\forall i, j,\ l_i \prec l_j \Leftrightarrow v(l_i) < v(l_j). \tag{6}$$

We denote by $\mathcal{V}$ the set of all representations of $\mathcal{L}$. Note that $\mathcal{V}$ is identified with a subset of $\mathbb{R}^k_{>0}$, that is

$$\mathcal{V} := \left\{ v \in \mathbb{R}^k_{>0} : v_{i+1} > v_i,\ \forall i \in [k-1] \right\}. \tag{7}$$

In the following, to simplify the notation, we will use v_i instead of $v(l_i)$ for a representation v.

The rank cardinality vector of an $x \in \mathcal{X}$ is defined as:

$$g(x) = (g_1(x), \ldots, g_k(x)) \tag{8}$$

where $g_j(x) = |\{i : x_i = 1 \text{ and } r_i = l_j\}|$. Hence, the j-th component of $g(x)$ is nothing but the total number of items in x with profit level l_j.

Let $v(x)$ be the profit of x with respect to a representation $v \in \mathcal{V}$. By definition,

$$v(x) = \sum_{i=1}^{n} x_i v(r_i). \tag{9}$$

We can also compute $v(x)$ via its rank cardinality vector as

$$v(x) = \sum_{i=1}^{k} g_i(x) v_i. \tag{10}$$

The preferences between feasible solutions crucially depend on the choice of v as illustrated in the next example.

Example 3. Consider the following KP with 5 items and $W = 6$. The profits and weights of items are given in Table 1. Let $x = (1, 1, 1, 0, 0)$ (selecting items 1,2

items	1	2	3	4	5
w	2	2	2	3	4
r	l_1	l_2	l_2	l_3	l_3

Table 1: Profits and weights of items

and 3) and $y = (0, 0, 1, 1, 0)$ (selecting items 3 and 4) be two feasible solutions. If a representation v is chosen such that $v(l_1) = 2, v(l_2) = 3, v(l_3) = 4$, x is preferred to y as $v(x) = 8 > v(y) = 7$. However, if v is chosen such that $v(l_1) = 2, v(l_2) = 3, v(l_3) = 6$, y is preferred to x as $v(x) = 8 < v(y) = 9$.

To avoid the issue encountered in Example 3, the preference between feasible solutions is defined as follows in [9]:

Definition 1. *Let $x, y \in \mathcal{X}$ be two feasible solutions. Then,*

1. *x weakly dominates y, denoted by $x \succeq y$, if for every $v \in \mathcal{V}$, it holds that $v(x) \geq v(y)$.*
2. *x dominates y, denoted by $x \succ y$, if x weakly dominates y and there exists $v^* \in \mathcal{V}$ such that $v^*(x) > v^*(y)$.*
3. *$x^* \in \mathcal{X}$ is called efficient or non-dominated, if there does not exist any $x \in \mathcal{X}$ such that $x \succ x^*$.*

In [9], it is shown that the relation $\succeq$ in Definition 1 is a preorder, *i.e.,* it is reflexive and transitive. At first glance, Definition 1 appears to require checking every representation of $\mathcal{L}$ to determine the dominance relation between two feasible solutions. However, there exists a rapid and straightforward test based on the following key result.

Lemma 1 (see [9]). *Let x, y be two feasible solutions. We have $x \succeq y$ iff $\sum_{i=j}^{k} g_i(x) \geq \sum_{i=j}^{k} g_i(y)$ for all $j \in [k]$.*

Lemma 1 is of great importance as it establishes the link between 01COP with qualitative levels and multi-objective optimization. This link was first observed for the KP in [9] and has been systematically studied in [5] for 01COP. Indeed, from Lemma 1 and material in Section 2.2, it is easy to see that x^* is an efficient solution according to Definition 1 if and only if it is an efficient solution of the following problem:

$$\max \left\{ \sum_{i=1}^{k} g_i(x), \sum_{i=2}^{k} g_i(x), \ldots, \sum_{i=k}^{k} g_i(x) \right\} \tag{11}$$

$$x \in \mathcal{X} \tag{12}$$

Note that Problem (11-12) can be rewritten so that its objective functions are linear. Indeed, for each $i \in [k]$, define vector $c^i \in \{0,1\}^n$ as follow:

$$c^i_j = 0 \text{ if } r_j \neq l_i \text{ and } c^i_j = 1 \text{ otherwise.} \tag{13}$$

Hence, c^i is nothing but a vector that records positions of the qualitative level l_i in r, and thus we have $(c^i)^T x = g_i(x)\ \forall i \in [k]$. Problem (11-12) is then rewritten as

$$\max \left\{ (\sum_{i=1}^{k} c^i)^T x, (\sum_{i=2}^{k} c^i)^T x, \ldots, (c^k)^T x \right\} \tag{14}$$

$$x \in \mathcal{X} \tag{15}$$

Therefore, methods in multi-objective optimization can be readily applied to find efficient solutions of Problem 01COP with qualitative profits.

Example 4 (Example 3 continued). In the KP in Example 3, the position vectors are $c^1 = (1,0,0,0,0), c^2 = (0,1,1,0,0)$, and $c^3 = (0,0,0,1,1)$. To find non-dominated solutions according to Definition 1 of the KP, we need to solve the following multi-objective optimization problem:

$$\max \ \{x_1 + x_2 + x_3 + x_4 + x_5,\ x_2 + x_3 + x_4 + x_5,\ x_4 + x_5\} \tag{16}$$

$$x \in \{0,1\}^5 : 2x_1 + 2x_2 + 2x_3 + 3x_4 + 4x_5 \leq 6 \tag{17}$$

4 01COPs with uncertain qualitative profits

In this section, we extend the approaches presented in [5,9] to address the case where profits are uncertain and qualitative. Note that Lemma 1 is originally proved for rank cardinality vectors in $\mathbb{Z}^k_{\geq 0}$ (as shown in the original proof in [9] or a simplified version in [5]). In our extended setting, we will require a generalized version of this lemma that can accommodate vectors in $\mathbb{R}^k_{\geq 0}$. Therefore, we present the generalized version here. Note that the proof in [9] can be easily modified to fit the generalized version. However, we present a new proof of Lemma 1 based on the duality theory of linear programming, which conceptually differs from the proofs presented in [5,9].

Let A be a $m \times n$ matrix. Let us recall from linear programming that if the primal problem is

$$\min \left\{ v^T b : v^T A \geq c, v \in \mathbb{R}^m_{\geq 0} \right\}. \tag{18}$$

then its dual problem is

$$\max\left\{c^T u : Au \le b, u \in \mathbb{R}^n_{\ge 0}\right\}. \tag{19}$$

Lemma 2. *Let $g(x), g(y)$ be two vectors in $\mathbb{R}^k_{\ge 0}$. Then,*

$$\sum_{i=1}^{k} g_i(x)v_i \ge \sum_{i=1}^{k} g_i(y)v_i \ \forall v \in \mathcal{V} \Leftrightarrow \sum_{i=j}^{k} g_i(x) \ge \sum_{i=j}^{k} g_i(y) \ \forall j \in [k]. \tag{20}$$

Proof. Let $f(v) = \sum_{i=1}^{k}(g_i(x) - g_i(y))v_i$. Then,

$$\sum_{i=1}^{k} g_i(x)v_i \ge \sum_{i=1}^{k} g_i(y)v_i \ \forall v \in \mathcal{V} \Leftrightarrow f(v) \ge 0 \ \forall v \in \mathcal{V} \tag{21}$$

$$\Leftrightarrow f(v) \ge 0 \ \forall v \in \overline{\mathcal{V}} := \left\{v \in \mathbb{R}^k_{\ge 0} : v_{i+1} \ge v_i, \ \forall i \in [k-1]\right\}, \tag{22}$$

since f is continuous and $\overline{\mathcal{V}}$ is the closure of $\mathcal{V}$. Let $z^* = \min\left\{f(v) : v \in \overline{\mathcal{V}}\right\}$. So z^* is the optimal value of the linear programming problem (P):

$$\begin{aligned} &\min \ \sum_{i=1}^{k}(g_i(x) - g_i(y))v_i \\ &v_1 \ge 0 \\ &v_{i+1} - v_i \ge 0, \ \forall i \in [k-1] \end{aligned} \tag{P}$$

Note that $f(v) \ge 0 \ \forall v \in \overline{\mathcal{V}}$ iff $z^* \ge 0$. Furthermore, $z^* \ge 0$ iff Problem (P) is bounded, *i.e.*, $z^* \ne -\infty$.

Indeed, for the sake of contradiction, suppose that Problem (P) is bounded, and yet there exists a v^* such that $f(v^*) = z^* < 0$. For any positive scalar λ, we have $\lambda v^* \in \overline{\mathcal{V}}$, and thus $f(\lambda v^*) = \lambda z^* < z^*$, which contradicts the optimality of z^*. By duality, we have $z^* \ne \infty$ iff the dual Problem (D) has the finite optimal value, or in this case Problem (D) is feasible:

$$\begin{aligned} &\max \ 0^T u \\ &u_i - u_{i+1} \le g_i(x) - g_i(y), \ \forall i \in [k-1] \\ &u_k \le g_k(x) - g_k(y) \\ &u \ge 0. \end{aligned} \tag{D}$$

It is easy to see that Problem (D) is feasible iff $\sum_{i=j}^{k} g_i(x) \ge \sum_{i=j}^{k} g_i(y) \ \forall j \in [k]$. Hence, we get the desired result. □

4.1 Under probabilistic uncertainty

In this section, we assume that information about the qualitative levels of items is given by a probability distribution P on a subset $\mathcal{R}$ of $\mathcal{L}^n$. Each $r \in \mathcal{R}$ is called a scenario. Given $v \in \mathcal{V}$, let $v^r(x)$ be the profit of x under scenario r. The expected utility of a feasible solution $x \in \mathcal{X}$ with respect to v is defined as:

$$E_P^v(x) := \sum_{r \in \mathcal{R}} P(r) v^r(x) \tag{23}$$

According to the Maximum Expected Utility principle [8], it is reasonable to compare solutions based on their expectations. Furthermore, for similar reasons as those that lead to Definition 1, i.e., the preference between two solutions x and y should not depend on the choice of v, we define, for any $x, y \in \mathcal{X}$,

$$x \succeq_P y \text{ iff } E_P^v(x) \geq E_P^v(y) \ \forall v \in \mathcal{V}. \tag{24}$$

Let $g_i^r(x)$ be the number of items in x with qualitative level l_i under scenario r. The next result shows how to check whether $x \succeq_P y$.

Proposition 1. *$x \succeq_P y \Leftrightarrow \sum_{i=j}^k \overline{g}_i(x) \geq \sum_{i=j}^k \overline{g}_i(y) \ \forall j \in [k]$, where $\overline{g}_i(x) := \sum_{r \in \mathcal{R}} P(r) g_i^r(x)$.*

Proof. By definition in Equation (23), we have

$$E_P^v(x) = \sum_{r \in \mathcal{R}} P(r) \sum_{i=1}^k g_i^r(x) v_i = \sum_{i=1}^k \left(\sum_{r \in \mathcal{R}} P(r) g_i^r(x) \right) v_i. \tag{25}$$

Equivalently,

$$E_P^v(x) = \sum_{i=1}^k \overline{g}_i(x) v_i. \tag{26}$$

Therefore, $x \succeq_P y \Leftrightarrow \sum_{i=1}^k \overline{g}_i(x) v_i \geq \sum_{i=1}^k \overline{g}_i(y) v_i \ \forall v \in \mathcal{V}$. The desired result follows by applying Lemma 2. ◻

Note that $\overline{g}_i(x)$ can be interpreted as the expected number of items in x with profit level i.

From Proposition 1, non-dominated solutions according to $\succeq_P$ are efficient solutions of the following problem:

$$\max \ \left\{ \sum_{i=1}^k \overline{g}_i(x), \sum_{i=2}^k \overline{g}_i(x), \ldots, \sum_{i=k}^k \overline{g}_i(x) \right\} \tag{27}$$

$$x \in \mathcal{X}. \tag{28}$$

Note that each objective of Problem (27-28) is still linear. Indeed, let $c^{ri} \in \{0,1\}^n$ be a vector that records positions of qualitative level l_i in scenario r, defined as:

$$c_j^{ri} = 0 \text{ if } r_j \neq l_i \text{ and } c_j^{ri} = 1 \text{ otherwise.} \tag{29}$$

Therefore,

$$g_i^r(x) = (c^{ri})^T x \tag{30}$$

and $\overline{g}_i(x) = \left(\sum_{r \in \mathcal{R}} P(r) c^{ri} \right)^T x$.

Example 5 (Example 3 continued). Assume now that the information about the profits of items in Example 3 are given by two scenarios r^1 and r^2 in Table 2 with $P(r^1) = 0.8$ and $P(r^2) = 0.2$. We can see that $c^{r^1 1} = (1,0,0,0,0), c^{r^1 2} = (0,1,1,0,0), c^{r^1 3} = (0,0,0,1,1), c^{r^2 1} = (0,0,1,0,1), c^{r^2 2} = (0,1,0,1,0)$ and $c^{r^2 3} = (1,0,0,0,1)$. For any feasible solution x, we have

items	1	2	3	4	5
w	2	2	2	3	4
r^1	l_1	l_2	l_2	l_3	l_3
r^2	l_3	l_2	l_1	l_2	l_1

Table 2: Profits of items under two scenarios

$$\overline{g}_1(x) = (0.8c^{r^1 1} + 0.2c^{r^2 1})^T x = 0.8x_1 + 0.2x_3 + 0.2x_5. \quad (31)$$

Similarly, $\overline{g}_2(x) = x_2 + 0.8x_3 + 0.2x_4$ and $\overline{g}_3(x) = 0.2x_1 + 0.8x_4 + 0.8x_5$. Hence, finding non-dominated solutions boils down to solving the following multi-objective KP.

$$\max \left\{ \begin{array}{c} x_1 + x_2 + x_3 + x_4 + x_5, \\ 0.2x_1 + x_2 + 0.8x_3 + x_4 + 0.8x_5, \\ 0.2x_1 + 0.8x_4 + 0.8x_5, \end{array} \right\} \quad (32)$$

$$x \in \{0,1\}^5 : 2x_1 + 2x_2 + 2x_3 + 3x_4 + 4x_5 \leq 6 \quad (33)$$

4.2 Under evidential uncertainty

A more general approach than the one in Section 4.1 is to use evidence theory to represent uncertainty. Let m be a mass function on a subset $\mathcal{R}$ of $\mathcal{L}^n$. Let $\mathcal{F}$ be the set of focal sets of m. Following [11], the lower and upper expected values of a feasible solution $x \in \mathcal{X}$ with respect to a $v \in \mathcal{V}$ are defined as:

$$\underline{E}^v(x) := \sum_{F \in \mathcal{F}} m(F) \min_{r \in F} v^r(x), \quad (34)$$

$$\overline{E}^v(x) := \sum_{F \in \mathcal{F}} m(F) \max_{r \in F} v^r(x). \quad (35)$$

For a fixed v, we may remark that the interval $\left[\underline{E}^v(x), \overline{E}^v(x)\right]$ is the range of $E^v_P(x)$ for all compatible probability measures P in $\mathcal{P}(m)$ [2].

As in [11], solutions can be compared according to the generalized Hurwicz criterion [2], defined by $H^v_\alpha(x) = \alpha\overline{E}^v(x) + (1-\alpha)\underline{E}^v(x)$ for some chosen optimism/pessimism degree $\alpha \in [0,1]$. Furthermore, as in Sections 3 and 4.1, we wish to compare solutions regardless of the choice of representation:

$$x \succeq^\alpha_{hu} y \text{ iff } H^v_\alpha(x) \geq H^v_\alpha(y) \ \forall v \in \mathcal{V}. \quad (36)$$

We first consider the case where the focal sets of m take a special form.

Rectangular focal sets A subset $F \subseteq \mathcal{L}^n$ is called a rectangle iff it can be expressed as the Cartesian product of sets, that is, $F = \times_{i=1}^n F^{\downarrow i}$, where $F^{\downarrow i} \subseteq \mathcal{L}$. Since $\mathcal{L}$ is a linear order, we can associate two scenarios RF, rF for each focal set F defined as:

$$RF_i = \max F^{\downarrow i} \text{ and } rF_i = \min F^{\downarrow i}, \ \forall i \in [n] \tag{37}$$

In this case, it is easy to compute $\overline{E}^v(x)$, $\underline{E}^v(x)$ for a given v as shown in the Proposition 2.

Proposition 2. *When focal sets of m are rectangles, for any $v \in \mathcal{V}$ we have*

$$\underline{E}^v(x) = \sum_{F \in \mathcal{F}} m(F) v^{rF}(x) \tag{38}$$

$$\overline{E}^v(x) = \sum_{F \in \mathcal{F}} m(F) v^{RF}(x). \tag{39}$$

Proof. For any $r \in F$, by (37) we have

$$v^{rF}(x) = \sum_{i=1}^n x_i v(rF_i) \leq v^r(x) = \sum_{i=1}^n x_i v(r_i) \leq \sum_{i=1}^n x_i v(RF_i) = v^{RF}(x) \tag{40}$$

Hence, inequality (40) together with Eqs (34)-(35) lead to the desired result. □

Similarly to the probabilistic case in Section 4.1, we are able to derive a characterization for $x \succeq_{hu}^{\alpha} y$:

Proposition 3. $x \succeq_{hu}^{\alpha} y \Leftrightarrow \sum_{i=j}^k \overline{g}_i^{\alpha}(x) \geq \sum_{i=j}^k \overline{g}_i^{\alpha}(y) \ \forall j \in [k]$ *where*

$$\overline{g}_i^{\alpha}(x) := \sum_{F \in \mathcal{F}} m(F) \left(\alpha g_i^{RF}(x) + (1-\alpha) g_i^{rF}(x) \right). \tag{41}$$

Proof. By Proposition 2, we have

$$H_{\alpha}^v(x) = \sum_{F \in \mathcal{F}} m(F) \left((1-\alpha) \sum_{i=1}^k g_i^{rF}(x) v_i + \alpha \sum_{i=1}^k g_i^{RF}(x) \right) v_i. \tag{42}$$

Exchanging the summation leads to

$$H_{\alpha}^v(x) = \sum_{i=1}^k \left[\sum_{F \in \mathcal{F}} m(F) \left(\alpha g_i^{RF}(x) + (1-\alpha) g_i^{rF}(x) \right) \right] v_i = \sum_{i=1}^k \overline{g}_i^{\alpha}(x) v_i. \tag{43}$$

Hence $H_{\alpha}^v(x) \geq H_{\alpha}^v(y) \ \forall v \Leftrightarrow \sum_{i=1}^k \overline{g}_i^{\alpha}(x) v_i \geq \sum_{i=1}^k \overline{g}_i^{\alpha}(y) v_i \ \forall v$. The result follows then from Lemma 2. □

From Proposition 3, we obtain that non-dominated solutions according to $\succeq_h^\alpha$ are efficient solutions of the following problem:

$$\max \left\{ \sum_{i=1}^{k} \overline{g}_i^\alpha(x), \sum_{i=2}^{k} \overline{g}_i^\alpha(x), \ldots, \sum_{i=k}^{k} \overline{g}_i^\alpha(x) \right\} \tag{44}$$

$$x \in \mathcal{X}. \tag{45}$$

Similar to Problem (27-28), each objective of Problem (44,45) is also linear. At first glance, the assumption that focal sets are rectangles may seem restrictive. Still, it can appear in numerous practical situations. In the next example, we provide such a situation.

Example 6 (Example 3 continued). Assume that the profits of items are unknown, and an expert predicts that the profit vector is $r = \{l_2, l_3, l_1, l_2, l_2\}$. However, the expert is not entirely reliable, and from results of his past predictions, we know that the probability of him being correct is 0.8. If the prediction is accurate, the profit vector is indeed r. On the other hand, when the prediction is wrong, we are completely ignorant about the true profit, which could be any vector in $\{l_1, l_2, l_3\}^5$. This piece of information can be naturally modeled using a mass function m with two focal sets: $F_1 = \{(l_2, l_3, l_1, l_2, l_2)\}$ with a mass of 0.8, and $F_2 = \{l_1, l_2, l_3\}^5$ with a mass of 0.2. Let us choose $\alpha = 0.5$. For any feasible

items	1	2	3	4	5
w	2	2	2	3	4
F_1	l_2	l_3	l_1	l_2	l_2
F_2	$\{l_1, l_2, l_3\}$	$\{l_1, l_2, l_3\}$	$\{l_1, l_2, l_3\}$	$\{l_1, l_2, l_3\}$	$\{l_1, l_2, l_3\}$.

Table 3: Profits of items in two focal sets

solution x, we can compute that

$$\begin{aligned} \overline{g}_1^\alpha(x) &= 0.1x_1 + 0.1x_2 + 0.9x_3 + 0.1x_4 + 0.1x_5 \\ \overline{g}_2^\alpha(x) &= 0.8x_1 + 0.8x_4 + 0.8x_5 \\ \overline{g}_3^\alpha(x) &= 0.1x_1 + 0.9x_2 + 0.1x_3 + 0.1x_4 + 0.1x_5. \end{aligned}$$

So, finding non-dominated solutions according to the $\succeq_{hu}^\alpha$ can be formulated as solving the following multi-objective KP:

$$\max \left\{ \begin{array}{c} x_1 + x_2 + x_3 + x_4 + x_5, \\ 0.9x_1 + 0.9x_2 + 0.1x_3 + 0.9x_4 + 0.9x_5, \\ 0.1x_1 + 0.9x_2 + 0.1x_3 + 0.1x_4 + 0.1x_5, \end{array} \right\} \tag{46}$$

$$x \in \{0,1\}^5 : 2x_1 + 2x_2 + 2x_3 + 3x_4 + 4x_5 \le 6 \tag{47}$$

Arbitrary focal sets In this case, it is hard to derive a similar result as in Lemma 2. As a first result in this direction, we give a sufficient condition for $x \succeq_{hu}^{\alpha} y$, with $x, y \in \mathcal{X}$. Let $\mathcal{R}^* := \{r \in \mathcal{R} : \exists F \in \mathcal{F} \text{ such that } r \in F\}$.

Proposition 4. *If for each $r \in \mathcal{R}^*$, we have $\sum_{i=j}^{k} g_i^r(x) \geq \sum_{i=j}^{k} g_i^r(y)$ for all $j \in [k]$ then $x \succeq_{hu}^{\alpha} y$.*

Proof. Immediate from (34-35). □

Clearly, the condition stated in Proposition 4 is very stringent as it requires that for each scenario in $\mathcal{R}^*$, x weakly dominates y. Hence, in future research, it would be valuable to find more relaxed conditions or, ideally, establish a characterization similar to Lemma 2.

5 Conclusion

In this paper, we have investigated 0-1 Combinatorial Optimization Problems (01COPs), where the profits of items can be both qualitative and uncertain. We have combined approaches from [5,9] with decision-making under uncertainty methodologies [2] to compare solutions. Our main result is that under probabilistic uncertainty and a special case of evidential uncertainty where focal sets are rectangles, we still can find non-dominated solutions by solving a multi-objective version of the original 01COP. Going forward, we plan to study deeper the case of evidential uncertainty with arbitrary focal sets, aiming to provide more comprehensive insights and understanding. Another interesting direction is to adapt the approach in [3] where the authors compared acts by means of Sugeno integrals.

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