Appendix to the invited talk "Reliability and dependence in information fusion" at the BELIEF 2024 conference

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Abstract

This document provides the proofs of the main (unpublished) results presented in [1], which can be seen as "normalized" versions of results previously published in [3, 2].

1 Reliability

1.1 General case

Theorem 1. $^{\mathcal{B}}m(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \frac{\sum_b P^{\mathcal{B}}(b) \sum_{B \otimes_b C = A} m_1(B) m_2(C)}{1 - \sum_b P^{\mathcal{B}}(b) \sum_{B \otimes_b C = \emptyset} m_1(B) m_2(C)} & \text{otherwise.} \end{cases}$

Proof. Let $P_{12\mathcal{B}} := P_{12} \times P^{\mathcal{B}}$. For all $A \in 2^{\Theta}$, we have

$$\begin{aligned} & P_{\mathcal{B}}(\{(\omega_1, \omega_2, b) \in \Omega_1 \times \Omega_2 \times \mathcal{B} : \Gamma(\omega_1, \omega_2, b) = A\}) \\ = \; & P_{12\mathcal{B}}(\{(\omega_1, \omega_2, b) \in \Omega_1 \times \Omega_2 \times \mathcal{B} : \Gamma(\omega_1, \omega_2, b) = A | \; \Theta_{\mathcal{B}}\}) \\ = \; & \frac{P_{12\mathcal{B}}(\{(\omega_1, \omega_2, b) \in \Omega_1 \times \Omega_2 \times \mathcal{B} : \Gamma(\omega_1, \omega_2, b) = A\} \cap \; \Theta_{\mathcal{B}})}{P_{12\mathcal{B}}(\Theta_{\mathcal{B}})} \end{aligned}$$

(assuming that $P_{12\mathcal{B}}(\Theta_{\mathcal{B}}) > 0$).

From $\{(\omega_1, \omega_2, b) \in \Omega_1 \times \Omega_2 \times \mathcal{B} : \Gamma(\omega_1, \omega_2, b) = \emptyset\} \cap \; \Theta_{\mathcal{B}} = \emptyset$, we obtain

$$P_{\mathcal{B}}(\{(\omega_1, \omega_2, b) \in \Omega_1 \times \Omega_2 \times \mathcal{B} : \Gamma(\omega_1, \omega_2, b) = \emptyset\}) = 0. \quad (1)$$

Now, for $A \neq \emptyset$, we have

$$\begin{aligned} & \{(\omega_1, \omega_2, b) \in \Omega_1 \times \Omega_2 \times \mathcal{B} : \Gamma(\omega_1, \omega_2, b) = A\} \cap \; \Theta_{\mathcal{B}} \\ = \; & \{(\omega_1, \omega_2, b) \in \Omega_1 \times \Omega_2 \times \mathcal{B} : \Gamma(\omega_1, \omega_2, b) = A\}. \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{P_{12\mathcal{B}}(\{(\omega_1,\omega_2,b)\in\Omega_1\times\Omega_2\times\mathcal{B}:\Gamma(\omega_1,\omega_2,b)=A\}\cap\ \Theta_{\mathcal{B}})}{P_{12\mathcal{B}}(\Theta_{\mathcal{B}})} \\ = & \frac{P_{12\mathcal{B}}(\{(\omega_1,\omega_2,b)\in\Omega_1\times\Omega_2\times\mathcal{B}:\Gamma(\omega_1,\omega_2,b)=A\})}{1-P_{12\mathcal{B}}(\{(\omega_1,\omega_2,b)\in\Omega_1\times\Omega_2\times\mathcal{B}:\Gamma(\omega_1,\omega_2,b)=\emptyset\})}. \end{aligned} \tag{2}$$

Furthermore, for all $A\in 2^{\Theta}$, we have

$$\begin{aligned} & P_{12\mathcal{B}}(\{(\omega_1,\omega_2,b)\in\Omega_1\times\Omega_2\times\mathcal{B}:\Gamma(\omega_1,\omega_2,b)=A\}) \\ = & \sum_{(\omega_1,\omega_2,b):\Gamma_1(\omega_1)\otimes_b\Gamma_2(s_2)=A} P_{12\mathcal{B}}(\omega_1,\omega_2,b) \\ = & \sum_{(\omega_1,\omega_2,b):\Gamma_1(\omega_1)\otimes_b\Gamma_2(s_2)=A} P_{12}(\omega_1,\omega_2)P^{\mathcal{B}}(b) \\ = & \sum_b P^{\mathcal{B}}(b) \sum_{(\omega_1,\omega_2):\Gamma_1(\omega_1)\otimes_b\Gamma_2(s_2)=A} P_{12}(\omega_1,\omega_2) \\ = & \sum_b P^{\mathcal{B}}(b) \sum_{(\omega_1,\omega_2):\Gamma_1(\omega_1)\otimes_b\Gamma_2(s_2)=A} P_1(\omega_1)P_2(s_2) \\ = & \sum_b P^{\mathcal{B}}(b) \sum_{B\otimes_b C=A} m_1(B)m_2(C). \end{aligned} \tag{3}$$

The theorem follows from (1), and from (2) and (3). □

1.2 Particular cases

1.2.1 Dependent reliabilities

Proposition 1. *Let $P^{\mathcal{R}}$ such that*

$$\begin{aligned} P^{\mathcal{R}}(rel_1, unrel_2) &= \alpha, \\ P^{\mathcal{R}}(unrel_1, rel_2) &= 1-\alpha, \end{aligned}$$

for some $\alpha\in[0,1]$.

We have

$${}^{\mathcal{R}}m = \alpha m_1 + (1-\alpha)m_2.$$

Proof. Let $b1$ be the connective induced by assumption $(rel_1, unrel_2)$ and let $b2$ be the connective induced by assumption $(unrel_1, rel_2)$.

${}^{\mathcal{R}}m$ is equivalently induced by the random set $(\Omega_1\times\Omega_2\times\mathcal{B}, P_{\mathcal{B}}, \Gamma^{\mathcal{B}})$ with $P_{\mathcal{B}}$ the probability measure such that $P^{\mathcal{B}}(b1)=\alpha$ and $P^{\mathcal{B}}(b2)=1-\alpha$.

The proposition follows from Theorem 1 and the fact that for all $(A,B)\in 2^{\Theta}\times 2^{\Theta}$, we have $A\otimes_{b1}B=A$ and $A\otimes_{b2}B=B$.

□

1.2.2 Independent reliabilities

Proposition 2. *Let* $P^{\mathcal{R}}$ *such that* $P^{\mathcal{R}} = P^{\mathcal{R}_1} \times P^{\mathcal{R}_2}$*, with* $P^{\mathcal{R}_i}(unrel_i) = \alpha_i$ *for some* $\alpha \in [0, 1]$.

We have

$$^{\mathcal{R}}m = {}^{\alpha_1}m_1 \oplus {}^{\alpha_2}m_2 \tag{4}$$

Proof. Let $b3$ be the connective induced by assumption (rel_1, rel_2) and let $b4$ be the connective induced by assumption $(unrel_1, unrel_2)$.

$^{\mathcal{R}}m$ is equivalently induced by the random set $(\Omega_1 \times \Omega_2 \times \mathcal{B}, P_{\mathcal{B}}, \Gamma^{\mathcal{B}})$ with $P_{\mathcal{B}}$ the probability measure such that

$$\begin{aligned}
P^{\mathcal{B}}(b3) &= (1-\alpha_1)(1-\alpha_2), \\
P^{\mathcal{B}}(b4) &= \alpha_1\alpha_2, \\
P^{\mathcal{B}}(b1) &= (1-\alpha_1)\alpha_2, \\
P^{\mathcal{B}}(b2) &= \alpha_1(1-\alpha_2).
\end{aligned}$$

From Theorem 1 and the fact that for all $(A, B) \in 2^{\Theta} \times 2^{\Theta}$, we have $A \otimes_{b3} B = A \cap B$ and $A \otimes_{b4} B = \Theta$, $^{\mathcal{R}}m$ is equal to:

$$^{\mathcal{R}}m(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \frac{(1-\alpha_1)(1-\alpha_2)m_1(\Theta)m_2(\Theta)+(1-\alpha_1)\alpha_2 m_1(\Theta)+\alpha_1(1-\alpha_2)m_2(\Theta)+\alpha_1\alpha_2}{1-(1-\alpha_1)(1-\alpha_2)\sum_{B\cap C=\emptyset} m_1(B)m_2(C)} & \text{if } A = \Theta, \\ \frac{(1-\alpha_1)(1-\alpha_2)\sum_{B\cap C=A} m_1(B)m_2(C)+(1-\alpha_1)\alpha_2 m_1(A)+\alpha_1(1-\alpha_2)m_2(A)}{1-(1-\alpha_1)(1-\alpha_2)\sum_{B\cap C=\emptyset} m_1(B)m_2(C)} & \text{otherwise.} \end{cases} \tag{5}$$

Now, let us consider the mass function m' resulting from the discounting of mass function m_i with discount rate α_i, $i = 1, 2$, followed by the combination by Dempster's rule of the resulting discounted mass functions, i.e., $m' := {}^{\alpha_1}m_1 \oplus {}^{\alpha_2}m_2$. We have $m'(\emptyset) = 0$ and, for $A \neq \emptyset$,

$$m'(A) = \frac{\sum_{B\cap C=A} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C)}{1 - \sum_{B\cap C=\emptyset} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C)}. \tag{6}$$

Now, for $A \neq \Theta$, Eq. (6) can be rewritten

$$
\begin{aligned}
& \frac{\sum_{\substack{B\neq\Theta, C\neq\Theta\\ B\cap C=A}} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C) + {}^{\alpha_1}m_1(\Theta)\, {}^{\alpha_2}m_2(A) + {}^{\alpha_1}m_1(A)\, {}^{\alpha_2}m_2(\Theta)}{1-\sum_{B\cap C=\emptyset} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C)} \\
= & \frac{(1-\alpha_1)(1-\alpha_2)\sum_{\substack{B\neq\Theta, C\neq\Theta\\ B\cap C=A}} m_1(B)m_2(C) + ((1-\alpha_1)m_1(\Theta)+\alpha_1)(1-\alpha_2)m_2(A) + (1-\alpha_1)m_1(A)((1-\alpha_2)m_2(\Theta)+\alpha_2)}{1-\sum_{B\cap C=\emptyset} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C)} \\
= & \frac{(1-\alpha_1)(1-\alpha_2)\sum_{\substack{B\neq\Theta, C\neq\Theta\\ B\cap C=A}} m_1(B)m_2(C) + (1-\alpha_1)(1-\alpha_2)m_1(\Theta)m_2(A) + \alpha_1(1-\alpha_2)m_2(A) + (1-\alpha_1)(1-\alpha_2)m_1(A)m_2(\Theta) + (1-\alpha_1)\alpha_2 m_1(A)}{1-\sum_{B\cap C=\emptyset} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C)} \\
= & \frac{(1-\alpha_1)(1-\alpha_2)\sum_{B\cap C=A} m_1(B)m_2(C) + \alpha_1(1-\alpha_2)m_2(A) + (1-\alpha_1)\alpha_2 m_1(A)}{1-\sum_{B\cap C=\emptyset} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C)} \qquad (7)
\end{aligned}
$$

and, for $A = \Theta$, Eq. (6) reduces to

$$
\begin{aligned}
& \frac{{}^{\alpha_1}m_1(\Theta)\, {}^{\alpha_2}m_2(\Theta)}{1-\sum_{B\cap C=\emptyset} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C)} \\
= & \frac{((1-\alpha_1)m_1(\Theta)+\alpha_1)((1-\alpha_2)m_2(\Theta)+\alpha_2)}{1-\sum_{B\cap C=\emptyset} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C)} \\
= & \frac{(1-\alpha_1)(1-\alpha_2)m_1(\Theta)m_2(\Theta) + (1-\alpha_1)\alpha_2 m_1(\Theta) + \alpha_1(1-\alpha_2)m_2(\Theta) + \alpha_1\alpha_2}{1-\sum_{B\cap C=\emptyset} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C)}
\end{aligned}
\qquad (8)
$$

Remark that if $B \cap C = \emptyset$ for some B and C such ${}^{\alpha_1}m_1(B) > 0$ and ${}^{\alpha_2}m_2(C) > 0$, it must be the case that $B \neq \Theta$ and $C \neq \Theta$. Therefore, we have

$$
\sum_{B\cap C=\emptyset} {}^{\alpha_1}m_1(B)\, {}^{\alpha_2}m_2(C) \quad = \quad (1-\alpha_1)(1-\alpha_2)\sum_{B\cap C=\emptyset} m_1(B)m_2(C) \qquad (9)
$$

and thus Eq. (7) is equal to the last case of (5), and Eq. (8) is equal to the second case of (5).

□

2 Dependence

Theorem 2. *Any mass function m on $\Theta = \{\theta_1, \ldots, \theta_K\}$ satisfies*

$$
m = \oplus_{\boldsymbol{\sigma}}(\overline{\{\theta_1\}}^{d_1}, \ldots, \overline{\{\theta_K\}}^{d_K})
$$

with d_i, $1 \leq i \leq K$, the means and $\boldsymbol{\sigma}$ the dependence vector of the K-variate Bernoulli distribution $P_{1\ldots K}$ such that

$$
P_{1\ldots K}(S_1 = \omega_1, \ldots, S_K = \omega_K) := m(A_{\boldsymbol{\omega}})
$$

with $A_{\boldsymbol{\omega}}$ the subset of Θ such that $\theta_i \in A_{\boldsymbol{\omega}}$ if $\omega_i = 1$ and $\theta_i \notin A_{\boldsymbol{\omega}}$ if $\omega_i = 0$, for all $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_K) \in \boldsymbol{\Omega}$.

Proof. Given the definition of $\oplus_{\boldsymbol{\sigma}}$, we have that $\oplus_{\boldsymbol{\sigma}}(\overline{\{\theta_1\}}^{d_1}, \ldots, \overline{\{\theta_K\}}^{d_K})$ is the mass function $m_{1\ldots K}$ induced by the random set $(\boldsymbol{\Omega}, P_\cap, \Gamma_\cap)$ with

- $\boldsymbol{\Omega} := \times_{i=1}^K \Omega_i$, $\Omega_i = \{0, 1\}$,
- $\Gamma_\cap(\boldsymbol{\omega}) := \bigcap_{i=1}^K \Gamma_i(\omega_i)$ for all $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N) \in \boldsymbol{\Omega}$, with $\Gamma_i(0) = \overline{\{\theta_i\}}$ and $\Gamma_i(1) = \Theta$.
- $P_\cap$ the probability distribution $P_{1\ldots K}$ conditioned on $\Theta_\cap$, where $P_{1\ldots K}$ is the distribution having $P_1, \ldots, P_K$, as marginals, with $P_i(1) = d_i$, and specified by vector $\boldsymbol{\sigma}$, i.e., it is the K-variate Bernoulli distribution with means d_i, $1 \leq i \leq K$, and dependence vector $\boldsymbol{\sigma}$.

Now, since a K-variate Bernoulli distribution is characterized by its means and dependence vector, given the statement of the theorem, we have that this distribution $P_{1\ldots K}$ satisfies $P_{1\ldots K}(S_1 = \omega_1, \ldots, S_K = \omega_K) = m(A_{\boldsymbol{\omega}})$, for all $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N) \in \boldsymbol{\Omega}$.

Furthermore, similarly as in the proof of [2, Proposition 1], we have, for all $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N) \in \boldsymbol{\Omega}$:

$$\begin{aligned}
\Gamma_\cap(\boldsymbol{\omega}) &= \bigcap_{i=1}^K \Gamma_i(\omega_i) \\
&= \left(\bigcap_{i:\omega_i=0} \Gamma_i(0)\right) \bigcap \left(\bigcap_{i:\omega_i=1} \Gamma_i(1)\right) \\
&= \bigcap_{i:\omega_i=0} \Gamma_i(0) \\
&= \bigcap_{i:\omega_i=0} \Theta \backslash \{\theta_i\} \\
&= \Theta \backslash \{\theta_i : i, \omega_i = 0\} \\
&= \{\theta_i : i, \omega_i = 1\} = A_{\boldsymbol{\omega}}. \qquad (10)
\end{aligned}$$

Remark that $\Gamma_\cap(\mathbf{0}) = A_{\mathbf{0}} = \emptyset$ and $\Gamma_\cap(\boldsymbol{\omega}) \neq \emptyset$ for all $\boldsymbol{\omega} \neq \mathbf{0}$, and that $P_{1\ldots K}(\mathbf{0}) = m(A_{\mathbf{0}}) = m(\emptyset) = 0$, hence we have, for all $\boldsymbol{\omega} \in \boldsymbol{\Omega}$:

$$\begin{aligned}
P_\cap(\boldsymbol{\omega}) &= P_{1\ldots K}(\boldsymbol{\omega}) \\
&= m(A_{\boldsymbol{\omega}}). \qquad (11)
\end{aligned}$$

From Eqs. (10) and (11), we have $m_{1\ldots K}(A_{\boldsymbol{\omega}}) = P_\cap(\{\boldsymbol{\omega} \in \boldsymbol{\Omega} : \Gamma_\cap(\boldsymbol{\omega}) = A_{\boldsymbol{\omega}}) = m(A_{\boldsymbol{\omega}})$, for all $\boldsymbol{\omega} \in \boldsymbol{\Omega}$. Hence, the mass function $\oplus_{\boldsymbol{\sigma}}(\overline{\{\theta_1\}}^{d_1}, \ldots, \overline{\{\theta_K\}}^{d_K})$ is the mass function m.

□

References

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[2] F. Pichon. Canonical decomposition of belief functions based on Teugels' representation of the multivariate Bernoulli distribution. *Information Sciences*, 428:76–104, 2018.

[3] F. Pichon, D. Dubois, and T. Denœux. Relevance and truthfulness in information correction and fusion. *International Journal of Approximate Reasoning*, 53(2):159–175, 2012.