Corrigendum to "Belief Functions Contextual Discounting and Canonical Decompositions" [International Journal of Approximate Reasoning 53 (2012) 146–158]

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Abstract

Proposition 4 and Theorem 1 of the article "Belief Functions Contextual Discounting and Canonical Decompositions" [International Journal of Approximate Reasoning 53 (2012) 146–158] provide an erroneous result. We give here the true result with a correct proof.

Keywords: Belief functions, Contextual Discounting.

We hereby correct Proposition 4 and Theorem 1 in [2], which contained erroneous results.

Let us first recall the problem. A source S of information provides to agent Ag a piece of information represented by a mass function m_S^Ω (with $\Omega = \{\omega_1, \ldots, \omega_K\}$), simply denoted by m in this corrigendum. Let $\mathcal{A}$ be a non empty set of subsets of Ω called contexts. Agent Ag owns a meta-knowledge regarding the reliability of S conditionally on each set $A \in \mathcal{A}$. Formally, for all $A \in \mathcal{A}$, we suppose that

$$\begin{cases} m_{Ag}^{\mathcal{R}}[A](\{R\}) & = & 1 - \alpha_A = \beta_A \\ m_{Ag}^{\mathcal{R}}[A](\mathcal{R}) & = & \alpha_A \ , \end{cases} \tag{1}$$

where $\alpha_A \in [0, 1]$ and $\mathcal{R} = \{R, NR\}$ (R meaning the source is reliable, NR otherwise), and the notation $m[\cdot]$ denotes conditioning.

With the same reasoning as in [1] (where $\mathcal{A}$ was supposed to form a partition of Ω), the knowledge m_{Ag}^Ω held by agent Ag on Ω, based on the in-

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formation m provided by S and his metaknowledge regarding S represented by (1) for all $A \in \mathcal{A}$, can be obtained by the following computation,

$$\left(m^{\Omega}[\{R\}]^{\Uparrow\Omega\times\mathcal{R}} \circledcap_{A\in\mathcal{A}} m^{\mathcal{R}}[A]^{\Uparrow\Omega\times\mathcal{R}}\right)^{\downarrow\Omega} \ , \tag{2}$$

where symbol $\Uparrow$ and $\downarrow$ denote, respectively, the deconditioning and projection operations, and $m^{\Omega}[\{R\}] = m$.

It is stated in [2] that, for $\mathcal{A} = 2^{\Omega}$ (Proposition 4) and more generally for any set $\mathcal{A}$ of contexts (Theorem 1), Equation (2) is equivalent to

$$m \circledcup \left(\circledcup_{A\in\mathcal{A}} A_{\beta_A}\right) \ . \tag{3}$$

This statement is incorrect. In the general case, for any non empty $\mathcal{A}$, Equation (2) is equivalent to

$$m \circledcup \left(\circledcap_{A\in\mathcal{A}} \overline{A}^{\alpha_A}\right) \ , \tag{4}$$

as shown by the following proof, which corrects Theorem 1 from [2]. The fact that, in general, (4) is not equivalent to (3) (and particularly when $\mathcal{A} = 2^{\Omega}$), and therefore (2) is not equivalent in general to (3), is shown below by Example 1.

Proof 1. *Let us denote by A_i, $i \in I = \{1, \ldots, n\}$, the contexts present in $\mathcal{A}$, and let us write β_{A_i} simply by β_i, for all $i \in I$. For all $A_i \in \mathcal{A}$, the deconditioning of $m^{\mathcal{R}}[A_i]$ over $\Omega \times \mathcal{R}$ is given by*

$$m^{\mathcal{R}}[A_i]^{\Uparrow\Omega\times\mathcal{R}}(A_i \times \{R\} \cup \overline{A_i} \times \mathcal{R}) = \beta_i, \tag{5a}$$

$$m^{\mathcal{R}}[A_i]^{\Uparrow\Omega\times\mathcal{R}}(\Omega \times \mathcal{R}) = \alpha_i. \tag{5b}$$

Moreover, for all $(A_i, A_j) \in \mathcal{A}^2$, such that $j \neq i$,

$$\begin{aligned} & (A_i \times \{R\} \cup \overline{A_i} \times \mathcal{R}) \cap (A_j \times \{R\} \cup \overline{A_j} \times \mathcal{R}) \\ = \ & (A_i \cap A_j) \times \{R\} \cup (A_i \cap \overline{A_j}) \times \{R\} \cup (\overline{A_i} \cap A_j) \times \{R\} \cup \overline{(A_i \cup A_j)} \times \mathcal{R} \\ = \ & (A_i \cup A_j) \times \{R\} \cup \overline{(A_i \cup A_j)} \times \mathcal{R}. \end{aligned}$$

With $\mathcal{A}$ composed of two elements denoted by A_i and A_j, we then have

$$\begin{cases} (m^{\mathcal{R}}[A_i]^{\Uparrow\Omega\times\mathcal{R}} \circledcap m^{\mathcal{R}}[A_j]^{\Uparrow\Omega\times\mathcal{R}})((A_i \cup A_j) \times \{R\} \cup \overline{(A_i \cup A_j)} \times \mathcal{R}) & = \beta_i\beta_j \\ (m^{\mathcal{R}}[A_i]^{\Uparrow\Omega\times\mathcal{R}} \circledcap m^{\mathcal{R}}[A_j]^{\Uparrow\Omega\times\mathcal{R}})(A_i \times \{R\} \cup \overline{A_i} \times \mathcal{R}) & = \beta_i\alpha_j \\ (m^{\mathcal{R}}[A_i]^{\Uparrow\Omega\times\mathcal{R}} \circledcap m^{\mathcal{R}}[A_j]^{\Uparrow\Omega\times\mathcal{R}})(A_j \times \{R\} \cup \overline{A_j} \times \mathcal{R}) & = \alpha_i\beta_j \\ (m^{\mathcal{R}}[A_i]^{\Uparrow\Omega\times\mathcal{R}} \circledcap m^{\mathcal{R}}[A_j]^{\Uparrow\Omega\times\mathcal{R}})(\Omega \times \mathcal{R}) & = \alpha_i\alpha_j \end{cases} .$$

In other words, all the focal elements of $\textcircled{\cap}_{A\in\mathcal{A}} m^{\mathcal{R}}[A]^{\Uparrow\Omega\times\mathcal{R}}$ *are the elements* $C\times\{R\}\cup\overline{C}\times\mathcal{R}$ *with* C *composed of a union of elements* A_i *in* $\mathcal{A}$*,* I' *being the set of indices of the* A_i*'s, which means with* $C=\cup_{i\in I'\subseteq I}A_i$*. Moreover, each focal element has a mass equal to* $\prod_{i\in I'}\beta_i\prod_{j\in I\setminus I'}\alpha_j$*. Let us note that this latter result is also true if* $\mathcal{A}$ *is composed of one element* $A\subseteq\Omega$ *(directly from Equations (5)).*

By induction, we can show that this property remains true with $\mathcal{A}$ *composed of* n *contexts* A_i*,* $i\in I=\{1,\ldots,n\}$*. Indeed, let us suppose the property true with* $\mathcal{A}$ *composed of* $n-1$ *contexts* A_i*,* $i\in I=\{1,\ldots,n-1\}$*, we then have for all focal elements* $C\times\{R\}\cup\overline{C}\times\mathcal{R}$ *of* $\textcircled{\cap}_{i\in I} m^{\mathcal{R}}[A_i]^{\Uparrow\Omega\times\mathcal{R}}$*, with* $C=\cup_{i\in I'\subseteq I}A_i$*,*

$$(\textcircled{\cap}_{i\in I} m^{\mathcal{R}}[A_i]^{\Uparrow\Omega\times\mathcal{R}}\textcircled{\cap} m^{\mathcal{R}}[A_n]^{\Uparrow\Omega\times\mathcal{R}})((C\cup A_n)\times\{R\}\cup\overline{(C\cup A_n)}\times\mathcal{R})$$
$$=\beta_n\prod_{i\in I'}\beta_i\prod_{j\in I\setminus I'}\alpha_j=\prod_{i\in I'\cup\{n\}}\beta_i\prod_{j\in(I\cup\{n\})\setminus(I'\cup\{n\})}\alpha_j\ ,$$

and

$$(\textcircled{\cap}_{i\in I} m^{\mathcal{R}}[A_i]^{\Uparrow\Omega\times\mathcal{R}}\textcircled{\cap} m^{\mathcal{R}}[A_n]^{\Uparrow\Omega\times\mathcal{R}})(C\times\{R\}\cup\overline{C}\times\mathcal{R})$$
$$=\alpha_n\prod_{i\in I'}\beta_i\prod_{j\in I\setminus I'}\alpha_j=\prod_{i\in I'}\beta_i\prod_{j\in(I\cup\{n\})\setminus I'}\alpha_j\ ,$$

which means that focal elements of $\textcircled{\cap}_{i\in\{1,\ldots,n-1\}} m^{\mathcal{R}}[A_i]^{\Uparrow\Omega\times\mathcal{R}}\textcircled{\cap} m^{\mathcal{R}}[A_n]^{\Uparrow\Omega\times\mathcal{R}}$ *are also of the form* $C\times\{R\}\cup\overline{C}\times\mathcal{R}$*, with* $C=\cup_{i\in I'\subseteq I}A_i$*,* $I=\{1,\ldots,n\}$*,* $A_i\in\mathcal{A}$*, and have for mass:* $\prod_{i\in I'}\beta_i\prod_{j\in I\setminus I'}\alpha_j$*.*

Besides, for all $B\subseteq\Omega$*,*

$$m^{\Omega}[\{R\}]^{\Uparrow\Omega\times\mathcal{R}}(B\times\{R\}\cup\Omega\times\{NR\})=m(B)\ ,$$

and, for all $B\subseteq\Omega$*, for all* $C=\cup_{i\in I'\subseteq I}A_i$*,*

$$(C\times\{R\}\cup\overline{C}\times\mathcal{R})\cap(B\times\{R\}\cup\Omega\times\{NR\})=B\times\{R\}\cup\overline{C}\times\{NR\}\ .$$

Therefore, after the projection on Ω*,* $\left(m^{\Omega}[\{R\}]^{\Uparrow\Omega\times\mathcal{R}}\textcircled{\cap}_{A\in\mathcal{A}} m^{\mathcal{R}}[A]^{\Uparrow\Omega\times\mathcal{R}}\right)^{\downarrow\Omega}$ *consists in transferring a part* $\prod_{i\in I'}\beta_i\prod_{j\in I\setminus I'}\alpha_j$ *of each mass* $m(B)$*,* $B\subseteq\Omega$*, from* B *to* $B\cup\overline{C}$*, for all* $C=\cup_{i\in I'\subseteq I}A_i$*.*

On the other hand, $m\textcircled{\cup}\left(\textcircled{\cap}_{A\in\mathcal{A}}\overline{A}^{\alpha_A}\right)$ *can be written as*

$$m\textcircled{\cup}\left(\textcircled{\cap}_{i\in I}\overline{A_i}^{\alpha_i}\right)=m\textcircled{\cup}\left(\textcircled{\cap}_{i\in I}\begin{cases}\Omega & \mapsto\ \alpha_i\\ \overline{A_i} & \mapsto\ \beta_i\end{cases}\right)\ .$$

As for all $(i,j) \in I^2$ s.t. $i \neq j$, $\overline{A_i} \cap \overline{A_j} = \overline{A_i \cup A_j}$, it can be shown (with an induction for example) that the focal elements of $\bigcirc\!\!\!\!\cap_{i \in I} \overline{A_i}^{\alpha_i}$ are the elements $\overline{C}$ with $C = \cup_{i \in I' \subseteq I} A_i$ and have a mass equal to $\prod_{i \in I'} \beta_i \prod_{j \in I \setminus I'} \alpha_j$.

Consequently, operation $m \bigcirc\!\!\!\!\cup \left(\bigcirc\!\!\!\!\cap_{i \in I} \overline{A_i}^{\alpha_i}\right)$ also consists in transferring a part $\prod_{i \in I'} \beta_i \prod_{j \in I \setminus I'} \alpha_j$ of each mass $m(B)$, $B \subseteq \Omega$, from B to $B \cup \overline{C}$, for all $C = \cup_{i \in I' \subseteq I} A_i$. We can then conclude that Equations (2) and (4) are equivalent for any non empty set of contexts $\mathcal{A}$.

□

Example 1. *Let us consider $\Omega = \{\omega_1, \omega_2\}$ and $\mathcal{A} = 2^\Omega$, and let us denote $\alpha_{\{\omega_1\}}$ by α_1, $\alpha_{\{\omega_2\}}$ by α_2, and α_Ω by α_{12}. Equation (4) gives*

$$
\begin{aligned}
& m \ \bigcirc\!\!\!\!\cup \ \left(\bigcirc\!\!\!\!\cap_{A \in \mathcal{A}} \overline{A}^{\alpha_A}\right) \\
= & m \ \bigcirc\!\!\!\!\cup \ \left(\overline{\emptyset}^{\alpha_\emptyset} \bigcirc\!\!\!\!\cap \overline{\{\omega_1\}}^{\alpha_1} \bigcirc\!\!\!\!\cap \overline{\{\omega_2\}}^{\alpha_2} \bigcirc\!\!\!\!\cap \overline{\Omega}^{\alpha_{12}}\right) \\
= & m \ \bigcirc\!\!\!\!\cup \ \left(\{\omega_2\}^{\alpha_1} \bigcirc\!\!\!\!\cap \{\omega_1\}^{\alpha_2} \bigcirc\!\!\!\!\cap \emptyset^{\alpha_{12}}\right) \\
= & m \ \bigcirc\!\!\!\!\cup \ \left(\begin{cases} \{\omega_2\} & \mapsto \beta_1 \\ \Omega & \mapsto \alpha_1 \end{cases} \bigcirc\!\!\!\!\cap \begin{cases} \{\omega_1\} & \mapsto \beta_2 \\ \Omega & \mapsto \alpha_2 \end{cases} \bigcirc\!\!\!\!\cap \begin{cases} \emptyset & \mapsto \beta_{12} \\ \Omega & \mapsto \alpha_{12} \end{cases}\right) \\
= & m \ \bigcirc\!\!\!\!\cup \ \begin{cases} \emptyset & \mapsto \beta_1\beta_2\alpha_{12} + \beta_{12} \\ \{\omega_1\} & \mapsto \alpha_1\beta_2\alpha_{12} \\ \{\omega_2\} & \mapsto \beta_1\alpha_2\alpha_{12} \\ \Omega & \mapsto \alpha_1\alpha_2\alpha_{12} \end{cases} .
\end{aligned}
$$

In contrast, Equation (3) leads to

$$
\begin{aligned}
& m \ \bigcirc\!\!\!\!\cup \ \left(\bigcirc\!\!\!\!\cup_{A \in \mathcal{A}} A_{\beta_A}\right) \\
= & m \ \bigcirc\!\!\!\!\cup \ \emptyset_{\beta_\emptyset} \bigcirc\!\!\!\!\cup \{\omega_1\}_{\beta_1} \bigcirc\!\!\!\!\cup \{\omega_2\}_{\beta_2} \bigcirc\!\!\!\!\cup \Omega_{\beta_{12}} \\
= & m \ \bigcirc\!\!\!\!\cup \ \{\omega_1\}_{\beta_1} \bigcirc\!\!\!\!\cup \{\omega_2\}_{\beta_2} \bigcirc\!\!\!\!\cup \Omega_{\beta_{12}} \\
= & m \ \bigcirc\!\!\!\!\cup \ \begin{cases} \emptyset & \mapsto \beta_1 \\ \{\omega_1\} & \mapsto \alpha_1 \end{cases} \bigcirc\!\!\!\!\cup \begin{cases} \emptyset & \mapsto \beta_2 \\ \{\omega_2\} & \mapsto \alpha_2 \end{cases} \bigcirc\!\!\!\!\cup \begin{cases} \emptyset & \mapsto \beta_{12} \\ \Omega & \mapsto \alpha_{12} \end{cases} \\
= & m \ \bigcirc\!\!\!\!\cup \ \begin{cases} \emptyset & \mapsto \beta_1\beta_2\beta_{12} \\ \{\omega_1\} & \mapsto \alpha_1\beta_2\beta_{12} \\ \{\omega_2\} & \mapsto \beta_1\alpha_2\beta_{12} \\ \Omega & \mapsto \alpha_1\alpha_2\beta_{12} + \alpha_{12} \end{cases} .
\end{aligned}
$$

To summarize, in [1], the equivalence was shown between (2) and (3) when $\mathcal{A}$ forms a partition of Ω. This corrigendum shows that this equivalence does not hold for any $\mathcal{A}$, and that (2) is actually equivalent to (4) for any (non empty) $\mathcal{A}$.

References

[1] D. Mercier, B. Quost and T. Denœux. Refined modeling of sensor reliability in the belief function framework using contextual discounting. *Information Fusion*, 9 (2008) 246–258.

[2] D. Mercier, É. Lefèvre and F. Delmotte. Belief Functions Contextual Discounting and Canonical Decompositions. *International Journal of Approximate Reasoning*, 53 (2012) 146–158.