Proposition and learning of some belief function contextual correction mechanisms*

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Abstract

Knowledge about the quality of a source can take several forms: it may for instance relate to its truthfulness or to its relevance, and may even be uncertain. Of particular interest in this paper is that such knowledge may also be contextual; for instance the reliability of a sensor may be known to depend on the actual object observed. Various tools, called correction mechanisms, have been developed within the theory of belief functions, to take into account knowledge about the quality of a source. Yet, only a single tool is available to account for contextual knowledge about the quality of a source, and precisely about the relevance of a source. There is thus some lack of flexibility since contextual knowledge about the quality of a source does not have to be restricted to its relevance. The first aim of this paper is thus to try and enlarge the set of tools available in belief function theory to deal with contextual knowledge about source quality. This aim is achieved by (1) providing an interpretation to each one of two contextual correction mechanisms introduced initially from purely formal considerations, and (2) deriving extensions – essentially by uncovering contextual forms – of two interesting and non contextual correction mechanisms. The second aim of this paper is related to the origin of contextual knowledge about the quality of a source: due to the lack of dedicated approaches, it is indeed not clear how to obtain such specific knowledge in practice. A sound, easy to interpret and computationally simple method is therefore provided to learn from data contextual knowledge associated with the contextual correction mechanisms studied in this paper.

Keywords: Dempster-Shafer theory, Belief functions, Information correction, Discounting.

1 Introduction

In today's society, a lot of information is accessible. Yet, for a piece of information to be useful, it must be interpreted with respect to the source that provides it, and in particular in the light of the quality of the source. Clearly, this is no easy task. First, the quality of a source may come in many guises: a source can for instance be biased, or even be totally irrelevant. Second, this quality may be only known with some uncertainty by the agent who has to interpret the piece of information [28].

The theory of belief functions [32, 39, 36] is a flexible framework to model and deal with uncertainty. Various tools have been developed within this framework to take into account uncertain knowledge about the quality of a source and to modify, or *correct* [19, 28], a piece of information provided by the source according to this knowledge. The most common, and historically the first, of such tools is the discounting operation [32, 33], which corresponds to the situation where the agent has some knowledge regarding the relevance of the source [28]. The discounting operation is central in numerous and diverse applications of belief function theory, such as classification [5] and information fusion [31, 16, 40] (see [27, Remarks 5 and 6] for more details on the role of discounting in these applications).

Since its inception, the discounting operation has been extended in different ways. Notably, its inverse, called de-discounting, is introduced and used in [7] to show that two well-known and apparently quite different classifiers based on belief functions, produce actually similar outputs in an important special case. This mechanism allows one to retract a discounting which is judged no longer valid or justified; it has the effect of strengthening, rather than weakening as is the case with discounting, a piece of information. It is applied successfully in a mailing address recognition system [19], where it is used in conjunction with discounting to correct outputs of postal address readers.

Another interesting extension is the correction mechanism proposed recently by Pichon *et al.* [28], in order to take into account knowledge about the truthfulness of a source, besides its relevance. Its interest resides in the fact that it offers a means to deal with sources that may lie, or that are biased in the case where the source is a sensor. As shown in [28], truthfulness assumptions are also quite interesting in that they can be used to reinterpret all connectives of Boolean logic, which in turn leads to generalize the unnormalized Dempster's rule [4, 32] to all Boolean connectives – this rule being the pivotal and most often used combination rule in belief function theory.

Of particular interest in this paper is the fact that the quality of a source may also be contextual; for instance[1], a thermometer is relevant to measure a temperature which falls within its range, but is typically useless if the temperature is outside of it; if we let $\mathcal{X} = \{-100°C, \ldots, 1000°C\}$ be the possible temperatures, then the context here is the range, which could be, *e.g.*, $\{-38°C, \ldots, 356°C\}$ (range of mercury thermometers). Furthermore, such contextual quality may also be known with some uncertainty; for instance one may believe to some degree that a source is relevant for a given context.

To deal with such contextual knowledge, yet another extension of discounting is introduced by Mercier *et al.* [23], who consider the case where one has some knowledge about the relevance of the source, conditionally on different subsets (contexts) A of $\mathcal{X}$ such that the set $\mathcal{A}$ of these contexts forms a partition of $\mathcal{X}$, leading to an operation called *contextual discounting based on a coarsening.* Formally, contextual discounting

[1] Other examples of contextual quality will be given in later sections of this paper.

based on a coarsening relies on the disjunctive rule of combination [9, 33] and is related to the canonical decomposition of a belief function [34] as highlighted in [20]. This contextual correction mechanism was extended recently by Mercier *et al.* [21]: the set of contexts $\mathcal{A}$ for which one has some knowledge about the relevance of the source can be arbitrary (it no longer needs to form a partition of $\mathcal{X}$).

Contextual discounting based on a coarsening [23] and its extension uncovered in [21] are, to the best of our knowledge, the only contextual correction mechanisms that have been thoroughly studied in the literature. There is thus clearly a lack of tools to deal with contextual quality, since it does not have to be restricted to contextual relevance. As a matter of fact, Mercier *et al.* [20] introduce two other contextual correction mechanisms, which are quite interesting from a formal point of view: the first one, referred simply as *contextual discounting* in [20, Theorem 1][2], can be viewed as a generalization of contextual discounting based on a coarsening in that it has the same formal definition as this latter mechanism except that the set $\mathcal{A}$ that appears in its definition can be arbitrary; the second one is a dual reinforcement process to contextual discounting, which has a similar definition as contextual discounting, except that it relies on the unnormalized Dempster's rule, and it can also be linked to the canonical decomposition of a belief function. However, Mercier *et al.* [20] do not provide an interpretation for this latter correction mechanism, nor do they provide an interpretation for contextual discounting as shown in [21], hence the practical usefulness of these two contextual correction mechanisms remains unknown.

As a first step toward enlarging the set of tools dedicated to handling contextual quality, one may thus try and provide an interpretation to each one of Mercier *et al.* [20] contextual discounting and reinforcement processes. Mimicking what has been done for discounting with the introduction of contextual discounting based on a coarsening, one may also try and derive contextual forms of correction mechanisms that have already proved interesting in their non contextual versions; in particular one may attempt to "contextualize" the two extensions of discounting recalled above that are the de-discounting operation and Pichon *et al.* [28] truthfulness-based correction mechanism. The first aim of this paper is to explore these different routes and to find out whether they can yield useful complements to contextual discounting based on a coarsening and its recent extension [21], with respect to the problem of handling contextual knowledge about the quality of a source. As will be seen, this exploration rests on a detailed analysis of Pichon *et al.* [28] truthfulness model.

In addition to the above issue of being able to take into account contextual knowledge about the quality of a source, an associated issue is the origin of such knowledge; it is indeed not totally clear how to obtain such specific knowledge in practice. Two different approaches [11, 23] have been proposed to find out the contextual quality of a source, and more precisely to discover it from available labelled data. Elouedi *et al.* [11] approach is based on the use of confusion matrices. Its simplicity makes it quite appealing. However, it is restricted to the case of contextual discounting based on a coarsening, where the coarsening is fixed to the partition of singletons. Besides, it

[2] The operation referred to as contextual discounting in [20, Theorem 1] was thought – erroneously as shown in [21] – to be the extension to an arbitrary set of contexts, of contextual discounting based on a coarsening, hence its name. To ensure consistency with previous published works, the same name is used for this operation in this paper, although the results in [21] suggest this name may be somewhat of a misnomer.

basically amounts to assuming that a source makes a correct prediction only when it is relevant, which is debatable (a non relevant source may provide correct information, see, *e.g.*, [28]). Mercier *et al.* [23] approach on the other hand, relies on the minimization of an error criterion. It is quite interesting since, in addition to learning the contextual quality of a source, it may be potentially useful to improve the performance of a source in, *e.g.*, a classification application. However, it is restricted to the case of contextual discounting based on a coarsening, where a partition (set of contexts) of $\mathcal{X}$ has been fixed beforehand. The second aim of this paper is therefore to alleviate this latter restriction and more generally to extend Mercier *et al.* [23] learning approach to the other contextual correction mechanisms studied in this paper, such as Mercier *et al.* [20] contextual discounting and reinforcement processes.

This paper is organized as follows. Necessary notions on belief function theory and on existing correction mechanisms are recalled in Section 2. A new framework for handling detailed assumptions about the truthfulness of a source is then obtained from a careful analysis of Pichon *et al.* [28] truthfulness model, which is carried out in two steps (Section 3 then Section 4). Using this framework, an interpretation for each one of Mercier *et al.* [20] contextual discounting and reinforcement processes is derived (Section 5). Contextual de-discounting is introduced and then used in conjunction with the canonical decomposition, to define an extension of contextual discounting (Section 6). A contextual version of Pichon *et al.* [28] truthfulness-based correction mechanism is uncovered in Section 7. Learning contextual correction mechanisms from labelled data is addressed in Section 8. Finally, Section 9 concludes the paper.

2 Belief function theory: necessary notions

In this section, we first recall basic concepts of belief function theory. Then, we present existing correction mechanisms that are of interest for this paper.

2.1 Basic concepts

We review in this section the following basic concepts: the representation, combination and canonical decomposition of beliefs.

2.1.1 Representation of beliefs

In this paper, we adopt Smets' Transferable Belief Model (TBM) [39, 36], where the beliefs held by an agent regarding the actual value taken by a parameter $\mathbf{x}$ defined on a finite domain, called *frame of discernment*, $\mathcal{X} = \{x_1, \ldots, x_K\}$, are modeled using a belief function [32] and represented using an associated *mass function*. A mass function (MF) on $\mathcal{X}$ is defined as a mapping $m : 2^{\mathcal{X}} \rightarrow [0, 1]$ verifying $\sum_{A \subseteq \mathcal{X}} m(A) = 1$. The mass $m(A)$ represents the subjective probability that the agent knows that the value of $\mathbf{x}$ lies somewhere in set A, and nothing more specific [3, 10].

Subsets A of $\mathcal{X}$ such that $m(A) > 0$ are called *focal sets* of m. A MF is said to be: *vacuous* if $\mathcal{X}$ is its only focal set, in which case it is denoted by $m_{\mathcal{X}}$; *inconsistent* if $\emptyset$ is its only focal set, in which case it is denoted by $m_{\emptyset}$; *dogmatic* if $\mathcal{X}$ is not a focal set; *normal* if $\emptyset$ is not a focal set. A non normal MF m can be transformed into a normal

MF m^* by the normalization operation defined as follows, for all $A \subseteq \mathcal{X}$:

$$m^*(A) = \begin{cases} k \cdot m(A) & \text{if } A \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

with $k = (1 - m(\emptyset))^{-1}$.

Equivalent representations of a MF m exist. In particular the *belief*, *plausibility*, *commonality* and *implicability* functions are defined, respectively, as:

$$bel\,(A) = \sum_{\emptyset \neq B \subseteq A} m\,(B)\,,$$

$$pl\,(A) = \sum_{B \cap A \neq \emptyset} m\,(B)\,,$$

$$q\,(A) = \sum_{B \supseteq A} m\,(B)\,,$$

and

$$b\,(A) = \sum_{B \subseteq A} m\,(B)\,,$$

for all $A \subseteq \mathcal{X}$. MF m can be recovered from any of these functions. In particular, we have:

$$m(A) = \sum_{B \supseteq A} (-1)^{|B|-|A|} q(B), \tag{2}$$

for all $A \subseteq \mathcal{X}$, with $|A|$ denoting the cardinality of A. The degree of belief $bel(A)$ evaluates to what extent event A is logically implied by the available evidence and $pl(A)$ evaluates to what extent event A is consistent with the available evidence [10]; the commonality and implicability functions play more of a technical role as will be seen in the next section.

2.1.2 Combination of beliefs

Beliefs can be aggregated using so-called combination rules. In particular, the conjunctive combination rule, or *conjunctive rule* for short, which is the unnormalized version of Dempster's rule [4], is defined as follows. Let m_1 and m_2 be two MFs, and let $m_{1 \textcircled{\cap} 2}$ be the MF resulting from their combination by the conjunctive rule denoted by $\textcircled{\cap}$. We have:

$$m_{1 \textcircled{\cap} 2}\,(A) \quad = \quad \sum_{B \cap C = A} m_1\,(B)\, m_2\,(C)\,, \quad \forall A \subseteq \mathcal{X}. \tag{3}$$

The conjunctive rule admits a simple expression in terms of commonality functions:

$$q_{1 \textcircled{\cap} 2}(A) = q_1\,(A) \cdot q_2\,(A)\,, \quad \forall A \subseteq \mathcal{X}, \tag{4}$$

where q_1, q_2 and $q_{1 \textcircled{\cap} 2}$ denote the commonality functions associated to m_1, m_2 and $m_{1 \textcircled{\cap} 2}$, respectively. The conjunctive rule is commutative, associative and admits the vacuous MF $m_{\mathcal{X}}$ as neutral element.

Assume now that $m_{1\textcircled{\cap}2}$ has been obtained by combining MFs m_1 and m_2, and then it appears that m_2 is actually not supported by evidence and should thus be removed from $m_{1\textcircled{\cap}2}$. This operation is possible using the inverse of the conjunctive rule [34, 6], which may be called the conjunctive decombination rule and denoted by $\textcircled{\not\cap}$. We have:

$$m_{1\textcircled{\cap}2}\textcircled{\not\cap}m_2 \quad = \quad m_1.$$

Let q_1 and q_2 be the commonality functions associated respectively to any two MFs m_1 and m_2, the conjunctive decombination rule is defined as:

$$q_{1\textcircled{\not\cap}2}(A) \quad = \quad \frac{q_1(A)}{q_2(A)}, \quad \forall A \subseteq \mathcal{X}. \tag{5}$$

This operation is well-defined as long as m_2 is non dogmatic (in which case we have $q_2(A) > 0$ for all A) and $m_{1\textcircled{\not\cap}2}$ is a MF (this is not necessarily the case since the quotient of two commonality functions is not always a commonality function).

Other combination rules of interest for this paper are the disjunctive rule $\textcircled{\cup}$ [9, 33] and the equivalence rule $\textcircled{\underline{\cap}}$ [35, 25]. Their definitions are similar to that of the conjunctive rule: one merely needs to replace $\cap$ in (3) by, respectively, $\cup$ and $\underline{\cap}$, where $\underline{\cap}$ denotes logical equality, i.e., $B\underline{\cap}C = (B \cap C) \cup (\overline{B} \cap \overline{C})$ for all $B, C \subseteq \mathcal{X}$, where $\overline{A}$ denotes the complement of some $A \subseteq \mathcal{X}$. The interpretations of these three rules are discussed in detail in [28].

The disjunctive rule has a simple expression in terms of implicability functions, which is the counterpart of (4):

$$b_{1\textcircled{\cup}2}(A) \quad = \quad b_1(A) \cdot b_2(A), \quad \forall A \subseteq \mathcal{X}.$$

The disjunctive rule is commutative, associative and admits the inconsistent MF $m_\emptyset$ as neutral element. Besides, as for the conjunctive rule, an inverse operation may be defined for $\textcircled{\cup}$ [6]:

$$b_{1\textcircled{\not\cup}2}(A) = \frac{b_1(A)}{b_2(A)}, \quad \forall A \subseteq \mathcal{X}.$$

This operation, referred to as disjunctive decombination, is well-defined as long as m_2 is non normal, since in this case we have $b_2(A) > 0$ for all A.

We may note that a similar expression as (4), *i.e.*, a simple pointwise product expression, exists for the rule $\textcircled{\underline{\cap}}$. We defer its introduction for clarity of presentation.

2.1.3 Canonical decompositions

Following Shafer [32, Chapter 4] (see also [34, 6]), a MF m may be called *conjunctively separable*, or, for short, $\textcircled{\cap}$-*separable*, if it can be obtained as the result of the combination by the conjunctive rule of so-called *simple* MFs, which are MFs having at most two focal sets, including the frame of discernment $\mathcal{X}$ [6].

A simple MF having focal sets $\mathcal{X}$ and $A \subset \mathcal{X}$, with respective masses w and $1 - w$, $w \in [0, 1]$, may be simply denoted by A^w; for instance, a MF defined on $\mathcal{X} = \{x_1, x_2, x_3\}$ and having focal sets $\mathcal{X}$ and $\{x_1, x_3\}$, with respective masses 0.7 and 0.3, may be denoted by $\{x_1, x_3\}^{0.7}$. Using this notation, every non dogmatic $\textcircled{\cap}$-separable MF m may be uniquely expressed as [34, 6]:

$$m \quad = \quad \textcircled{\cap}_{A \subset \mathcal{X}} A^{w(A)}, \tag{6}$$

with $w(A) \in (0, 1]$ for all $A \subset \mathcal{X}$. Example 1 illustrates Equation (6).

Example 1. *Let $\mathcal{X} = \{x_1, x_2, x_3\}$ and m be a non dogmatic MF defined on $\mathcal{X}$ by:*

$$\begin{aligned} m(\{x_1\}) &= 0.6, \\ m(\{x_1, x_3\}) &= 0.12, \\ m(\mathcal{X}) &= 0.28. \end{aligned}$$

We have

$$m = \emptyset^1 \circledcap \{x_1\}^{0.4} \circledcap \{x_2\}^1 \circledcap \{x_3\}^1 \circledcap \{x_1, x_2\}^1 \circledcap \{x_1, x_3\}^{0.7} \circledcap \{x_2, x_3\}^1,$$

or, equivalently, $m = \circledcap_{A \subset \mathcal{X}} A^{w(A)}$, with

$$w(A) = \begin{cases} 0.4 & \text{if } A = \{x_1\}, \\ 0.7 & \text{if } A = \{x_1, x_3\}, \\ 1 & \forall A \in 2^{\mathcal{X}} \setminus \{\{x_1\}, \{x_1, x_3\}, \mathcal{X}\}, \end{cases}$$

Hence, m is a $\circledcap$-separable MF.

For any non dogmatic $\circledcap$-separable MF m uniquely expressed as (6), let us define the sets $\mathcal{C} = \{A | A \subset \mathcal{X}, w(A) < 1\}$ and $\mathcal{W} = \{A^{w(A)} | A \in \mathcal{C}\}$; for instance, for the MF m in Example 1, we have $\mathcal{C} = \{\{x_1\}, \{x_1, x_3\}\}$ and $\mathcal{W} = \{\{x_1\}^{0.4}, \{x_1, x_3\}^{0.7}\}$. We will refer to set $\mathcal{C}$ associated to a non dogmatic $\circledcap$-separable MF m, as its *conjunctive core*. Clearly, since A^1 is equivalent to the vacuous mass function $m_{\mathcal{X}}$ for any $A \subset \mathcal{X}$ and as $m_{\mathcal{X}}$ is a neutral element for $\circledcap$, Equation (6) reduces to

$$m = \circledcap_{A \in \mathcal{C}} A^{w(A)}, \tag{7}$$

for any non dogmatic $\circledcap$-separable MF m such that $m \neq m_{\mathcal{X}}$[3], *i.e.*, m can be uniquely expressed as the conjunctive combination of the simple MFs in $\mathcal{W}$. For instance, for MF m in Example 1, we have $m = \{x_1\}^{0.4} \circledcap \{x_1, x_3\}^{0.7}$.

Smets [34] further shows that in fact any non dogmatic MF can be obtained from simple MFs. More precisely, let m be a non dogmatic MF, then it may be uniquely expressed as the conjunctive decombination of two non dogmatic $\circledcap$-separable MFs, that is, as:

$$m = m^c \oslash m^d, \tag{8}$$

where m^c and m^d are non dogmatic $\circledcap$-separable MFs, such that their conjunctive cores denoted respectively by $\mathcal{C}^c$ and $\mathcal{C}^d$ satisfy $\mathcal{C}^c \cap \mathcal{C}^d = \emptyset$, as illustrated by Example 2 below. This decomposition into simple MFs of a non dogmatic MF m is referred to as the *conjunctive canonical decomposition* of m. The m^c and m^d components in (8) are called the *confidence and diffidence components*, respectively, of m by Smets [34], who proposed to view m^c as representing positive evidence ("good reasons to believe") in some propositions $A \subseteq \mathcal{X}$, and m^d as representing negative evidence ("good reasons not to believe") in some other propositions.

Example 2 (Based on Example 2 of [34])**.** *Let $\mathcal{X} = \{x_1, x_2, x_3\}$ and m be a MF defined on $\mathcal{X}$ by:*

$$\begin{aligned} m(\{x_1, x_2\}) &= 1/3, \\ m(\{x_1, x_3\}) &= 1/3, \\ m(\mathcal{X}) &= 1/3. \end{aligned}$$

[3] If $m = m_{\mathcal{X}}$, *i.e.*, m is vacuous, then we have $w(A) = 1$ for all $A \subset \mathcal{X}$ and thus $\mathcal{C} = \emptyset$.

We have $m = m^c \bigcirc\!\!\!\!\!\!\not\cap\, m^d$*, with* $m^c = \bigcirc\!\!\!\!\!\cap_{A \subset \mathcal{X}} A^{w^c(A)}$ *where*

$$w^c(A) = \begin{cases} 0.5 & \text{if } A = \{x_1, x_2\}, \\ 0.5 & \text{if } A = \{x_1, x_3\}, \\ 1 & \forall A \in 2^{\mathcal{X}} \setminus \{\{x_1, x_2\}, \{x_1, x_3\}, \mathcal{X}\}, \end{cases}$$

and with $m^d = \bigcirc\!\!\!\!\!\cap_{A \subset \mathcal{X}} A^{w^d(A)}$ *where*

$$w^d(A) = \begin{cases} 0.75 & \text{if } A = \{x_1\}, \\ 1 & \forall A \in 2^{\mathcal{X}} \setminus \{\{x_1\}, \mathcal{X}\}. \end{cases}$$

Therefore, $\mathcal{C}^c = \{\{x_1, x_2\}, \{x_1, x_3\}\}$*,* $\mathcal{C}^d = \{\{x_1\}\}$ *and* $\mathcal{C}^c \cap \mathcal{C}^d = \emptyset$*.*

As shown in [6], it is possible to obtain a relation based on the disjunctive rule $\bigcirc\!\!\!\!\!\cup$ that is the counterpart to (8). Let us call $\bigcirc\!\!\!\!\!\cup$*-separable* a MF that can be obtained as the result of the combination by the disjunctive rule of so-called *negative simple* MFs, which are MFs having at most two focal sets, including the empty set $\emptyset$ [6]. A negative simple MF having focal sets $\emptyset$ and $A \neq \emptyset$, with respective masses v and $1-v$, $v \in [0,1]$, may be simply denoted by A_v; for instance, a MF defined on $\mathcal{X} = \{x_1, x_2, x_3\}$ and having focal sets $\emptyset$ and $\{x_1, x_3\}$, with respective masses 0.7 and 0.3, may be denoted by $\{x_1, x_3\}_{0.7}$. Every non normal $\bigcirc\!\!\!\!\!\cup$-separable MF m may then be uniquely expressed as [6]:

$$m = \bigcirc\!\!\!\!\!\cup_{A \neq \emptyset} A_{v(A)}, \tag{9}$$

with $v(A) \in (0,1]$ for all $A \neq \emptyset$. For any non normal $\bigcirc\!\!\!\!\!\cup$-separable MF m uniquely expressed as (9), we refer to the set $\mathcal{C}^{disj} = \{A | A \neq \emptyset, v(A) < 1\}$ as the *disjunctive core* of m.

Then, any non normal MF m may be uniquely expressed as the disjunctive decombination of two non normal $\bigcirc\!\!\!\!\!\cup$-separable MFs, that is, as:

$$m = m^{c,disj} \bigcirc\!\!\!\!\!\!\not\cup\, m^{d,disj}, \tag{10}$$

where $m^{c,disj}$ and $m^{d,disj}$ are non normal $\bigcirc\!\!\!\!\!\cup$-separable MFs, such that their disjunctive cores denoted respectively by $\mathcal{C}^{c,disj}$ and $\mathcal{C}^{d,disj}$ satisfy $\mathcal{C}^{c,disj} \cap \mathcal{C}^{d,disj} = \emptyset$. This decomposition into negative simple MFs of a non normal MF is referred to as its *disjunctive canonical decomposition.*

2.2 Correction mechanisms

Knowledge about the reliability of a source is classically taken into account in the TBM through the *discounting* operation [32, 33]. Suppose a source S providing a piece of information represented by a MF m_S. Let β, with $\beta \in [0,1]$, be the agent's degree of belief that the source is reliable. The agent's belief m on $\mathcal{X}$ is then defined by:

$$m(A) = \beta\, m_S(A) + (1-\beta) m_{\mathcal{X}}(A), \quad \forall A \subseteq \mathcal{X}. \tag{11}$$

Remarkably, Equation (11) is also obtained if the agent assumes that the source is reliable with mass β and not reliable with mass $1-\beta$, rather than if he assumes that the source is reliable with degree of belief β [23, 28]. This alternative interpretation of discounting may be sometimes more instructive when discounting needs to be compared with other correction mechanisms.

Denœux and Smets [7] introduce a correction mechanism, called *de-discounting*, which is basically the inverse of the discounting operation. Assume an agent that receives a mass function m_S from a source, m_S being the result of a discounting with degree β, $1 - m_S(\mathcal{X}) \leq \beta \leq 1$, of some mass function m. Assume further that the agent believes that this discounting is not valid. He can then recover m using de-discounting:

$$m(A) = \frac{m_S(A) - (1-\beta)m_{\mathcal{X}}(A)}{\beta}, \quad \forall A \subseteq \mathcal{X}. \tag{12}$$

Mercier *et al.* [23] consider the case where the agent has some knowledge about the reliability of a source, conditionally on different subsets (contexts) A of $\mathcal{X}$ such that the set of these contexts forms a partition of $\mathcal{X}$. Precisely, let β_A, with $\beta_A \in [0,1]$, be the agent's degree of belief that the source is reliable in context $A \subseteq \mathcal{X}$ and let $\mathcal{A}$ be the set of contexts for which the agent possesses such contextual knowledge, where $\mathcal{A}$ forms a partition of $\mathcal{X}$. The agent's belief m on $\mathcal{X}$ is then defined by [23]

$$m = m_S \circledcup (\circledcup_{A \in \mathcal{A}} A_{\beta_A}),$$

or, for short (with an obvious abuse of notation already used in [20]),

$$m = m_S \circledcup_{A \in \mathcal{A}} A_{\beta_A}, \tag{13}$$

where A_{β_A} denotes the negative simple MF having focal sets $\emptyset$ and A with respective masses β_A and $1 - \beta_A$. Equation (13) is known as *contextual discounting based on a coarsening*. It extends discounting defined by (11), which can be expressed as [23]:

$$m = m_S \circledcup \mathcal{X}_\beta, \tag{14}$$

where $\mathcal{X}_\beta$ denotes the negative simple MF having focal sets $\emptyset$ and $\mathcal{X}$ with respective masses β and $1-\beta$, *i.e.*, discounting is a particular case of contextual discounting based on a coarsening, which is recovered for $\mathcal{A} = \{\mathcal{X}\}$. Moreover, Mercier *et al.* [20] provide an equivalent representation for (13) using the fact that the term $\circledcup_{A \in \mathcal{A}} A_{\beta_A}$ in (13) constitutes a MF whose canonical decomposition into negative simple MFs is direct (from its definition one can see that it is a $\circledcup$-separable MF). This other representation is based on the so-called disjunctive weight function [6] (we refer the interested reader to [20] for details on this other representation).

Recently, Mercier *et al.* [20, 21] consider the more general case where the agent has some knowledge about the reliability of a source, conditionally on different subsets (contexts) A of $\mathcal{X}$, but where the set of these contexts can be arbitrary, that is do not need to form a partition of $\mathcal{X}$. Let β_A, with $\beta_A \in [0,1]$, be the agent's degree of belief that the source is reliable in context $A \subseteq \mathcal{X}$ and let $\mathcal{A}$ be the set of contexts for which the agent possesses such contextual knowledge. The agent's belief m on $\mathcal{X}$ is then defined by [21]:

$$m = m_S \circledcup (\circledcap_{A \in \mathcal{A}} \overline{A}^{1-\beta_A}). \tag{15}$$

In the correction schemes recalled above, the reliability of a source is assimilated to its relevance as explained in [28]. In [28], Pichon *et al.* assume that the reliability of a source involves in addition another dimension: its truthfulness. Pichon *et al.* [28] note that there exists various forms of lack of truthfulness for a source. However, Pichon *et al.* [28] study only the crudest description of the lack of truthfulness, where a non truthful source is a source that declares the contrary of what it knows. According to this definition, from a piece of information of the form $\mathbf{x} \in B$ for some $B \subseteq \mathcal{X}$ provided by a relevant source S, one must conclude that $\mathbf{x} \in B$ or $\mathbf{x} \in \overline{B}$, depending on whether the source S is assumed to be truthful or not. More generally, suppose that S provides a piece of information represented by a MF m_S and that the agent thinks that the source is truthful with mass β and non truthful with mass $1-\beta$. Then, his belief m on $\mathcal{X}$ is defined by [28]:

$$m(A) = \beta \cdot m_S(A) + (1-\beta) \cdot \overline{m}_S(A), \quad \forall A \subseteq \mathcal{X}, \tag{16}$$

where $\overline{m}_S$ denotes the *negation* of MF m_S defined as $\overline{m}_S(A) = m_S(\overline{A})$, $\forall A \subseteq \mathcal{X}$ [9]. The operation defined by (16) may be called *negating* of a belief function, since m becomes closer to the negation $\overline{m}_S$ of m_S as β approaches 0. This is in contrast with the discounting operation, for which m becomes closer to the vacuous mass function $m_{\mathcal{X}}$ as β approaches 0.

Finally, Mercier *et al.* [20] introduce formally two contextual correction mechanisms, called *contextual discounting* (CD) and *contextual reinforcement* (CR) hereafter. They are defined as follows. Let m_S be a MF provided by a source S. Then, the CD of m_S is the MF m defined by:

$$m = m_S \bigcirc\!\!\!\!\!\cup_{A \in \mathcal{A}} A_{\beta_A}, \tag{17}$$

and the CR of m_S is the MF m defined by:

$$m = m_S \bigcirc\!\!\!\!\!\cap_{A \in \mathcal{A}} A^{\beta_A}, \tag{18}$$

with $\beta_A \in [0,1]$, $A \in \mathcal{A}$, for some subset $\mathcal{A}$ of $2^{\mathcal{X}}$. CD (17) is clearly a straightforward formal generalization of contextual discounting based on a coarsening (13), the mere difference being that $\mathcal{A}$ in (17) do not need to form a partition of $\mathcal{X}$ contrary to that in (13). Mercier *et al.* [20] show that CR amounts to the negation of the CD of the negation of m_S. However, they do not go further in providing a clear explanation as to what knowledge about the behavior of the source this correction of m_S correspond, nor do they provide an interpretation for CD as shown by [21]. One of the main results of this paper is to provide an interpretation for both CD and CR; it relies on an extension of Pichon *et al.* [28] truthfulness model, which is introduced in two steps (Section 3 then Section 4).

3 Contextual truthfulness

As recalled in Section 2.2, Pichon *et al.* [28] study only a rudimentary form of non truthfulness. In this section, a detailed analysis of Pichon *et al.* [28] truthfulness model is first conducted. This analysis then leads naturally to a refined model of source truthfulness that allows the integration of more subtle knowledge about the lack of truthfulness of an information source.

3.1 Analysis of Pichon *et al.* truthfulness model

Let us review and analyze in some details Pichon *et al.* [28] truthfulness model. This analysis will be informed by the following example (Example 3) borrowed from [14], which will be subsequently adapted to try and provide new insights on Pichon *et al.* truthfulness model.

Example 3 (Example 1[4] of [14])**.** *Suppose a murder has been committed. There are three suspects: $Peter$, $John$, and $Mary$. In the belief function framework, the set $\mathcal{X} = \{Peter, John, Mary\}$ can be seen as the frame of discernment associated to the parameter $\mathbf{x}$ representing the murderer.*

Suppose a witness who is aware that the three suspects are $Peter$, $John$, and $Mary$, and that tells that the murderer was a man. This piece of information is equivalent to an agent who receives it, to the testimony $\mathbf{x} \in B$, with $B = \{Peter, John\}$. Hence, based on this evidence, the following MF representing the agent's belief on the murderer can be constructed:

$$m(\{Peter, John\}) = 1. \tag{19}$$

This kind of example is quite common in the literature on belief functions and is often used to illustrate notions of the framework (see in particular the original problem "The murder of Mr. Jones" in Smets and Kennes [39]). Example 3 is admittedly quite simple, yet if we take a closer look at it, a fact that seems trivial but that will nonetheless be instrumental for our analysis, can be noticed. Indeed, by remarking that the witness is actually providing information on the parameter of interest $\mathbf{x}$ via an auxiliary variable $\mathbf{y}$ with domain $\mathcal{Y} = \{male, \neg male\}$, we can notice that an implicit assumption is made in this example: the witness must have the *same* view as the agent on who is a man among the suspects, *i.e.*, $Peter$ and $John$, for the conclusion reached by the agent to be proper, that is, formally, they must both think that the variables $\mathbf{y}$ and $\mathbf{x}$ are related by the mapping $\rho : \mathcal{Y} \to 2^{\mathcal{X}}$ defined by:

$$\rho(\{man\}) = \{Peter, John\}, \quad \rho(\{\neg man\}) = \{Mary\},$$

and known as a *refining* [32]. Obviously, in this example, the witness most certainly has the same view as the agent on who is a man, thus this assumption can be safely kept implicit and the conclusion reached by the agent is sound.

Now, as recalled in Section 2.2, Pichon *et al.* [28] consider a truthfulness model where an agent should deduce that $\mathbf{x} \in \overline{B}$ from a piece of information $\mathbf{x} \in B$ provided by a non truthful source, assuming that a non truthful source is a source that declares the contrary of what it knows. For instance, if the witness in Example 3 had declared that the murderer was $Mary$ or $John$, and if the agent had believed that the witness was non truthful, then the agent should have deduced that the murderer was $Peter$. Let us further note that Pichon *et al.* [28] also call a non truthful source, a *lying* source, and they explicitly write that they "use the term *lying* as a synonym of *not telling the truth*, irrespective of the existence of any intention of a source to deceive" [28]. The same applies in this paper, where we use this term to ensure continuity with previous related works. Yet, we may also use the term *biased*, which is perhaps a more appropriate and

[4]This is actually only an excerpt of [14, Example 1], which has furthermore been slightly modified here to fit the formalism and terminology used in the present paper, and to serve our purpose.

less connoted term. Hence, a source that is considered to be lying may equivalently be said to be biased.

We can see at least two practical situations where one would need to use such a truthfulness model and negate the information provided by a source. The first one is perhaps the most obvious and easy to understand: when a source lies intentionally, *i.e.*, has the intention to deceive, and chooses the most simple and common strategy to deceive, *i.e.*, the crudest kind of lie, which is to tell the contrary of what it knows. The second situation is one where the source is non truthful unintentionally, which may be the case when the source has a *different* (precisely opposite) view from that of the agent on the relation between an auxiliary variable and the parameter of interest, as illustrated by Example 4.

Example 4. *Suppose a murder has been committed. There are four male suspects: Eloy, Conrad, Linus and Aeneas. Let* $\mathcal{X} = \{Eloy, Conrad, Linus, Aeneas\}$ *be the domain associated to the parameter* $\mathbf{x}$ *representing the murderer.*

Suppose an agent who sees the suspects and that only Eloy and Conrad have a beard. Assume further that a witness, Jane, is aware that the suspects are Eloy, Conrad, Linus and Aeneas, and that tells that the murderer had a beard. This piece of information provided by Jane is thus equivalent to the agent to the testimony $\mathbf{x} \in B$*, with* $B = \{Eloy, Conrad\}$*.*

Suppose the agent learns some time after receiving this testimony that Jane has actually not met the suspects recently and in particular she is not aware that since she has last seen them, each of the suspects has changed his beard situation (those that did not have a beard have grown one and those that had one have cut it). In other words, her knowledge ρ_{Jane} *about the relation between the auxiliary variable* $\mathbf{y}$ *defined on* $\mathcal{Y} = \{beard, \neg beard\}$ *and the parameter of interest* $\mathbf{x}$ *is outdated and is actually the opposite of the agent's knowledge* ρ_{Ag} *on this relation:*

$$\begin{aligned} \rho_{Jane}(\{beard\}) &= \rho_{Ag}(\{\neg beard\}) = \{Linus, Aeneas\}, \\ \rho_{Jane}(\{\neg beard\}) &= \rho_{Ag}(\{beard\}) = \{Eloy, Conrad\}. \end{aligned}$$

This means that through the piece of information "the murderer had a beard", that was equivalent to the agent to the testimony $\mathbf{x} \in B = \{Eloy, Conrad\}$*, Jane mislead (unintentionally) the agent about what she knows of the guilt of each suspect, and in particular she told the opposite of what she knows, since she actually knows that* $\mathbf{x} \in \{Linus, Aeneas\}$*.*

Hence, from his knowledge on her bias with respect to the beard situation of all the suspects, and in particular that she tells the opposite of what she knows, i.e.*, is non truthful, the agent should deduce from Jane's piece of information* $\mathbf{x} \in B = \{Eloy, Conrad\}$ *that in fact* $\mathbf{x} \in \overline{B} = \{Linus, Aeneas\}$*, which is indeed what Jane actually knows about the murderer.*

Further insight on Pichon *et al.* [28] truthfulness model may be gained by examining precisely what happens when one deduces $\mathbf{x} \in \overline{B}$ from assuming that a source providing testimony $\mathbf{x} \in B$ is lying, that is, tells the contrary of what it knows. In details, it means that the source is assumed to be lying, *i.e.*, telling the contrary of what it knows, whatever it is telling concerning each of the possible values $x \in \mathcal{X}$ that admits parameter $\mathbf{x}$, since one must invert what the source tells for each of these values, as illustrated by Example 5.

Example 5. *Without lack of generality, assume for instance* $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ *and that the source tells* $\mathbf{x} \in B = \{x_3, x_4\}$*,* i.e.*, it tells that* x_3 *and* x_4 *are possible values for* $\mathbf{x}$ *and it tells that* x_1 *and* x_2 *are not possible values for* $\mathbf{x}$*, then one must deduce that* $\mathbf{x} \in \overline{B} = \{x_1, x_2\}$*,* i.e.*,* x_3 *and* x_4 *are not possible values for* $\mathbf{x}$ *and* x_1 *and* x_2 *are possible values for* $\mathbf{x}$*.*

This leads us to introduce the following definition.

Definition 1. *A source is said to be truthful (resp. non truthful)* for a value $x \in \mathcal{X}$*, when it tells what it knows (resp. the contrary of what it knows) for this value.*

According to this new terminology, a non truthful source in Pichon *et al.* [28] truthfulness model is then a source that is non truthful for *ALL* values $x \in \mathcal{X}$ (and a truthful source is a source that is truthful for all values $x \in \mathcal{X}$).

This analysis allows us to highlight that the crude form of non truthfulness studied in [28] is actually a quite strong model of the lack of truthfulness of an information source, and as such might only be suitable for a limited number of practical situations such as the ones discussed above. It seems thus interesting to study more subtle variants of this model, and in particular to relax its assumption about the truthfulness of the source for each $x \in \mathcal{X}$: a source could be non truthful only for *SOME* values $x \in \mathcal{X}$ (and truthful for all other values $x \in \mathcal{X}$).

This alternative should add some flexibility in terms of knowledge that can be taken into account about a source lack of truthfulness, and could thus be interesting from an applicative point of view. Such a study is carried out in the next section.

3.2 Contextual liar

Let us consider the case of a source assumed to be non truthful for *some* values $x \in \mathcal{X}$, and to be truthful for all other values $x \in \mathcal{X}$. Let $A \subseteq \mathcal{X}$ be the set of values for which the source is assumed to be truthful, and $\overline{A}$ the set of values for which it is assumed to be non truthful. For short, we may say that the source is truthful in A and non truthful in $\overline{A}$, or even more simply, when no confusion is possible, that the source is non truthful in $\overline{A}$ (or biased in $\overline{A}$) – it will then be implicit that the source is truthful in A.

Definition 2 (Non truthful in $\overline{A}$)**.** *A source is said to be* non truthful in $\overline{A}$ *if it is truthful for all* $x \in A$*, and non truthful for all* $x \in \overline{A}$*. This state of the source is denoted by* ℓ_A*.*

Intuitively, this state corresponds simply to a source that lies only for a subset of values, that is, it tells the opposite of what it knows for each value in this set, and tells what it knows for the values outside of this set. If we call this latter set a *context*, then the source may be seen and referred to as a contextual liar.

A sensible question is then: what must one conclude about $\mathbf{x}$ when the source tells $\mathbf{x} \in B$ and is assumed to be in state ℓ_A? The answer is provided by Proposition 1.

Proposition 1. *If a source tells* $\mathbf{x} \in B$ *and is assumed to be non truthful in* $\overline{A}$*, one must deduce that* $\mathbf{x} \in B \bigcap A$*.*

Proof. To prove this proposition, one merely needs to look in turn at each $x \in \mathcal{X}$ and to find which one of the four following cases applies:

1. If the source tells x is possibly the actual value of $\mathbf{x}$, *i.e.*, the information $\mathbf{x} \in B$ provided by the source is such that $x \in B$,
   (a) And if the source is assumed to be truthful for x, *i.e.*, $x \in A$, then one must conclude that x is possibly the actual value of $\mathbf{x}$;
   (b) And if the source is assumed to be non truthful for x, *i.e.*, $x \in \overline{A}$, then one must conclude that x is not a possibility for the actual value of $\mathbf{x}$;
2. If the source tells x is not a possibility for the actual value of $\mathbf{x}$, *i.e.*, $x \notin B$,
   (a) And if the source is assumed to be truthful for x, *i.e.*, $x \in A$, then one must conclude that x is not a possibility for the actual value of $\mathbf{x}$;
   (b) And if the source is assumed to be non truthful for x, *i.e.*, $x \in \overline{A}$, , then one must conclude that x is possibly the actual value of $\mathbf{x}$.

Table 1 synthesizes these four cases: it lists exhaustively, *i.e.*, for all possible cases with respect to the membership of a given value $x \in \mathcal{X}$ to the sets B and A, whether one should deduce that this value x is possibly the actual value of $\mathbf{x}$ or not – the former is indicated by a 1 and the latter by a 0 in column ℓ_A. According to Table 1, when the

Table 1: Non truthful in $\overline{A}$

$x \in B$	$x \in A$	ℓ_A
0	0	1
0	1	0
1	0	0
1	1	1

source is assumed to be in state ℓ_A, then one should deduce that $x \in \mathcal{X}$ is a possible value for $\mathbf{x}$ iff x belongs to both B and A or does not belong to both B and A (which corresponds to logical equality), and therefore, since this holds for all $x \in \mathcal{X}$, one should deduce that $\mathbf{x} \in (B \cap A) \cup (\overline{B} \cap \overline{A}) = B \sqcap A$. □

Example 6. *As an illustration of Proposition 1, assume for instance* $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ *and that the source tells* $\mathbf{x} \in B = \{x_3, x_4\}$. *Furthermore, assume the source is in state* $\ell_{\{x_1,x_3\}}$, i.e., *is non truthful for* x_2 *and for* x_4. *Then, one should deduce that* $\mathbf{x} \in \{x_3, x_4\} \sqcap \{x_1, x_3\} = \{x_2, x_3\}$.

Remark 1. *The non truthful state considered by Pichon* et al. *[28], which corresponds to a source that is non truthful for all values* $x \in \mathcal{X}$, *is equivalent to the state* $\ell_\emptyset$, *and is thus a particular case of the states* ℓ_A, $A \subseteq \mathcal{X}$.

Remark 2. *The states* ℓ_A, $A \subseteq \mathcal{X}$, *correspond to the states used by Pichon [25] to provide an interpretation for the* α*-conjunctions [35, 37], which is to our knowledge the first one to consider such states (but they are not introduced and discussed with as many details in [25] as they are in this paper).*

In practice, similarly to what we have done in Section 3.1 for the non truthful state considered by Pichon *et al.* [28], we can distinguish two situations where states ℓ_A may be useful. First, the source may be intentionally non truthful in $\overline{A}$, simply to deceive an agent, yet in a more subtle way than what is allowed by Pichon *et al.* [28] non truthful state. Indeed, the source may think that to better deceive the agent, it is going to lie, *i.e.*, tell the opposite of what it knows, but only for a subset of values rather than for all possible values that can take the parameter of interest. Second, in the unintentional case, such a state ℓ_A can be explained by a *difference* between the source and the agent who receives the piece of information provided by the source, on the relation they respectively believe holds between an auxiliary variable and the parameter of interest. More precisely, such a state can be encountered when the source is "wrong" on this relation – wrong with respect to the agent's knowledge – but only for a subset of values in $\mathcal{X}$, as illustrated by Example 7.

Example 7 (Example 4 continued)**.** *Consider again the setting of Example 4, but this time assume the agent learns some time after receiving Jane's testimony that she has actually not met recently SOME of the suspects, and in particular she is unaware that since she has last seen these suspects, each of them has changed his beard situation. In other words, her knowledge ρ_{Jane} about the relation between $\mathbf{y}$ and $\mathbf{x}$ is partially outdated and is actually partially the opposite of the agent's knowledge ρ_{Ag} on this relation. Indeed, let Conrad and Linus be these suspects that she has not met recently, we have then on the one hand*

$$\rho_{Ag}(\{beard\}) = \{Eloy, Conrad\}, \quad \rho_{Ag}(\{\neg beard\}) = \{Linus, Aeneas\},$$

and on the other hand

$$\rho_{Jane}(\{beard\}) = \{Eloy, Linus\}, \quad \rho_{Jane}(\{\neg beard\}) = \{Conrad, Aeneas\}.$$

This means that through the piece of information "the murderer had a beard", that was equivalent to the agent to the testimony $\mathbf{x} \in B = \{Eloy, Conrad\}$, Jane mislead unintentionally (because of her partially outdated knowledge about the beard situation of the suspects) the agent about what she knows of the guilt of the two suspects Conrad and Linus, since she actually knows that $\mathbf{x} \in \{Eloy, Linus\}$. Indeed, she told the opposite of what she knows for these two suspects, since, e.g.*, she told that Conrad was possibly the murderer ($Conrad \in B$) whereas she knows that he is not. On the contrary, she told the truth about Eloy and Aeneas, since,* e.g.*, she told that Eloy was possibly the murderer ($Eloy \in B$) and she knows that he is possibly the murderer.*

*Hence, since she told the truth for the suspects in $A = \{Eloy, Aeneas\}$ and lied (*i.e.*, was non truthful) for the suspects in $\overline{A}$, or in other words is in state $\ell_{\{Eloy,Aeneas\}}$, the agent should deduce from Jane's piece of information $\mathbf{x} \in B$ that in fact*

$$\mathbf{x} \in B \,\underline{\cap}\, A \quad = \quad \{Eloy, Conrad\} \,\underline{\cap}\, \{Eloy, Aeneas\} = \{Eloy, Linus\},$$

which is indeed what the source actually knows about the murderer.

This section has introduced a refined model of source truthfulness, which allows one to account for a contextual lack of truthfulness of a source. This model has been obtained by relaxing a strong assumption underlying Pichon *et al.* [28] truthfulness model, which was brought to light by a detailed analysis of this latter model. Next section will show that it is possible to push this analysis further and reveal another strong assumption underlying Pichon *et al.* [28] truthfulness model.

4 Polarized truthfulness

In this section, the analysis of Pichon *et al.* [28] truthfulness model started in Section 3 is pursued. This analysis then yields a further refined model of source truthfulness.

4.1 Analysis of Pichon *et al.* truthfulness model (continued)

Let us consider again Example 5: according to Pichon *et al.* [28] truthfulness model, when a source tells $\mathbf{x} \in B = \{x_3, x_4\}$ and is assumed to be non truthful, one must deduce that $\mathbf{x} \in \overline{B} = \{x_1, x_2\}$, that is:

- the source tells that x_3 is a possible value for $\mathbf{x}$, and one must deduce that x_3 is actually not a possible value for $\mathbf{x}$;
- it tells that x_4 is a possible value for $\mathbf{x}$, and one must deduce that it is not;
- it tells that x_1 is not a possible value for $\mathbf{x}$, and one must deduce that it is;
- it tells that x_2 is not a possible value for $\mathbf{x}$, and one must deduce that it is.

To characterize what is at stake in the above reasoning, we introduced Definition 1, that is, the notion of truthfulness for a value $x \in \mathcal{X}$, and revealed that a non truthful source in Pichon *et al.* [28] sense, is a source that is non truthful for each value $x \in \mathcal{X}$ since it amounts to assuming that it tells the contrary of what it knows for each of those values.

Actually, one can be even more specific about the assumptions underlying Pichon *et al.* [28] truthfulness model, by distinguishing between positive clauses and negative clauses, also known as clauses having positive polarity and negative polarity (in a grammatical sense; see, *e.g.*, [13, Chapter 8]), told by the source. For instance, when the source tells that x_3 is a possible value for $\mathbf{x}$, this is a positive clause told by the source, and when the source tells that x_1 is not a possible value for $\mathbf{x}$, it is a negative clause.

We may then characterize more finely the truthfulness of the source for each $x \in \mathcal{X}$, that is, with respect to the polarity of the clauses it tells.

Definition 3. *A source is said to be* positively *truthful (resp.* positively *non truthful) for a value $x \in \mathcal{X}$, when it tells that x is a possible value for $\mathbf{x}$ and knows that it is (resp. it is not) a possible value for $\mathbf{x}$.*

Definition 4. *A source is said to be* negatively *truthful (resp.* negatively *non truthful) for a value $x \in \mathcal{X}$, when it tells that x is not a possible value for $\mathbf{x}$ and knows that it is not (resp. it is) a possible value.*

According to this terminology, a source which is assumed to be non truthful for $x \in \mathcal{X}$ (Definition 1), is assumed to be positively *AND* negatively non truthful for x, since whatever it may tell about x (be it a positive clause or a negative clause), it is assumed to tell the contrary of what it knows. Most importantly, a non truthful source in Pichon *et al.* [28] truthfulness model is then a source that is assumed to be positively *AND* negatively non truthful, for *ALL* values $x \in \mathcal{X}$ (and a truthful source is a source that is positively and negatively truthful, for all values $x \in \mathcal{X}$).

This finer analysis reinforces the statement made in Section 3.1: the crude form of non truthfulness studied in [28] is actually a rather strong model of the lack of truthfulness of an information source. It make two assumptions, one on the context (set of values) concerned by the lack of truthfulness and one on the polarity of the lack of truthfulness, both of which are strong : the values concerned by the lack of truthfulness are *all* the values of the frame, and *both* polarities (positive and negative) are concerned by the lack of truthfulness.

Here again, it seems interesting to search for and study more subtle variants of this crude model, to obtain more versatility. This amounts to relaxing further the model, that is, relaxing the two above assumptions about the truthfulness of the source for each $x \in \mathcal{X}$: a source could be positively OR negatively non truthful for SOME values $x \in \mathcal{X}$ (and truthful for all other values $x \in \mathcal{X}$).

Concretely, such relaxation comes down to three cases: a source could be

1. positively and negatively non truthful for some values $x \in \mathcal{X}$ (and truthful for all other values $x \in \mathcal{X}$);
2. positively non truthful and negatively truthful for some values $x \in \mathcal{X}$ (and truthful for all other values $x \in \mathcal{X}$);
3. positively truthful and negatively non truthful for some values $x \in \mathcal{X}$ (and truthful for all other values $x \in \mathcal{X}$).

The first case was the object of the study carried out in Section 3.2, since, as already mentioned above, a source which is non truthful for $x \in \mathcal{X}$ (Definition 1) is more precisely said to be, using Definitions 3 and 4, positively and negatively non truthful for x. Cases 2 and 3 are treated in Sections 4.2 and 4.3, respectively.

4.2 Positive contextual liar

Let us turn our attention to case 2 above, *i.e.*, the case where a source is assumed to be positively non truthful and negatively truthful for some values $x \in \mathcal{X}$, and to be truthful for all other values $x \in \mathcal{X}$. Let $A \subseteq \mathcal{X}$ be the set of values for which the source is assumed to be truthful, and $\overline{A}$ the set of values for which it is assumed to be positively non truthful and negatively truthful. For short, we may say that the source is truthful in A, and positively non truthful and negatively truthful in $\overline{A}$, or even more simply, when no confusion is possible, that the source is positively non truthful in $\overline{A}$ (or positively biased in $\overline{A}$), hence mentioning explicitly only the situation where the source commits a lie.

Definition 5 (Positively non truthful in $\overline{A}$)**.** *A source is said to be* positively non truthful in $\overline{A}$ *if it is truthful for all $x \in A$, and positively non truthful and negatively truthful for all $x \in \overline{A}$. This state of the source is denoted by p_A.*

This state corresponds to a source that lies (*i.e.*, tells the contrary of what it knows) only for a subset of values and only when it tells for any of these values that it is a possibility for the actual value of $\mathbf{x}$.

Let us note that this is yet again a more elaborate, thus more interesting, strategy for a source to deceive an agent than to simply tell the opposite of what it knows as in Pichon *et al.* [28], and thus state p_A may be useful when faced with intentionally deceitful sources (the potential usefulness of state p_A in the unintentional case will be commented later in this section). A source lying in this way may be referred to as a positive contextual liar in the sequel.

Proposition 2. *If a source tells* $\mathbf{x} \in B$ *and is assumed to be positively non truthful in* $\overline{A}$*, one must deduce that* $\mathbf{x} \in B \cap A$*.*

Proof. The proof is similar to the proof of Proposition 1 and based on the fact that when the source is in state p_A, the four possible cases with respect to the membership of a given value $x \in \mathcal{X}$ to the sets B and A must be treated according to Table 2.

Table 2: Positively non truthful in $\overline{A}$

$x \in B$	$x \in A$	p_A
0	0	0
0	1	0
1	0	0
1	1	1

□

Example 8. *As an illustration of Proposition 2, assume for instance* $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ *and that the source tells* $\mathbf{x} \in B = \{x_3, x_4\}$*. Furthermore, assume the source is in state* $p_{\{x_1,x_3\}}$, i.e., *is positively non truthful for* x_2 *and for* x_4*. Then, one should deduce that* $\mathbf{x} \in \{x_3, x_4\} \cap \{x_1, x_3\} = \{x_3\}$*.*

In the unintentional case, similarly as state ℓ_A, state p_A can be explained by a difference between the source and the agent, on the relation they respectively believe holds between an auxiliary variable and the parameter of interest, as illustrated by Example 9.

Example 9 (Example 4 continued)**.** *Consider again the setting of Example 4, but this time assume the agent learns some time after receiving Jane's testimony that she has actually not met recently some of the suspects, and more specifically she is unaware that among these suspects that she has not met recently, those that have changed their beard situation are only those that have a beard.*

Let Conrad and Linus be these suspects that she has not met recently. Among these two suspects, only Conrad has a beard, and thus he is the only one who has changed his beard situation since Jane last saw these two suspects, which means that both Conrad and Linus did not have a beard when Jane last saw them. We have then

$$\rho_{Ag}(\{beard\}) = \{Eloy, Conrad\}, \quad \rho_{Ag}(\{\neg beard\}) = \{Linus, Aeneas\},$$

and

$$\rho_{Jane}(\{beard\}) = \{Eloy\}, \quad \rho_{Jane}(\{\neg beard\}) = \{Conrad, Linus, Aeneas\}.$$

This means that through the piece of information "the murderer had a beard", that was equivalent to the agent to the testimony $\mathbf{x} \in B = \{Eloy, Conrad\}$*, Jane mislead unintentionally the agent about what she actually knows of the guilt of the suspects. Precisely, due to her partially outdated knowledge on the beard situation of the suspects, Jane told what she knows for the suspects in* $A = \{Eloy, Aeneas\}$*, and told the opposite* *of what she knows for each of the suspects in $\overline{A}$ only when she told that he is possibly the murderer.*

Hence, since she was truthful for the suspects in $A = \{Eloy, Aeneas\}$ and was positively non truthful and negatively truthful for the suspects in $\overline{A}$, or in other words is in state $p_{\{Eloy,Aeneas\}}$, the agent should deduce from Jane's piece of information $\mathbf{x} \in B$ that in fact

$$\mathbf{x} \in B \cap A \quad = \quad \{Eloy, Conrad\} \cap \{Eloy, Aeneas\} = \{Eloy\},$$

which is indeed what the source actually knows about the murderer.

4.3 Negative contextual liar

To provide a full picture, case 3 mentioned at the end of Section 4.1, is studied in this section, yet more briefly since it is quite similar to case 2.

Let us recall that case 3 corresponds to assuming that a source is positively truthful and negatively non truthful for some values $x \in \mathcal{X}$, and is truthful for all other values $x \in \mathcal{X}$. Let $A \subseteq \mathcal{X}$ be the set of values for which the source is assumed to be positively truthful and negatively non truthful, and $\overline{A}$ the set of values for which it is assumed to be truthful[5]. For short, the source may be said to be negatively non truthful in A (or negatively biased in A).

Definition 6 (Negatively non truthful in A)**.** *A source is said to be* negatively non truthful in A *if it is positively truthful and negatively non truthful for all $x \in A$, and truthful for all $x \in \overline{A}$. This state is denoted by n_A.*

This state corresponds to a source that lies only for a subset of values and only when it tells for any of these values that it is not a possibility for the actual value of $\mathbf{x}$. A source lying in this way may therefore be called a negative contextual liar.

Proposition 3. *If a source tells $\mathbf{x} \in B$ and is assumed to be negatively non truthful in A, one must deduce that $\mathbf{x} \in B \cup A$.*

Proof. The proof is similar to the proof of Proposition 1 and based on the fact that when the source is in state n_A, the four possible cases with respect to the membership of a given value $x \in \mathcal{X}$ to the sets B and A must be treated according to Table 3.

Table 3: Negatively non truthful in A

$x \in B$	$x \in A$	n_A
0	0	0
0	1	1
1	0	1
1	1	1

□

[5] Contrary to the states ℓ_A and p_A, where A is the set of values for which the source is assumed to be truthful. This slight difference in denoting which set is the set where the source is truthful, is useful to present more elegantly results in the next sections, but it has no fundamental consequence.

Example 10. *As an illustration of Proposition 3, assume for instance* $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ *and that the source tells* $\mathbf{x} \in B = \{x_3, x_4\}$*. Furthermore, assume the source is in state* $n_{\{x_1,x_3\}}$, i.e., *is negatively non truthful for* x_1 *and for* x_3*. Then, one should deduce that* $\mathbf{x} \in \{x_3, x_4\} \cup \{x_1, x_3\} = \{x_1, x_3, x_4\}$.

For the same reason as state p_A, state n_A may be useful when faced with intentionally deceitful sources. A source may also happen to be a negative contextual liar unintentionally, as illustrated by Example 11.

Example 11 (Example 4 continued)**.** *Consider again the setting of Example 4, but this time assume the agent learns some time after receiving Jane's testimony that she has actually not met recently some of the suspects, and more specifically she is unaware that among these suspects that she has not met recently, those that have changed their beard situation are only those that do not have a beard.*

Let Conrad and Linus be these suspects that she has not met recently. Among these two suspects, only Linus does not have a beard, and thus he is the only one who has changed his beard situation since Jane last saw these two suspects, which means that both Conrad and Linus had a beard when Jane last saw them. We have then

$$\rho_{Ag}(\{beard\}) = \{Eloy, Conrad\}, \quad \rho_{Ag}(\{\neg beard\}) = \{Linus, Aeneas\},$$

and

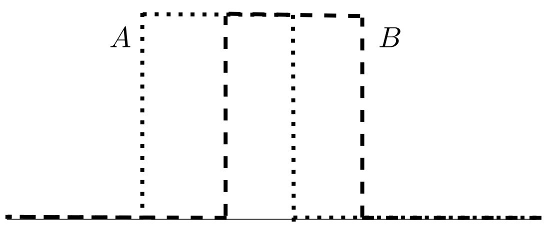
$$\rho_{Jane}(\{beard\}) = \{Eloy, Conrad, Linus\}, \quad \rho_{Jane}(\{\neg beard\}) = \{Aeneas\}.$$

Jane told the opposite of what she knows for each of the suspects in $A = \{Conrad, Linus\}$ *only when she told that he is not the murderer, and she told what she knows for the suspects in* $\overline{A}$*. In other words, she was positively truthful and negatively non truthful for the suspects in* A*, and truthful for the suspects in* $\overline{A}$*. Since she was thus in state* $n_{\{Conrad,Linus\}}$*, the agent should deduce from Jane's piece of information* $\mathbf{x} \in B$ *that in fact*

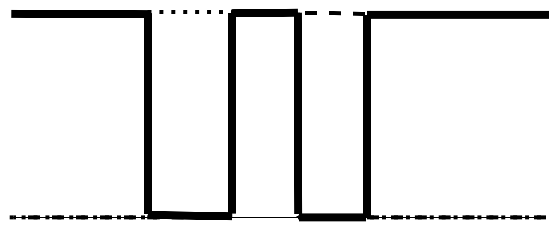
$$\mathbf{x} \in B \cup A \quad = \quad \{Eloy, Conrad\} \cup \{Conrad, Linus\} = \{Eloy, Conrad, Linus\},$$

which is indeed what the source actually knows about the murderer.

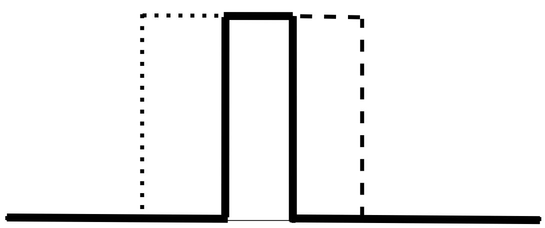
The three kinds – contextual liar, positive contextual liar, and negative contextual liar – of liar studied so far are summarized and illustrated on Figure 1. They constitute natural relaxations of the strong assumptions underlying the state of non truthfulness of Pichon *et al.* [28]. They seem at least as interesting as this state when considering intentionally deceitful sources and may also be used to account for various ways a source may lack truthfulness unintentionally. Yet, the setting considered is quite basic: the testimony provided by the source is crisp ($\mathbf{x} \in B$) and the state of the source is assumed to be known precisely. More generally, both the testimony provided by the source and the knowledge of the agent about the source truthfulness (referred to as *meta-knowledge* in [28]) may be uncertain. This is next section topic, which in addition uses this more general setting to provide an interpretation for contextual discounting as well as an interpretation for contextual reinforcement.



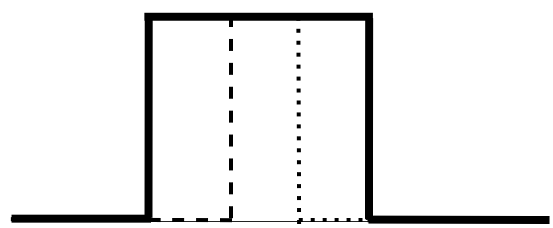
(a) Testimony $\mathbf{x} \in B$ and a context A.



(b) Non truthful in $\overline{A}$: $B \underline{\cap} A$.



(c) Positive liar in $\overline{A}$: $B \cap A$.



(d) Negative liar in A: $B \cup A$.

Figure 1: The three kinds of contextual biases of interest in this paper. (1a) Indicator functions of a testimony $\mathbf{x} \in B \subseteq \mathcal{X}$ and of a context $A \subseteq \mathcal{X}$. (1b) In bold, result of the transformation of the testimony $\mathbf{x} \in B$ given contextual lie ℓ_A; in such state, the source is believed truthful in A, hence whatever it says about $\mathbf{x}$ within A should be kept as is, and non truthful in $\overline{A}$, hence whatever it says about $\mathbf{x}$ outside of A should be reversed. (1c) Contextual lie p_A (whenever the source says that a value outside of A is possible, it lies, and whatever it says within A should be kept as is). (1d) Contextual lie n_A (whenever the source says that a value within A is not possible, it lies, and whatever it says outside of A should be kept as is).

5 Interpretation of CD and of CR

In this section, uncertainty is first added to the setting considered in Sections 3 and 4, resulting in a general framework able to handle various situations with respect to knowledge about the contextual biases of a source. Then, an interpretation for CR is proposed using this framework. In addition, it is shown that it is possible to provide a similar perspective on CD.

5.1 Uncertain testimony and meta-knowledge

Let $\mathcal{H}$ denote the possible states of a source S with respect to its polarized contextual truthfulness, *i.e.*, $\mathcal{H} = \mathcal{H}_\ell \cup \mathcal{H}_p \cup \mathcal{H}_n$, where $\mathcal{H}_\ell = \{\ell_A | A \subseteq \mathcal{X}\}$, $\mathcal{H}_p = \{p_A | A \subseteq \mathcal{X}\}$ and $\mathcal{H}_n = \{n_A | A \subseteq \mathcal{X}\}$.

Following [28], we can define a multivalued mapping Γ_B from $\mathcal{H}$ to $\mathcal{X}$ that encodes the three kinds of contextual lies studied in Sections 3 and 4:

$$\Gamma_B(\ell_A) = B \sqcap A, \tag{20}$$
$$\Gamma_B(p_A) = B \cap A, \tag{21}$$
$$\Gamma_B(n_A) = B \cup A, \tag{22}$$

for all $A \subseteq \mathcal{X}$. $\Gamma_B(h)$ indicates how to interpret the piece of information $\mathbf{x} \in B$ provided by the source, when the source is assumed to be in some state $h \in \mathcal{H}$. In addition, if the knowledge about the source state is imprecise and given by $H \subseteq \mathcal{H}$, then one should deduce that $\mathbf{x} \in \Gamma_B(H)$, where $\Gamma_B(H)$ denotes the image of H by Γ_B, defined by $\Gamma_B(H) := \bigcup_{h \in H} \Gamma_B(h)$.

Remark 3. *We have*

$$\Gamma_B(p_\mathcal{X}) = \Gamma_B(n_\emptyset) = \Gamma_B(\ell_\mathcal{X}) = B, \quad \forall B \subseteq \mathcal{X}.$$

This is so because these three states correspond actually to the same assumption of a truthful source. As such they may be simply denoted by t in the sequel (in accordance with the notation used for this state in the original paper [28]).

Remark 4. *States ℓ_A, p_A, n_A, $A \subseteq \mathcal{X}$, and their associated transformations* (20) *(logical equality),* (21) *(conjunction),* (22) *(disjunction), of a testimony $\mathbf{x} \in B$, are particular cases of a more general formal model of truthfulness assumptions yielding all possible binary Boolean connectives, as shown in Appendix A.*

As already mentioned, both the testimony of the source and the meta-knowledge of the agent may be uncertain. Let m_S be the uncertain testimony and $m^\mathcal{H}$ the uncertain meta-knowledge. In such case, the *Behavior-Based Correction* (BBC) procedure[6] introduced by Pichon *et al.* [28], can be used to derive the knowledge of the agent on $\mathcal{X}$. It is represented by the MF m defined for all $C \subseteq \mathcal{X}$ as [28]:

$$m(C) = \sum_{H \subseteq \mathcal{H}} m^\mathcal{H}(H) \sum_{B : \Gamma_B(H) = C} m_S(B). \tag{23}$$

[6]The BBC procedure is a general mechanism allowing one to derive an agent's knowledge on $\mathcal{X}$ from an uncertain testimony m_S, when the agent has some uncertain meta-knowledge $m^\mathcal{H}$ about the source, and where $\mathcal{H}$ may represent various state spaces, not necessarily related to the notion of truthfulness. See [28] for details.

For convenience, we may denote by $f_{m^{\mathcal{H}}}(m_S)$ the Behavior-Based Correction of MF m_S according to meta-knowledge $m^{\mathcal{H}}$, *i.e.*, we have $m = f_{m^{\mathcal{H}}}(m_S)$ with m the MF defined by (23). The BBC procedure is illustrated by Example 12.

Example 12. *Let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$. Assume a source S provides the following uncertain testimony:*

$$\begin{aligned} m_S(\{x_1, x_3\}) &= 0.7, \\ m_S(\{x_1, x_2, x_3\}) &= 0.3. \end{aligned}$$

Suppose further the following uncertain knowledge about the quality of the source:

$$\begin{aligned} m^{\mathcal{H}}\left(\{p_{\{x_3,x_4\}}\}\right) &= 0.4, \\ m^{\mathcal{H}}\left(\{p_{\{x_3,x_4\}}, n_{\{x_1,x_4\}}\}\right) &= 0.6, \end{aligned}$$

that is, the source is assumed to be positively biased in $\{x_1, x_2\}$ with mass 0.4, and positively biased in $\{x_1, x_2\}$ or negatively biased in $\{x_1, x_4\}$ with mass 0.6.

Since

$$\begin{aligned} \Gamma_{\{x_1,x_3\}}\left(p_{\{x_3,x_4\}}\right) &= \{x_1, x_3\} \cap \{x_3, x_4\} = \{x_3\}, \\ \Gamma_{\{x_1,x_3\}}\left(\{p_{\{x_3,x_4\}}, n_{\{x_1,x_4\}}\}\right) &= \Gamma_{\{x_1,x_3\}}\left(p_{\{x_3,x_4\}}\right) \bigcup \Gamma_{\{x_1,x_3\}}\left(n_{\{x_1,x_4\}}\right) \\ &= (\{x_3\}) \bigcup (\{x_1, x_3\} \cup \{x_1, x_4\}) = \{x_1, x_3, x_4\}, \\ \Gamma_{\{x_1,x_2,x_3\}}\left(p_{\{x_3,x_4\}}\right) &= \{x_1, x_2, x_3\} \cap \{x_3, x_4\} = \{x_3\}, \\ \Gamma_{\{x_1,x_2,x_3\}}\left(\{p_{\{x_3,x_4\}}, n_{\{x_1,x_4\}}\}\right) &= (\{x_3\}) \bigcup (\{x_1, x_2, x_3\} \cup \{x_1, x_4\}) = \mathcal{X}, \end{aligned}$$

the agent knowledge m on $\mathcal{X}$ is defined, according to the BBC procedure (23), by:

$$\begin{aligned} m(\{x_3\}) &= m^{\mathcal{H}}\left(\{p_{\{x_3,x_4\}}\}\right) \cdot (m_S(\{x_1, x_3\}) + m_S(\{x_1, x_2, x_3\})) \\ &= 0.4 \cdot (0.7 + 0.3) = 0.4, \\ m(\{x_1, x_3, x_4\}) &= m^{\mathcal{H}}\left(\{p_{\{x_3,x_4\}}, n_{\{x_1,x_4\}}\}\right) \cdot m_S(\{x_1, x_3\}) \\ &= 0.6 \cdot 0.7 = 0.42, \\ m(\{x_1, x_2, x_3, x_4\}) &= m^{\mathcal{H}}\left(\{p_{\{x_3,x_4\}}, n_{\{x_1,x_4\}}\}\right) \cdot m_S(\{x_1, x_2, x_3\}) \\ &= 0.6 \cdot 0.3 = 0.18. \end{aligned}$$

Next section will show how BBC may be used to provide an interpretation for CR.

5.2 Contextual reinforcement

Let us consider a particular kind of contextual lie among those introduced in Sections 3 and 4: the states p_A, $A \subseteq \mathcal{X}$, corresponding to the assumptions that the source is a positive liar in $\overline{A}$.

Next proposition, which is based on theses states and that uses the notation introduced in Remark 3, will be instrumental to provide our interpretation of CR.

Proposition 4. *Let m_S be the MF provided by a source S and let $m^{\mathcal{H}}_{A,\cap}$ be our meta-knowledge on the source defined by*

$$m^{\mathcal{H}}_{A,\cap}(\{t\}) = \beta_A, \quad m^{\mathcal{H}}_{A,\cap}(\{p_A\}) = 1 - \beta_A, \tag{24}$$

i.e., *with mass* β_A *the source is truthful, and with mass* $1-\beta_A$ *it is a positive liar in* $\overline{A}$*.*

We have

$$f_{m^{\mathcal{H}}_{A,\cap}}(m_S) = m_S \circledcap A^{\beta_A}.$$

Proof. From the definition of $\circledcap$, we have that, for all $B \subseteq \mathcal{X}$, the quantity $m_S(B) \cdot (A^{\beta_A})(\mathcal{X}) = m_S(B) \cdot \beta_A$ is allocated to set $B \cap \mathcal{X} = B$ and the quantity $m_S(B) \cdot A^{\beta_A}(A) = m_S(B) \cdot (1-\beta_A)$ is allocated to set $B \cap A$.

Similarly, from the definition of the BBC procedure, the quantity $m_S(B) \cdot \beta_A$ is allocated to set $\Gamma_B(t) = B$ and the quantity $m_S(B) \cdot (1-\beta_A)$ is allocated to set $\Gamma_B(p_A) = B \cap A$, for all $B \subseteq \mathcal{X}$.

□

We may then show one of our main results.

Proposition 5. *Let* m_S *be a MF. We have,* $\forall \mathcal{A}$ *and with* $\beta_A \in [0,1]$*,* $\forall A \in \mathcal{A}$*:*

$$m_S \circledcap_{A \in \mathcal{A}} A^{\beta_A} = (\circ_{A \in \mathcal{A}} \, f_{m^{\mathcal{H}}_{A,\cap}})(m_S), \tag{25}$$

where $\circ$ *denotes function composition and where mass functions* $m^{\mathcal{H}}_{A,\cap}$*,* $A \in \mathcal{A}$*, are defined by* (24)*.*

Proof. Without lack of generality, let us index the elements in $\mathcal{A}$: $A_1, ..., A_N$, where $N = |\mathcal{A}|$. Thus, Equation (25) can be rewritten as

$$m_S \circledcap_{i=1}^{N} A_i^{\beta_{A_i}} = (\circ_{i=1}^{N} \, f_{m^{\mathcal{H}}_{A_i,\cap}})(m_S), \tag{26}$$

From Proposition 4, we have: $m_S \circledcap A_1^{\beta_{A_1}} = f_{m^{\mathcal{H}}_{A_1,\cap}}(m_S)$. Hence (26) holds for $N = 1$.

Assume now that (26) holds for $N = Q$. To prove this proposition, it suffices then to show that (26) holds for $N = Q + 1$.

We have:

$$\begin{aligned} m_S \circledcap_{i=1}^{Q+1} A_i^{\beta_{A_i}} &= m_S \circledcap_{i=1}^{Q} A_i^{\beta_{A_i}} \circledcap A_{Q+1}^{\beta_{A_{Q+1}}} \\ &= (\circ_{i=1}^{Q} \, f_{m^{\mathcal{H}}_{A_i,\cap}})(m_S) \circledcap A_{Q+1}^{\beta_{A_{Q+1}}}. \end{aligned}$$

From Proposition 4, we obtain:

$$\begin{aligned} (\circ_{i=1}^{Q} \, f_{m^{\mathcal{H}}_{A_i,\cap}})(m_S) \circledcap A_{Q+1}^{\beta_{A_{Q+1}}} &= f_{m^{\mathcal{H}}_{A_{Q+1},\cap}}((\circ_{i=1}^{Q} \, f_{m^{\mathcal{H}}_{A_i,\cap}})(m_S)) \\ &= (\circ_{i=1}^{Q+1} \, f_{m^{\mathcal{H}}_{A_i,\cap}})(m_S). \end{aligned}$$

□

Proposition 5 is important in that it constitutes the first known interpretation for CR. It shows that CR, which appears on the left side of (25), corresponds to independent behavior-based corrections – one for each context $A \in \mathcal{A}$ – where for each context A, the source is assumed to be truthful with mass β_A, and to be a positive liar in $\overline{A}$ with mass $1-\beta_A$.

Proposition 5 is illustrated by Examples 13 and 14, which show that CR may be encountered when dealing with intentionally and unintentionally lying sources, respectively.

Example 13. *Let m_S be an uncertain testimony provided by a source S on $\mathcal{X} = \{x_1, x_2, x_3\}$. An agent believes that S lies intentionally, and more specifically that it positively lies for x_3 with mass 0.4, and, independently, positively lies for x_1 with mass 0.2.*

In other words, S is subject to independent contextual lies of the form "positively non truthful in $\overline{A}$" for contexts $\mathcal{A} = \{A_1, A_2\}$ with $A_1 = \{x_1, x_2\}$ and $A_2 = \{x_2, x_3\}$, and with masses $1 - \beta_{A_1} = 0.4$ and $1 - \beta_{A_2} = 0.2$.

From Proposition 5, the agent's belief m on $\mathcal{X}$ is then defined by

$$m \quad = \quad m_S \circledcap \{x_1, x_2\}^{0.6} \circledcap \{x_2, x_3\}^{0.8}.$$

Example 14 (Example 9 continued)**.** *Consider again the setting of Example 4, but this time assume the agent learns some time after receiving Jane's testimony that she may actually not have met recently some of the suspects, and more specifically she is unaware that among these suspects that she may not have met recently, those that have changed their beard situation are only those that have a beard. Besides, he is unsure of who are these suspects: it coud be Conrad and Linus with mass 0.4, or independently, with mass 0.2, Linus and Eloy.*

From Proposition 5, the agent's belief m on who is the murderer is then obtained by

$$m \quad = \quad \{Eloy, Conrad\}^0 \circledcap \{Eloy, Aeneas\}^{0.6} \circledcap \{Conrad, Aeneas\}^{0.8},$$

where $\{Eloy, Conrad\}^0$ is the MF representing testimony $\mathbf{x} \in B = \{Eloy, Conrad\}$.

5.3 Contextual discounting

We show in this section that it is possible to obtain a similar perspective on CD as the interpretation proposed for CR in Section 5.2.

Let us consider another kind of contextual lie in this section: the states n_A, $A \subseteq \mathcal{X}$, corresponding to the assumptions that the source is a negative liar in A.

Proposition 6. *Let m_S be the MF provided by a source S and let $m^{\mathcal{H}}_{A,\cup}$ be our meta-knowledge on the source defined by*

$$m^{\mathcal{H}}_{A,\cup}(\{t\}) = \beta_A, \quad m^{\mathcal{H}}_{A,\cup}(\{n_A\}) = 1 - \beta_A, \tag{27}$$

i.e., *with mass β_A the source is truthful, and with mass $1 - \beta_A$ it is a negative liar in A.*

We have

$$f_{m^{\mathcal{H}}_{A,\cup}}(m_S) = m_S \circledcup A_{\beta_A}.$$

Proof. The proof is similar to that of Proposition 4. □

Proposition 7. *Let m_S be a MF. We have, $\forall \mathcal{A}$ and with $\beta_A \in [0, 1]$, $\forall A \in \mathcal{A}$:*

$$m_S \circledcup_{A \in \mathcal{A}} A_{\beta_A} = (\circ_{A \in \mathcal{A}}\, f_{m^{\mathcal{H}}_{A,\cup}})(m_S), \tag{28}$$

where mass functions $m^{\mathcal{H}}_{A,\cup}$, $A \in \mathcal{A}$, are defined by (27).

Proof. The proof is similar to that of Proposition 5, using Proposition 6 instead of Proposition 4. □

Proposition 7 shows that, similarly to CR, CD (left side of (28)) amounts to independent BBCs – one for each context – corresponding to simple contextual biases. The only difference between the two correction mechanisms is what is assumed with mass $1 - \beta_A$: with the former that the source is a positive liar in $\overline{A}$, whereas with the latter that the source is a negative liar in A. This latter finding concerning the difference between CR and CD suggests that CR seems as interesting as CD to handle contextual knowledge about the quality of a source, since the truthfulness assumptions associated to CR seem as useful in practice as those associated to CD.

Remark 5. *This interpretation of CD implies that classical discounting* (11) *amounts simply to assuming that the source is truthful with mass β, and negatively non truthful in $\mathcal{X}$ with mass $1 - \beta$, or for short, truthful with mass β and* negatively *non truthful with mass $1 - \beta$. Hence, discounting* (11) *may be seen as a relaxation of negating* (16)*, since this latter correction amounts to assuming that the source is truthful with mass β, and (positively* and *negatively) non truthful with mass $1 - \beta$.*

Remark 6. *This interpretation of CD provides a new perspective on contextual discounting based on a coarsening, which is a particular case of CD. Moreover, it brings a new element to the discussion entertained in [20, Section 5.2], where Mercier* et al. *distinguish two kinds of contextual knowledge about the quality of a source: one may have some knowledge on the quality of the source given that the true value of the parameter of interest* $\mathbf{x}$ *is in some set $A \subseteq \mathcal{X}$, which is the kind of contextual knowledge used in the original derivation of contextual discounting based on a coarsening; and one may have some knowledge on the quality of the source with respect to what the source declares about* $\mathbf{x}$*, which is the kind of contextual knowledge at play when one considers the interpretation of CD uncovered in this section.*

Remark 7. *Each of CD and CR can also be viewed as a single BBC corresponding to some particular knowledge on the truthfulness of the information source, which is basically the one obtained by combining together, that is for all $A \in \mathcal{A}$, the simple pieces of meta-knowledge $m^{\mathcal{H}}_{A,\cup}$* (27) *and $m^{\mathcal{H}}_{A,\cap}$* (24)*, respectively, as shown in Appendix B.*

This section has provided an interpretation for CR as well as an interpretation for CD using our proposed refined model of source truthfulness. From a formal point of view, these new results were also obtained from the very definitions of these two mechanisms: both of these contextual corrections amount to the combination with some separable MF (conjunctive combination with a $\textcircled{\cap}$-separable MF in the case of CR, and disjunctive combination with a $\textcircled{\cup}$-separable MF in the case of CD), and the simple MFs composing those separable MFs are directly related to the assumptions on the truthfulness of the source made by CR and CD as shown by Propositions 4 and 6. In the next section, we study the possibility of extending CD and CR by going beyond the combination with separable MFs.

6 Canonical decompositions and contextual corrections

In this section, a more general form of CD is derived by exploiting the canonical decomposition and by contextualizing the de-discounting operation. As will be seen, this generalization may be useful in that it allows one to take into account even more situations with respect to knowledge about the quality of a source. First, the de-discounting operation is contextualized. Then, contextual de-discounting is used to extend CD. Similar results are also obtained for CR.

6.1 Contextual de-corrections

De-discounting (12) is the inverse of discounting (11), which can also be expressed as (14). As recalled in Section 2.2, de-discounting is useful to remove a discounting that is considered no longer valid. To remove a discounting, one merely needs to use the disjunctive decombination rule. Indeed, we have

$$m \textcircled{\cup} \mathcal{X}_\beta \textcircled{\not\cup} \mathcal{X}_\beta = m.$$

In other words, de-discounting defined by (12) admits a simple expression as:

$$m = m_S \textcircled{\not\cup} \mathcal{X}_\beta. \tag{29}$$

Now, much as discounting is a particular case of CD, it is natural to view de-discounting as a particular case of the following operation that may be called *contextual de-discounting* (CdD).

Definition 7 (Contextual de-discounting)**.** *Let m_S be a MF. Its correction using contextual de-discounting, given a set $\mathcal{A}$ of contexts with associated parameters $\beta_A \in (0,1]$, for all $A \in \mathcal{A}$, is defined as the following MF m:*

$$m = m_S \textcircled{\not\cup}_{A\in\mathcal{A}} A_{\beta_A}. \tag{30}$$

The interpretation of CdD is similar to that of de-discounting and of CD: it amounts to the removals of $|\mathcal{A}|$ independent BBCs, where for each context A the source was assumed to be truthful with mass β_A and a negative liar in A with mass $1-\beta_A$. Example 15 illustrates CdD.

Example 15. *Let m_S be an uncertain testimony provided by a source S on $\mathcal{X} = \{Peter, John, Mary\}$. An agent Ag believes that S lies intentionally, and more specifically that it negatively lies for $Peter$ and $John$ with mass* 0.4*.*

In other words, Ag assumes that S is subject to a contextual lie of the form "negatively non truthful in A" for context $\{Peter, John\}$ with mass $1 - \beta_{\{Peter,John\}} = 0.4$. From Proposition 7, the agent's belief m_{Ag} on $\mathcal{X}$ is then defined by:

$$m_{Ag} = m_S \textcircled{\cup} \{Peter, John\}_{0.6}.$$

Suppose that an agent Ag_2 receives from Ag the piece of information m_{Ag}. Suppose further that Ag_2 does not know what S told to Ag, i.e., *he does not know m_S, but he knows Ag's meta-knowledge on the source, and he thinks that it is wrong, since he believes that the source tells a negative lie for $Peter$ and $John$ with mass* 0.2*.*

This amounts to applying to m_{Ag} a contextual de-discounting given context $\{Peter, John\}$ with parameter $\beta_{\{Peter,John\}} = 0.6$ (in order to remove the correction performed by Ag on the testimony of the source), and then a contextual discounting given context $\{Peter, John\}$ with parameter $\beta_{\{Peter,John\}} = 0.8$, i.e.,

$$m_{Ag_2} = (m_{Ag} \textcircled{\not\cup} \{Peter, John\}_{0.6}) \textcircled{\cup} \{Peter, John\}_{0.8},$$

or, equivalently, applying to m_{Ag} a contextual de-discounting given context $\{Peter, John\}$ with parameter $\beta_{\{Peter,John\}} = 0.75$, i.e.,

$$m_{Ag_2} = m_{Ag} \oslash\!\!\!\!\bigcirc \{Peter, John\}_{0.75},$$

since for any two functions[7] $A_{v_1} : 2^{\mathcal{X}} \to \mathbb{R}$ *and* $A_{v_2} : 2^{\mathcal{X}} \to \mathbb{R}$ *defined by, for* $i = 1, 2$*:*

$$\begin{aligned} A_{v_i} : A &\mapsto 1 - v_i, \\ \emptyset &\mapsto v_i, \\ B &\mapsto 0 \quad \forall B \in 2^{\mathcal{X}} \setminus \{A, \emptyset\}, \end{aligned} \tag{31}$$

for some $A \supset \emptyset$ and some $v_i \in (0, +\infty)$, we have [6]:

$$A_{v_1} \bigcirc\!\!\!\!\!\cup\, A_{v_2} = A_{v_1 \cdot v_2}, \tag{32}$$

$$A_{v_1} \bigcirc\!\!\!\!\!\not\cup\, A_{v_2} = A_{v_1 / v_2}. \tag{33}$$

Let us now consider CR. CR for $\mathcal{A} = \{\emptyset\}$ amounts to $m = m_S \bigcirc\!\!\!\!\!\cap\, \emptyset^{\beta}$, *i.e.*, a process that redistributes a portion $1 - \beta$ of the masses given to the non empty sets, to the empty set. In other words, CR for $\mathcal{A} = \{\emptyset\}$ is the dual of discounting and may thus simply be called *reinforcement* hereafter. Its inverse is defined by $m = m_S \bigcirc\!\!\!\!\!\not\cap\, \emptyset^{\beta}$, and may be called *de-reinforcement*.

Remark 8. *Normalization* (1) *may be expressed as [6]: $m^* = m \bigcirc\!\!\!\!\!\not\cap\, \emptyset^{1-m(\emptyset)}$ and corresponds thus to the de-reinforcement of MF m with degree $\beta = 1 - m(\emptyset)$. Its dual is called* maximal *de-discounting [7], and corresponds to setting $\beta = 1 - m_S(\mathcal{X})$ in* (29)*.*

Similarly as for de-discouting and CdD, de-reinforcement is a particular case of the following operation that may be called contextual de-reinforcement (CdR):

Definition 8 (Contextual de-reinforcement)**.** *Let m_S be a MF. Its correction using contextual de-reinforcement, given a set $\mathcal{A}$ of contexts with associated parameters $\beta_A \in (0, 1]$, for all $A \in \mathcal{A}$, is defined as the following MF m:*

$$m = m_S \bigcirc\!\!\!\!\!\not\cap_{A \in \mathcal{A}} A^{\beta_A}. \tag{34}$$

The interpretation of CdR is similar to that of CdD, and CdR may be illustrated using a similar example as Example 15.

Remark 9. *As for other computations involving decombination rules $\bigcirc\!\!\!\!\!\not\cap$ and $\bigcirc\!\!\!\!\!\not\cup$, we note that CdD* (30) *and CdR* (34) *may not always yield a belief function, hence they should be used with care. For instance, as recalled in Section 2.2, for de-discounting* (29) *to yield a belief function, it is necessary that $1 - m_S(\mathcal{X}) \leq \beta \leq 1$.*

Figure 2 synthesizes the relations between CD, CR, CdD and CdR.

[7] Such function is called a negative generalized MF in [6]. It is used here only as a formal and useful tool to simplify the presentation.

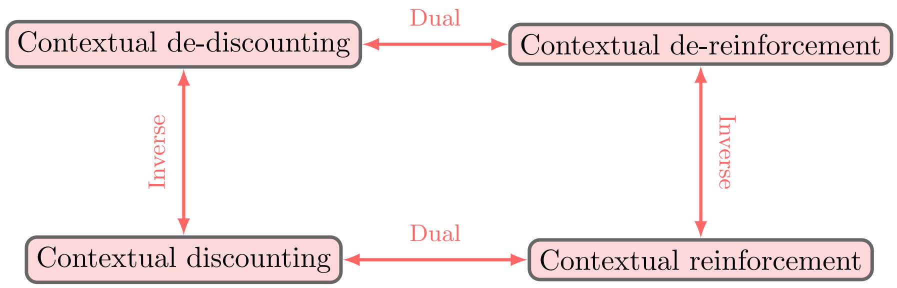


Figure 2: Relationships between the four contextual correction mechanisms.

6.2 Contextual corrections based on the canonical decompositions

Let us now consider a correction of a MF m_S resulting in a MF m, and involving both a CD (17) and a CdD (30), with associated sets of contexts $\mathcal{A}^c$ and $\mathcal{A}^d$, respectively, and associated degrees of belief $\beta_A^c \in (0,1), A \in \mathcal{A}^c$, and $\beta_A^d \in (0,1), A \in \mathcal{A}^d$, respectively, *i.e.*, the operation

$$m \quad = \quad m_S \bigcirc\!\!\!\!\!\!\cup_{A\in\mathcal{A}^c} A_{\beta_A^c} \bigcirc\!\!\!\!\!\!\not\cup_{A\in\mathcal{A}^d} A_{\beta_A^d}, \tag{35}$$

such that $\mathcal{A}^c \cap \mathcal{A}^d = \emptyset$ and such that these CD and CdD together form a disjunctive canonical decomposition, *i.e.*, the function $m_\cup$ defined by

$$m_\cup = \bigcirc\!\!\!\!\!\!\cup_{A\in\mathcal{A}^c} A_{\beta_A^c} \bigcirc\!\!\!\!\!\!\not\cup_{A\in\mathcal{A}^d} A_{\beta_A^d},$$

is a (non normal) mass function.

Equation (35) defines clearly an extension of CD, which is recovered if $\mathcal{A}^d = \emptyset$. It will be referred to as *contextual discounting based on the canonical decomposition* in the remainder of this paper and abbreviated by CD+.

Definition 9 (Contextual discounting based on the canonical decomposition)**.** *Let m_S be a MF. Its correction using contextual discounting based on the canonical decomposition, given sets of contexts $\mathcal{A}^c$ and $\mathcal{A}^d$ and associated degrees $\beta_A^c \in (0,1), A \in \mathcal{A}^c$, and $\beta_A^d \in (0,1), A \in \mathcal{A}^d$, such that $\mathcal{A}^c \cap \mathcal{A}^d = \emptyset$[8] and such that*

$$m_\cup = \bigcirc\!\!\!\!\!\!\cup_{A\in\mathcal{A}^c} A_{\beta_A^c} \bigcirc\!\!\!\!\!\!\not\cup_{A\in\mathcal{A}^d} A_{\beta_A^d},$$

is a non normal mass function, is defined as the following MF m:

$$m \quad = \quad m_S \bigcirc\!\!\!\!\!\!\cup m_\cup.$$

[8]This condition is not strictly needed. It is imposed in this definition, so that each couple $(\mathcal{A}^c, \mathcal{A}^d)$ with associated degrees $\beta_A^c, A \in \mathcal{A}^c$, and $\beta_A^d, A \in \mathcal{A}^d$, uniquely defines a CD+ correction of a MF m_S. Indeed, for each couple $(\mathcal{A}^c, \mathcal{A}^d)$ such that $\mathcal{A}^c \cap \mathcal{A}^d = \emptyset$, there exists an infinity of couples $(\mathcal{A}^{c'}, \mathcal{A}^{d'})$ with associated degrees $\beta_A^{c'} \in (0,1), A \in \mathcal{A}^{c'}$, and $\beta_A^{d'} \in (0,1), A \in \mathcal{A}^{d'}$, such that $\mathcal{A}^{c'} \cap \mathcal{A}^{d'} \neq \emptyset$, satisfying $\bigcirc\!\!\!\!\!\!\cup_{A\in\mathcal{A}^c} A_{\beta_A^c} \bigcirc\!\!\!\!\!\!\not\cup_{A\in\mathcal{A}^d} A_{\beta_A^d} = \bigcirc\!\!\!\!\!\!\cup_{A\in\mathcal{A}^{c'}} A_{\beta_A^{c'}} \bigcirc\!\!\!\!\!\!\not\cup_{A\in\mathcal{A}^{d'}} A_{\beta_A^{d'}}$, which is a consequence of (33). Besides, for any two couples $(\mathcal{A}^c, \mathcal{A}^d)$ and $(\mathcal{A}^{c'}, \mathcal{A}^{d'})$ such that $\mathcal{A}^c \cap \mathcal{A}^d = \emptyset$ and $\mathcal{A}^{c'} \cap \mathcal{A}^{d'} = \emptyset$, we have $\bigcirc\!\!\!\!\!\!\cup_{A\in\mathcal{A}^c} A_{\beta_A^c} \bigcirc\!\!\!\!\!\!\not\cup_{A\in\mathcal{A}^d} A_{\beta_A^d} = \bigcirc\!\!\!\!\!\!\cup_{A\in\mathcal{A}^{c'}} A_{\beta_A^{c'}} \bigcirc\!\!\!\!\!\!\not\cup_{A\in\mathcal{A}^{d'}} A_{\beta_A^{d'}}$ iff $\mathcal{A}^c = \mathcal{A}^{c'}$, $\mathcal{A}^d = \mathcal{A}^{d'}$, $\beta_A^c = \beta_A^{c'}$ for each $A \in \mathcal{A}^c$ and $\beta_A^d = \beta_A^{d'}$ for each $A \in \mathcal{A}^d$, which follows from the uniqueness of the disjunctive canonical decomposition.

Let us note that, by definition, a given CD based on the canonical decomposition can be applied to any MF m_S since it amounts simply to the disjunctive combination of m_S with another (non normal) MF, contrary to a given contextual de-discounting, which validity is dependent on the MF m_S to be corrected (cf Remark 9).

CD+ is a correction that is relevant if an uncertain testimony must be both discounted for some contexts and de-discounted for some other contexts, which may happen when one believes that this testimony is the result of an initial piece of information that has not been properly contextually discounted as illustrated by Example 16.

Example 16 (Example 15 continued)**.** *Let us consider again the setting of Example 15, but this time assumes that Ag_2 thinks Ag is wrong about the behavior of S, and in particular that the source tells actually a negative lie for $Peter$ with mass 0.6, and independently a negative lie for $John$ with mass 0.7.*

This amounts to applying to m_{Ag} a contextual de-discounting given context $\{Peter, John\}$ with parameter $\beta_{\{John\}} = 0.6$ (in order to remove the correction performed by Ag on the testimony of the source), and then a contextual discounting given contexts $\mathcal{A} = \{\{Peter\}, \{John\}\}$ with associated parameters $\beta_{\{Peter\}} = 0.4$ and $\beta_{\{John\}} = 0.3$, i.e.,

$$m_{Ag_2} = (m_{Ag} \circledcirc\!\!\!\!/\ \{Peter, John\}_{0.6}) \circledcup \{Peter\}_{0.4} \circledcup \{John\}_{0.3},$$

or, equivalently, combining disjunctively m_{Ag} with a non normal MF $m_\cup$:

$$m_{Ag_2} = m_{Ag} \circledcup m_\cup,$$

with $m_\cup$ defined by

$$m_\cup = (\{Peter\}_{0.4} \circledcup \{John\}_{0.3}) \circledcirc\!\!\!\!/\ (\{Peter, John\}_{0.6}).$$

Remark 10. *From a formal point of view, CD+* (35) *can be equivalently presented as, using* (32) *and* (33)*:*

$$m = m_S \circledcup_{A \in \mathcal{A}} A_{\beta_A}, \tag{36}$$

with $\mathcal{A} = \mathcal{A}^c \cup \mathcal{A}^d$ and with $\beta_A = \beta_A^c$ if $A \in \mathcal{A}^c$ and $\beta_A = 1/\beta_A^d$ if $A \in \mathcal{A}^d$, for all $A \in \mathcal{A}$ (hence $\beta_A \in (0, +\infty)$). In other words, CD+ has the same definition as CD (17)*, except that we may have $\beta_A > 1$, for some $A \in \mathcal{A}$. This technical remark will be useful later for a technical result (Remark 12 in Section 8).*

Of course, a similar reasoning can be followed to introduce the dual notion of *contextual reinforcement based on the canonical decomposition* denoted by CR+: formally, it is simply the combination by $\circledcap$ of MF m_S with a non dogmatic MF $m_\cap$.

Definition 10 (Contextual reinforcement based on the canonical decomposition)**.** *Let m_S be a MF. Its correction using contextual reinforcement based on the canonical decomposition, given sets of contexts $\mathcal{A}^c$ and $\mathcal{A}^d$ and associated degrees $\beta_A^c \in (0,1), A \in \mathcal{A}^c$, and $\beta_A^d \in (0,1), A \in \mathcal{A}^d$, such that $\mathcal{A}^c \cap \mathcal{A}^d = \emptyset$ and such that*

$$m_\cap = \circledcap_{A \in \mathcal{A}^c} A^{\beta_A^c} \circledcirc\!\!\!\!/\ _{A \in \mathcal{A}^d} A^{\beta_A^d},$$

is a non dogmatic mass function, is defined as the following MF m:

$$m = m_S \circledcap m_\cap.$$

CR+ extends CR since CR amounts to the combination with a $\textcircled{\cap}$-separable MF, that is, CR is a CR+ such that $\mathcal{A}^d = \emptyset$. CR+ is a correction that is sensible if MF m_S provided by the source must be both reinforced for some contexts $\mathcal{A}^c$ and de-reinforced for some other contexts $\mathcal{A}^d$, and may be illustrated similarly as CD+ (cf Example 16 for the illustration of CD+).

Remark 11. *A counterpart to Remark 10 exists for CR+. Indeed, by extending the notation A^w to the case where $w \in (0, +\infty)$, similarly as notation A_v is extended in* (31) *to $v \in (0, +\infty)$, and using the $\textcircled{\cap}$-counterparts of* (32) *and* (33) *shown in [34], CR+ can be equivalently presented as:*

$$m = m_S \textcircled{\cap}_{A \in \mathcal{A}} A^{\beta_A}, \tag{37}$$

with $\mathcal{A} = \mathcal{A}^c \cup \mathcal{A}^d$ and with $\beta_A = \beta_A^c$ if $A \in \mathcal{A}^c$ and $\beta_A = 1/\beta_A^d$ if $A \in \mathcal{A}^d$, for all $A \in \mathcal{A}$ (hence $\beta_A \in (0, +\infty)$). CR+ has then the same definition as CR, except that we may have $\beta_A > 1$, for some $A \in \mathcal{A}$. This will be useful later for a technical result (Remark 13 in Section 8).

This section has introduced extensions of CD and CR, by exploiting the fact that the assumptions made by a given CD or CR, can be readily seen through the definition of this CD or CR: the assumptions correspond to the simple MFs in said definition. Hence, thanks to the inverse rules $\textcircled{\not\cup}$ and $\textcircled{\not\cap}$, it becomes possible to remove such assumptions if one believes that they are no longer tenable, which amounts to a so-called contextutal de-correction. In the next section, yet another way of extending CD and CR is studied: it is based on exploiting the state ℓ_A introduced in Section 3.

7 Contextual negating

In this section, we introduce a new contextual correction scheme, which is formally similar to the two existing ones and that is related to the negating operation.

7.1 Non truthful in $\overline{A}$

As shown in Section 5, CD and CR result from corrections induced by simple pieces of meta-knowledge $m^{\mathcal{H}}_{A,\cup}$ (27) and $m^{\mathcal{H}}_{A,\cap}$ (24) relying on contextual biases n_A and p_A, respectively. In practice, these two states transform a testimony $\mathbf{x} \in B$ into $B \cup A$ and $B \cap A$, respectively.

In Section 3, a third natural contextual lie, state ℓ_A, which corresponds to assuming that the source is non truthful in $\overline{A}$, was brought to light and studied. This state yields $\mathbf{x} \in B \sqcap A$ from a testimony $\mathbf{x} \in B$. Interestingly, the properties satisfied by $\sqcap$ (associativity, commutativity, neutral element) allow us to obtain similar propositions as those obtained for CR and CD (the proofs of those propositions are similar to the ones of CR and CD, and are thus omitted).

Proposition 8. *Let m_S be the MF provided by a source S and let $m^{\mathcal{H}}_{A,\sqcap}$ be our meta-knowledge on the source defined by*

$$m^{\mathcal{H}}_{A,\sqcap}(\{t\}) = \beta_A, \quad m^{\mathcal{H}}_{A,\sqcap}(\{\ell_A\}) = 1 - \beta_A, \tag{38}$$

i.e., *with mass* β_A *the source is truthful, and with mass* $1-\beta_A$ *it is non truthful in* $\overline{A}$.

We have

$$f_{m^{\mathcal{H}}_{A,\Cap}}(m_S) = m_S \circledcap A^{\beta_A}.$$

Proposition 9. *Let* m_S *be a MF. We have,* $\forall \mathcal{A}$ *and with* $\beta_A \in [0,1]$, $\forall A \in \mathcal{A}$:

$$m_S \circledcap_{A\in\mathcal{A}} A^{\beta_A} = (\circ_{A\in\mathcal{A}}\, f_{m^{\mathcal{H}}_{A,\Cap}})(m_S), \tag{39}$$

where mass functions $m^{\mathcal{H}}_{A,\Cap}$, $A \in \mathcal{A}$, *are defined by* (38).

Equation (39) is the $\Cap$ counterpart to Equations (28) and (25), which are based on $\cup$ and $\cap$, respectively. It constitutes a contextual correction, which, similarly to CD and CR, amounts to independent BBCs – one for each context – corresponding to simple contextual biases. The only difference with CD and CR is what is assumed with mass $1-\beta_A$: that the source is non truthful in $\overline{A}$.

An interesting fact can be brought to light about this contextual correction.

Proposition 10. *For any MF* m_S, *we have*

$$m_S \circledcap \emptyset^{\beta} \;=\; \beta \cdot m_S + (1-\beta)\cdot \overline{m}_S. \tag{40}$$

Proof. This proposition follows from Proposition 8 and Remark 1. □

This proposition shows that negating (16) is a particular case of the contextual correction (39): it is recovered for $\mathcal{A} = \{\emptyset\}$, as should be since contextual correction (39) reduced to $\mathcal{A} = \{\emptyset\}$ corresponds to assuming that the source is truthful with mass β and non truthful with mass $1-\beta$, which is the meta-knowledge associated to the negating operation. Contextual correction (39) constitutes thus a similar extension with respect to negating, than CD is to discounting and CR is to reinforcement. It may therefore be seen as a contextual version of negating and be called *contextual negating* (CN).

Definition 11 (Contextual negating)**.** *Let* m_S *be a MF. Its correction using contextual negating, given a set* $\mathcal{A}$ *of contexts with associated parameters* $\beta_A \in [0,1]$, *for all* $A \in \mathcal{A}$, *is defined as the following MF* m:

$$m \;=\; m_S \circledcap_{A\in\mathcal{A}} A^{\beta_A}.$$

We note that the computational complexity of CN is similar to that of CD and CR: it merely corresponds to the complexity of applying $|\mathcal{A}|$ combinations by the $\circledcap$ rule.

Furthermore, similar examples as Examples 13 and 14 can easily be constructed to illustrate situations where CN may be needed.

7.2 Discussion

Let us briefly emphasize the similarities and differences between CD and the new contextual correction mechanism uncovered above that is CN.

In their non contextual version, they reduce to discounting and negating, respectively. Discounting is the correction mechanism derived from basic assumptions about the relevance of the source, whereas negating originates from basic assumptions about the truthfulness of the source. Alternatively, as pointed out by Remark 5, discounting can also be viewed as a relaxed form of negating, in that it may also be recovered from assumptions about the truthfulness of the source that are less strong than those yielding negating. These two mechanisms differ partially in how they treat a testimony $\mathbf{x} \in B$: they both keep mass β on B, and discounting transfers mass $1 - \beta$ to $\mathcal{X}$ whereas negating transfers it to $\overline{B}$; this comes from the fact that we deduce either $\mathbf{x} \in \mathcal{X}$ or $\mathbf{x} \in \overline{B}$ depending on whether we think the source is non relevant (or negatively non truthful, see Remark 5) or non truthful. As extensions of discounting and negating, CD and CN are thus clearly fundamentally different operations.

Furthermore, in essence, given a context A and testimony $\mathbf{x} \in B$, both CD and CN keep mass β_A on B, and CD transfers mass $1 - \beta_A$ to $B \cup A$, whereas CN transfers it to $B \underline{\cap} A$, so that in practice the difference between the two mechanisms is what happens with mass $1 - \beta_A$: with CD, it is assumed that at least one of the pieces of information $\mathbf{x} \in B$ and $\mathbf{x} \in A$ is true, whereas with CN, we have that either both or none of these pieces of information are true.

Let us finally remark that, similarly as inverses of $\textcircled{\cap}$ and $\textcircled{\cup}$ can be defined from pointwise divisions of commonality and implicability functions, respectively, it is in principle possible to define the inverse of the rule $\textcircled{\underline{\cap}}$ from pointwise division of the so-called 0-commonality function [26, 24], which is the counterpart of the commonality and implicability functions for the rule $\textcircled{\underline{\cap}}$. Hence, in principle, it is possible to define a notion of contextual de-negating as well as a more general form of CN similar to the more general forms of CD and CR obtained in Section 6. However, for the definition of the inverse of the rule $\textcircled{\underline{\cap}}$ to be usable in practice, one needs to know under which simple conditions the 0-commonality function does not equal to 0, since divisions by zeros must be avoided, similarly as it is known that the commonality and implicability functions are different from 0 as long as the MF is non dogmatic or non normal, respectively. Unfortunately, we have not been able so far to find such simples conditions for the 0-commonality function. This is left for further research.

This section has introduced a new contextual correction, which is formally similar to CD and CR and that is related to the negating operation. All three of these mechanisms stem from uncertain knowledge about the behavior of the source, which reduce to a set of context $\mathcal{A}$ and associated parameters β_A, $A \in \mathcal{A}$. Next section presents a method to derive such knowledge from labelled data.

8 Learning contextual biases of a source from labelled data

For contextual correction mechanisms to be useful in applications, one needs practical means to choose the set of contexts $\mathcal{A}$ and to determine the associated vector $\boldsymbol{\beta} = (\beta_A, A \in \mathcal{A})$. As already mentioned in Section 1, the set $\mathcal{A}$ and vector $\boldsymbol{\beta}$ could be learnt from available labelled data, and in particular using methods based on the minimization of an error criterion. This latter type of methods is considered in this section.

8.1 Description of the learning process

Let us assume that a training set describing the outputs of a source (expressed in the form of a MF) regarding the classes in $\mathcal{X} = \{x_1, \ldots, x_K\}$ of n objects o_i, $i \in \{1, \ldots, n\}$, is available. Small illustrative examples of such training sets are given in Table 4 for two

Table 4: Ouputs of two sensors regarding the classes of 4 objects which can be airplanes (a), helicopters (h) or rockets (r). Data come from [12, Table 1].

		a	h	r	$\{a,h\}$	$\{a,r\}$	$\{h,r\}$	$\mathcal{X}$	Ground truth
Sensor 1	$m_{S_1}\{o_1\}$	0	0	0.5	0	0	0.3	0.2	a
	$m_{S_1}\{o_2\}$	0	0.5	0.2	0	0	0	0.3	h
	$m_{S_1}\{o_3\}$	0	0.4	0	0	0.6	0	0	a
	$m_{S_1}\{o_4\}$	0	0	0	0	0.6	0.4	0	r
Sensor 2	$m_{S_2}\{o_1\}$	0	0	0	0.7	0	0	0.3	a
	$m_{S_2}\{o_2\}$	0.3	0	0	0.4	0	0	0.3	h
	$m_{S_2}\{o_3\}$	0.2	0	0	0	0	0.6	0.2	a
	$m_{S_2}\{o_4\}$	0	0	0	0	0	1	0	r

sensors in charge of recognizing flying objects which can be airplanes (a), helicopters (h) or rockets (r).

Inspired from previous work in pattern recognition [41], Elouedi *et al.* [12] propose a method to automatically compute, from such a training set, the degree of reliability $\beta \in [0,1]$ of the classical discounting operation (11). This scalar β is chosen as the one which minimizes the following measure of discrepancy between the corrected source outputs and the reality:

$$E_{bet}(\beta) = \sum_{i=1}^{n} \sum_{k=1}^{K} (BetP\{o_i\}(x_k) - \delta_{i,k})^2, \tag{41}$$

where $\forall i \in \{1,\ldots,n\}$, $BetP\{o_i\}$ is the pignistic probability [38] associated with the MF $m\{o_i\}$ obtained from a discounting with a degree of reliability β of the output $m_S\{o_i\}$ of the source S regarding the class of object o_i, and $\delta_{i,k}$ is a binary variable that indicates the class of object o_i as follows: $\forall k \in \{1,\ldots,K\}$, $\delta_{i,k} = 1$ if object o_i belongs to the class x_k, and $\delta_{i,k} = 0$ otherwise.

The main idea is to find the reliability degree $\beta \in [0,1]$ that will bring on average, after correction, the outputs of a source closer to the reality.

In [23], it has been shown that this measure of discrepancy (41) can serve as well to learn the vector $\boldsymbol{\beta} = (\beta_A \in [0,1], A \in \mathcal{A})$, $\mathcal{A}$ forming a partition of $\mathcal{X}$, of reliability degrees of a contextual discounting based on a coarsening, once a partition (a set of contexts) $\mathcal{A}$ has been fixed.

Moreover, it has also been proposed in [23] to learn this latter vector $\boldsymbol{\beta}$ using another measure of discrepancy based on the plausibility function and defined by:

$$E_{pl}(\boldsymbol{\beta}) = \sum_{i=1}^{n} \sum_{k=1}^{K} (pl\{o_i\}(\{x_k\}) - \delta_{i,k})^2 \ , \tag{42}$$

where $\forall i \in \{1,\ldots,n\}$, $pl\{o_i\}$ is the plausibility function obtained from a contextual discounting based on a coarsening $\mathcal{A}$ of $\mathcal{X}$, with a vector $\boldsymbol{\beta} = (\beta_A \in [0,1], A \in \mathcal{A})$ of reliability degrees, of $m_S\{o_i\}$. This measure allows one to express the problem of reliability degrees computation in a more efficient way than (41) does, specifically a constrained least-squares problem [23, Section 5.1]. However, the problem of finding the optimal partition $\mathcal{A}$ of $\mathcal{X}$ for a given source, that is the one minimizing (42), was left open.

In the remainder of this section, we propose to study and refine the above process to automatically learn CD, CR and CN. Precisely, in Section 8.2, we consider the learning of CD with an arbitrary set of contexts $\mathcal{A}$ (17) (*i.e*, the more general CD that does not require the set of contexts $\mathcal{A}$ to form a partition of $\mathcal{X}$), and of CD+ (35) (*i.e.*, CD based on the canonical decomposition), using the measure of discrepancy E_{pl} (42). In Sections 8.3 and 8.4, similar investigations are pursued for the learning of CR and CN respectively. Learning of CD, CR and CN is then illustrated and commented in Section 8.5. Finally, we show in Section 8.6 that this learning approach may be useful to improve the performance of a source in a classification application.

We remark beforehand that our choice to use measure E_{pl} (42) in our investigations is explained by four main reasons. First, this measure was the preferred one in the approach proposed in [23], which we are clearly extending with this current work. Second, using the plausibility on singletons is in accordance with the Shafer [32] and Smets [39] *singular* [8] interpretation of belief functions adopted in this paper, where one searches to know the actual value of $\mathbf{x}$. Third, as will be seen later, it is possible to obtain simple analytical expressions showing how the parameters β_A of CD, CR and CN, affect the plausibility of singletons, which can be quite helpful when analyzing the respective capacities of these mechanisms to improve a source performance. At last, as it will also be seen later, it can be shown that there exists actually an optimal set of contexts for each of CD, CR and CN, that ensures the minimization of the measure, and that finding this minimum amounts to a computationally simple optimization problem (a constrained least-squares problem with $|\mathcal{X}|$ unknowns). To sum up, we chose E_{pl} to ensure: continuity with previous works, conformance with the belief function theory interpretation used in this paper, ease of analysis and ease of optimization. Yet, we note that other measures of discrepancy could be used, *e.g.*, the measure E_{bet} (41) or a measure based on a distance [15], but then it is neither guaranteed that their minimization can be performed efficiently nor guaranteed that it will be easy to analyse how CD, CR and CN affect them. Let us finally mention that if the measure of discrepancy is used to optimize some decision system, then this measure should be related to the chosen decision rule; for instance, the measure E_{bet} (41) should be used in conjunction with decisions based on pignistic probability, and E_{pl} (42) should be used, as done in Section 8.6, for decisions based on the plausibility transformation [2], which transforms a belief function into a probability distribution by normalizing the plausibilities on singletons.

8.2 Learning contextual discounting

In this section, the learning of CD according to the plausibility based measure of discrepancy (42) is studied, first for the case of CD defined by (17), which means a contextual discounting of the following form: $m = m_S \circledcup m_C$, with m_C a $\circledcup$-separable MF.

The main issue here is to decide which set $\mathcal{A}$ to consider and to find the associated vector $\boldsymbol{\beta}$ that minimize measure (42). To try and solve this issue, we can first remark that this latter measure requires that the plausibilities on singletons after having applied such a CD defined by (17), be known. These plausibilities are given in the following proposition.

Proposition 11. *Let* $m = m_S \textcircled{\cup}_{A \in \mathcal{A}} A_{\beta_A}$*,* $\beta_A \in [0,1]$*, for all* $A \in \mathcal{A}$*, be the CD of a MF* m_S*. The plausibility function associated with* m *is defined for all* $x \in \mathcal{X}$ *by:*

$$pl(\{x\}) = 1 - (1 - pl_S(\{x\})) \prod_{A \in \mathcal{A}, x \in A} \beta_A \ . \tag{43}$$

Proof. As $m = m_S \textcircled{\cup} m_C$, with $m_C = \textcircled{\cup}_{A \in \mathcal{A}} A_{\beta_A}$, CD is given in terms of implicability functions by:

$$\begin{aligned} b &= b_S \cdot b_C \\ &= b_S \prod_{A \in \mathcal{A}} b_{\beta_A}, \end{aligned} \tag{44}$$

with, for all $B \subseteq \mathcal{X}$, $b_{\beta_A}(B) = 1$ if $A \subseteq B$, $b_{\beta_A}(B) = \beta_A$ otherwise. Thus, for all $B \subseteq \mathcal{X}$:

$$b(B) = b_S(B) \prod_{A \in \mathcal{A}, A \not\subseteq B} \beta_A. \tag{45}$$

Consequently, for all $x \in \mathcal{X}$:

$$\begin{aligned} pl(\{x\}) &= 1 - b(\overline{\{x\}}) = 1 - b_S(\overline{\{x\}}) \prod_{A \in \mathcal{A}, A \not\subseteq \overline{\{x\}}} \beta_A \quad \text{(From (45))} \\ &= 1 - b_S(\overline{\{x\}}) \prod_{A \in \mathcal{A}, x \in A} \beta_A = 1 - (1 - pl_S(\{x\})) \prod_{A \in \mathcal{A}, x \in A} \beta_A. \end{aligned} \tag{46}$$

□

Proposition 11 is illustrated by Example 17.

Example 17. *Let* $\mathcal{X} = \{x_1, x_2, x_3\}$ *and consider the CD of a MF* m_S *with various sets of contexts* $\mathcal{A}$*:*

- *If* $\mathcal{A} = \{\{x_1\}, \{x_2\}, \{x_3\}\}$*, then we have:*

$$\begin{aligned} pl(\{x_1\}) &= 1 - (1 - pl_S(\{x_1\}))\beta_{\{x_1\}}, \\ pl(\{x_2\}) &= 1 - (1 - pl_S(\{x_2\}))\beta_{\{x_2\}}, \\ pl(\{x_3\}) &= 1 - (1 - pl_S(\{x_3\}))\beta_{\{x_3\}}. \end{aligned}$$

- *If* $\mathcal{A} = 2^{\mathcal{X}}$*, then we have:*

$$\begin{aligned} pl(\{x_1\}) &= 1 - (1 - pl_S(\{x_1\}))\beta_{\{x_1\}}\beta_{\{x_1,x_2\}}\beta_{\{x_1,x_3\}}\beta_{\{x_1,x_2,x_3\}}, \\ pl(\{x_2\}) &= 1 - (1 - pl_S(\{x_2\}))\beta_{\{x_2\}}\beta_{\{x_1,x_2\}}\beta_{\{x_2,x_3\}}\beta_{\{x_1,x_2,x_3\}}, \\ pl(\{x_3\}) &= 1 - (1 - pl_S(\{x_3\}))\beta_{\{x_3\}}\beta_{\{x_1,x_3\}}\beta_{\{x_2,x_3\}}\beta_{\{x_1,x_2,x_3\}}. \end{aligned}$$

- *If* $\mathcal{A} = \{\{x_2\}\}$*, then we have:*

$$\begin{aligned} pl(\{x_1\}) &= pl_S(\{x_1\}), \\ pl(\{x_2\}) &= 1 - (1 - pl_S(\{x_2\}))\beta_{\{x_2\}}, \\ pl(\{x_3\}) &= pl_S(\{x_3\}). \end{aligned}$$

Next proposition indicates that the minimization of E_{pl} when CD has been applied, is obtained using the vector $\boldsymbol{\beta}$ composed of the K parameters $\beta_{\{x_k\}}$, which means the parameters associated with the singletons of $\mathcal{X}$. Moreover the minimization of E_{pl} using this vector constitutes a constrained least-squares problem which can then be solved efficiently using standard algorithms.

Proposition 12. *The minimization of E_{pl} with CD is obtained using the vector $\boldsymbol{\beta} = (\beta_{\{x_k\}} \in [0,1], k \in \{1,\ldots,K\})$ and constitutes a constrained least-squares problem as (42) can then be rewritten as:*

$$E_{pl}(\boldsymbol{\beta}) = \|\boldsymbol{Q}\boldsymbol{\beta} - \boldsymbol{d}\|^2 \text{ with } \boldsymbol{Q} = \begin{bmatrix} diag(\boldsymbol{pl_1} - 1) \\ \vdots \\ diag(\boldsymbol{pl_n} - 1) \end{bmatrix} \text{ and } \boldsymbol{d} = \begin{bmatrix} \boldsymbol{\delta_1} - 1 \\ \vdots \\ \boldsymbol{\delta_n} - 1 \end{bmatrix}, \quad (47)$$

with $diag(\boldsymbol{v})$ a square diagonal matrix with the elements of vector $\boldsymbol{v}$ on the main diagonal, and with $\boldsymbol{pl_i} = (pl_S\{o_i\}(\{x_1\}), \ldots, pl_S\{o_i\}(\{x_K\}))^T$, and $\boldsymbol{\delta_i} = (\delta_{i,1}, \ldots, \delta_{i,K})^T$ the column vector of 0-1 class indicator variables for object o_i.

Proof. From Proposition 11, after having applied CD on m_S, the discrepancy measure E_{pl} (42) can be written: $E_{pl}(\boldsymbol{\beta}) = \sum_{k=1}^{K} E_{pl}(\boldsymbol{\beta}, x_k)$, with for all $k \in \{1,\ldots,K\}$:

$$E_{pl}(\boldsymbol{\beta}, x_k) \;:=\; \sum_{i=1}^{n} \left(\left(1 - (1 - pl_S\{o_i\}(\{x_k\})) \prod_{A \in \mathcal{A}, x_k \in A} \beta_A \right) - \delta_{i,k} \right)^2 . \quad (48)$$

As $E_{pl}(\boldsymbol{\beta}, x_k) \geq 0$ for all $k \in \{1,\ldots,K\}$, the minimum value of $E_{pl}(\boldsymbol{\beta})$ is obtained when each $E_{pl}(\boldsymbol{\beta}, x_k)$ reaches its minimum.

Besides, from (44), (45) and (46), the product $\prod_{A \in \mathcal{A}, x_k \in A} \beta_A$ of coefficients β_A in $E_{pl}(\boldsymbol{\beta}, x_k)$ (48) is equal to $b_C(x_k)$ for all $k \in \{1,\ldots,K\}$, b_C being the implicability function associated with $m_C = \circledcup_{A \in \mathcal{A}} A_{\beta_A}$, and belongs then to $[0,1]$ and can be denoted by a variable $\beta_k \in [0,1]$. Hence, for each $k \in \{1,\ldots,K\}$, the minimum of $E_{pl}(\boldsymbol{\beta}, x_k)$ is reached for a particular value of β_k.

Now, we can remark that each coefficient $\beta_{\{x_k\}}$, $k \in \{1,\ldots,K\}$, only appears in the expression of $E_{pl}(\boldsymbol{\beta}, x_k)$ (48), $k \in \{1,\ldots,K\}$. Hence, choosing $\beta_k = \beta_{\{x_k\}}$ for all k (which means choosing $\mathcal{A}$ composed of the set of singletons of $\mathcal{X}$) constitutes then a solution, *i.e.*, a set of contexts for which the minimum value of $E_{pl}(\boldsymbol{\beta})$ is reached.

Each value of E_{pl} is then reachable using the vector $\boldsymbol{\beta}$ of coefficients $\beta_k := \beta_{\{x_k\}}$, $k \in \{1,\ldots,K\}$, and as already mentioned in [23, Section 5.1], the computation of the coefficient $\boldsymbol{\beta}$ with CD based on the singletons is a constrained least-squares problem. Indeed, for all $k \in \{1,\ldots,K\}$, and for all $i \in \{1,\ldots,n\}$:

$$\begin{aligned} pl\{o_i\}(\{x_k\}) - \delta_{i,k} &= 1 - (1 - pl_S\{o_i\}(\{x_k\}))\beta_k - \delta_{i,k} \\ &= (pl_S\{o_i\}(\{x_k\}) - 1)\beta_k - (\delta_{i,k} - 1) . \end{aligned}$$

Then (42) can be rewritten as (47).

□

This answers a prospect given in [23] concerning the study of the set of contexts which yields the best possible value for the measure of discrepancy E_{pl}. The answer given here is that there will be no smaller value reachable for E_{pl} than the one obtained with the set of the singletons of $\mathcal{X}$ with associated coefficients $\boldsymbol{\beta} = (\beta_{\{x_k\}}, k \in \{1,\ldots,K\})$.

Remark 12. *If the more general CD based on the canonical decomposition (cf Section 6.2) is applied, namely a CD defined by: $m = m_S \textcircled{\cup} m_C$ with m_C a non normal MF, the results of this learning (optimisation in the sense of (42)) will still be the same, that is to say a constrained least-squares problem of the form* (47) *with $|\mathcal{X}|$ unknowns $\beta_{\{x_k\}} \in [0,1]$. Indeed, from* (44), (45) *and* (46), *the product $\prod_{A \in \mathcal{A}, x_k \in A} \beta_A$ is equal to $b_C(x_k)$, b_C being the implicability function associated with m_C, and belongs then to $[0,1]$ and thus Proposition 12 also holds for this more general CD. This allows us to remark that the degrees of freedom added by the possibility of having $\beta_A > 1$ are actually useless*[9] *to improve the measure (42).*

8.3 Learning contextual reinforcement

With the same idea but in a reinforcement context, we study in this section the learning of CR according to the plausibility based measure of discrepancy (42).

Plausibilities on the singletons after having applied CR are given in next proposition.

Proposition 13. *Let $m = m_S \textcircled{\cap}_{A \in \mathcal{A}} A^{\beta_A}$, $\beta_A \in [0,1]$, for all $A \in \mathcal{A}$, be the CR of a MF m_S. The plausibility function associated with m is defined for all $x \in \mathcal{X}$ by:*

$$pl(\{x\}) \quad = \quad pl_S(\{x\}) \prod_{A \in \mathcal{A}, x \notin A} \beta_A \ .$$

Proof. As $m = m_S \textcircled{\cap} m_C$, with $m_C = \textcircled{\cap}_{A \in \mathcal{A}} A^{\beta_A}$, the CR is determined in terms of commonality functions by:

$$\begin{aligned} q &= q_S \cdot q_C &(49)\\ &= q_S \prod_{A \in \mathcal{A}} q^{\beta_A} \end{aligned}$$

with, for all $B \subseteq \mathcal{X}$, $q^{\beta_A}(B) = 1$ if $B \subseteq A$, $q^{\beta_A}(B) = \beta_A$ otherwise. Then, for all $B \subseteq \mathcal{X}$:

$$q(B) = q_S(B) \prod_{A \in \mathcal{A}, B \not\subseteq A} \beta_A, \tag{50}$$

which means that after having applied CR, plausibilities on singletons are defined, for all $x \in \mathcal{X}$, by:

$$\begin{aligned} pl(\{x\}) &= q(\{x\}) = q_S(\{x\}) \prod_{A \in \mathcal{A}, x \notin A} \beta_A \\ &= pl_S(\{x\}) \prod_{A \in \mathcal{A}, x \notin A} \beta_A. &(51) \end{aligned}$$

□

Example 18. *Let $\mathcal{X} = \{x_1, x_2, x_3\}$ and consider the CR of a MF m_S with various sets of contexts $\mathcal{A}$:*

[9] However, this more general form of CD may be useful considering other discrepancy measures (and it may be, of course, potentially useful outside of a learning context, to model richer knowledge about the behavior of the source than what is allowed by the less general forms of CD).

- *If* $\mathcal{A} = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$*, then we have:*

$$\begin{aligned} pl(\{x_1\}) &= pl_S(\{x_1\})\beta_{\{x_2,x_3\}}, \\ pl(\{x_2\}) &= pl_S(\{x_2\})\beta_{\{x_1,x_3\}}, \\ pl(\{x_3\}) &= pl_S(\{x_3\})\beta_{\{x_1,x_2\}}. \end{aligned}$$

- *If* $\mathcal{A} = 2^{\mathcal{X}}$*, then we have:*

$$\begin{aligned} pl(\{x_1\}) &= pl_S(\{x_1\})\beta_{\emptyset}\beta_{\{x_2\}}\beta_{\{x_3\}}\beta_{\{x_2,x_3\}}, \\ pl(\{x_2\}) &= pl_S(\{x_2\})\beta_{\emptyset}\beta_{\{x_1\}}\beta_{\{x_3\}}\beta_{\{x_1,x_3\}}, \\ pl(\{x_3\}) &= pl_S(\{x_3\})\beta_{\emptyset}\beta_{\{x_1\}}\beta_{\{x_2\}}\beta_{\{x_1,x_2\}}. \end{aligned}$$

Proposition 14. *The minimization of* E_{pl} *with CR is obtained using the vector* $\boldsymbol{\beta} = (\beta_{\overline{\{x_k\}}} \in [0,1], k \in \{1, \ldots, K\})$ *and constitutes a constrained least-squares problem as (42) can then be written as:*

$$E_{pl}(\boldsymbol{\beta}) = \|\boldsymbol{P}\boldsymbol{\beta} - \boldsymbol{\delta}\|^2, \text{ with } \boldsymbol{P} = \begin{bmatrix} diag(\boldsymbol{pl_1}) \\ \vdots \\ diag(\boldsymbol{pl_n}) \end{bmatrix} \text{ and } \boldsymbol{\delta} = \begin{bmatrix} \boldsymbol{\delta_1} \\ \vdots \\ \boldsymbol{\delta_n} \end{bmatrix}, \tag{52}$$

with the same notations as in Proposition 12.

Proof. From Proposition 13, for each $k \in \{1, \ldots, K\}$, coefficient $\beta_{\overline{\{x_k\}}}$ only appears in $pl(x_k)$ when a CR has been applied. Then, with the same reasoning as for the CD case, the minimum value of E_{pl} with CR can be reached using the set of contexts $\{\overline{x_k} = \mathcal{X} \setminus \{x_k\}, k \in \{1, \ldots, K\}\}$.

The minimization of E_{pl} with CR based on the vector $\boldsymbol{\beta} = (\beta_k := \beta_{\overline{\{x_k\}}}, k \in \{1, \ldots, K\})$ is also a constrained least-squares problem as (42) can be written as (52) (as $\forall k \in \{1, \ldots, K\}$ and $\forall i \in \{1, \ldots, n\}$, $pl\{o_i\}(\{x_k\}) - \delta_{i,k} = pl_S\{o_i\}(\{x_k\})\beta_k - \delta_{i,k}$). □

Remark 13. *As was the case for CD (Remark 12), if CR based on the canonical decomposition is used, namely:* $m = m_S \bigcirc\!\!\!\!\cap\, m_C$ *with* m_C *a non dogmatic MF, the learning in the sense of (42) will still yield to the learning of* $|\mathcal{X}|$ *parameters* $\beta_{\overline{\{x_k\}}} \in [0,1]$*, as from* (49)*,* (50) *and* (51)*, for all* $x \in \mathcal{X}$*,* $\prod_{A \in \mathcal{A}, x \notin A} \beta_A = q_C(\{x\}) \in [0,1]$*, and thus Proposition 14 also holds for this more general form of CR. .*

8.4 Learning contextual negating

The learning of CN is explored in this section.

The plausibilities on the singletons after having applied CN are given in the next proposition.

Proposition 15. *Let* $m = m_S \bigcirc\!\!\!\!\cap_{A \in \mathcal{A}} A^{\beta_A}$ *with* $\beta_A \in [0,1]$*, for all* $A \in \mathcal{A}$*, be the CN of a MF* m_S*. The plausibility function associated with* m *is defined for all* $x \in \mathcal{X}$ *by:*

$$pl(\{x\}) = 0.5 + 0.5 \cdot (2 \cdot pl_S(\{x\}) - 1) \prod_{A \in \mathcal{A}, x \notin A} (2 \cdot \beta_A - 1). \tag{53}$$

Proof. See Appendix C. □

Proposition 16. *The minimization of E_{pl} with CN is obtained using the vector $\boldsymbol{\beta} = (\beta_{\overline{\{x_k\}}} \in [0,1], k \in \{1,\ldots,K\})$ and constitutes a constrained least-squares problem as (42) can then be written as:*

$$E_{pl}(\boldsymbol{\beta}) = \|\boldsymbol{D}\boldsymbol{\beta} - \boldsymbol{z}\|^2, \text{ with } \boldsymbol{D} = \begin{bmatrix} diag(2\,\boldsymbol{pl_1} - 1) \\ \vdots \\ diag(2\,\boldsymbol{pl_n} - 1) \end{bmatrix} \text{ and } \boldsymbol{z} = \begin{bmatrix} \boldsymbol{pl_1} + \boldsymbol{\delta_1} - 1 \\ \vdots \\ \boldsymbol{pl_n} + \boldsymbol{\delta_n} - 1 \end{bmatrix}, \quad (54)$$

with the same notations as in Proposition 12.

Proof. The proof is similar to those of Propositions 12 and 14, and uses the fact that from Proposition 15 we obtain $pl\{o_i\}(\{x_k\}) = 0.5 + (pl_S\{o_i\}(\{x_k\}) - 0.5)(2\beta_{\overline{\{x_k\}}} - 1)$, $\forall k \in \{1,\ldots,K\}$ and $\forall i \in \{1,\ldots,n\}$, which allows us to rewrite (42) as (54). □

8.5 Comments and illustration

In this section, some comments are first made on the respective correction capacities of CD, CR and CN, with respect to the discrepancy measure (42) and taking into consideration the findings of Propositions 12, 14 and 16. Then, an illustrative example of the proposed learning of CD, CR and CN, is given. This example is also useful to make insightful additional remarks on CD, CR and CN, with respect to source performance improvement.

8.5.1 Correction capacities

The plausibility ranges on singletons after having applied CD, CR and CN are given by Remarks 14, 15 and 16, respectively.

Remark 14. *With CD, as $pl(\{x\}) = 1 - (1 - pl_S(\{x\}))\beta_{\{x\}}$ for each $x \in \mathcal{X}$, with $\beta_{\{x\}}$ varying in $[0,1]$, $pl(\{x\})$ can take any values in the interval $[pl_S(\{x\}), 1]$. It means that with CD, the plausibility $pl_S(\{x\})$ on each singleton can be shifted as close to 1 as required.*

Remark 15. *With CR, as $pl(\{x\}) = pl_S(\{x\})\beta_{\overline{\{x\}}}$ for each $x \in \mathcal{X}$, with $\beta_{\overline{\{x\}}}$ varying in $[0,1]$, $pl(\{x\})$ can take any values in $[0, pl_S(\{x\})]$. In other words, with CR, the plausibility $pl_S(\{x\})$ on each singleton can be carried as close to 0 as necessary.*

Remark 16. *With CN, as $pl(\{x\}) = 0.5 + (pl_S(\{x\}) - 0.5)(2\beta_{\overline{\{x\}}} - 1)$ for each $x \in \mathcal{X}$, with $\beta_{\overline{\{x\}}}$ varying in $[0,1]$, $pl(\{x\})$ can take any values in the interval $[\min(pl_S(\{x\}), 1 - pl_S(\{x\})), \max(pl_S(\{x\}), 1 - pl_S(\{x\}))]$. It is of particular interest if $pl_S(\{x\})$ is close to 0 or on the contrary close to 1 meaning that in these cases $pl_S(\{x\})$ can be moved to any values in $[0,1]$. However if $pl_S(\{x\})$ is close to 0.5, CN is not able to change its value which is confined around 0.5.*

The following example illustrates these different capacities of adjustment on simple scenarios.

Table 5: Attainable plausibilities with CD, CR and CN for three sources outputs.

	Ground truth	Source n°1	CD	CR	CN	Source n°2	CD	CR	CN	Source n°3	CD	CR	CN
pl({a})	1	0	1	0	1	0	1	0	1	1	1	1	1
pl({b})	0	1	1	0	0	0	0	0	0	1	1	0	0
pl({c})	0	1	1	0	0	1	1	0	0	0	0	0	0
		CD: $E_{pl} = 2$ CR: $E_{pl} = 1$ CN: $E_{pl} = 0$				CD: $E_{pl} = 1$ CR: $E_{pl} = 1$ CN: $E_{pl} = 0$				CD: $E_{pl} = 1$ CR: $E_{pl} = 0$ CN: $E_{pl} = 0$			

Example 19. *Let $\mathcal{X} = \{a, b, c\}$ and suppose, without lack of generality, that the ground truth is a.*

Suppose a source n°1 outputs a mass $m_S(\{b, c\}) = 1$ which means that $pl_S(\{a\}) = 0$ and $pl_S(\{b\}) = pl_S(\{c\}) = 1$. From Remarks 14, 15 and 16, to bring source n°1 output closer to the ground truth: CD can increase $pl_S(\{a\})$ to 1; CR can decrease $pl_S(\{b\})$ to 0 and $pl_S(\{c\})$ to 0; CN can increase $pl_S(\{a\})$ to 1 and decrease $pl_S(\{b\})$ to 0 and $pl_S(\{c\})$ to 0.

This case is presented again in Table 5, where two more situations are considered: a source n°2 giving $m_S(\{c\}) = 1$, that is $pl_S(\{a\}) = pl_S(\{b\}) = 0$ and $pl_S(\{c\}) = 1$, and a source n°3 giving $m_S(\{a, b\}) = 1$, that means $pl_S(\{a\}) = pl_S(\{b\}) = 1$ and $pl_S(\{c\}) = 0$.

As it can be observed in Table 5, CD can improve only one value of plausibility: the plausibility on the ground truth by increasing it as close as possible to 1, whereas CR can improve the other plausibility values (the ones not associated with the ground truth) by decreasing them as near as possible to 0. In contrast, CN can improve all plausibility values, by increasing the plausibility on the ground truth up to 1 and by decreasing the other possibilities down to 0. CR has then more degrees of flexibility than CD to improve the plausibility output of the source, and CN has in turn one more degree of flexibility than CR, on these three particular cases. As a result, we can see in Table 5 that for each source, CN has a lower (or equal) E_{pl} than CR, which in turn has a lower (or equal) E_{pl} than CD.

Let us note however that there exist situations where CD may be of more help than CR and CN, in particular those where all the plausibilities on singletons which are not the ground truth are below 0.5, i.e., $pl_S(\{b\}) < 0.5$ and $pl_S(\{c\}) < 0.5$, and where the plausibility on the ground truth is above 0.5, i.e., $pl_S(\{a\}) > 0.5$. In such a situation, using Remark 14, the lowest possible E_{pl} for CD, denoted by E_{pl}^{CD}, is attained with vector $\boldsymbol{\beta} = (0, 1, 1)$, in which case we have $E_{pl}^{CD} = (pl_S(\{b\})^2 + (pl_S(\{c\})^2$. Similarly, the lowest possible E_{pl} for CR is attained with, using Remark 15, vector $\boldsymbol{\beta} = (1, 0, 0)$, in which case $E_{pl}^{CR} = (1 - pl_S(\{a\}))^2$. Concerning CN, using Remark 16, the lowest possible E_{pl} is $E_{pl}^{CN} = (pl_S(\{b\})^2 + (pl_S(\{c\})^2 + (1 - pl_S(\{a\}))^2$ which is attained for $\boldsymbol{\beta} = (1, 1, 1)$. We thus see that in such a situation, we have necessarily $E_{pl}^{CD} < E_{pl}^{CN}$ and $E_{pl}^{CR} < E_{pl}^{CN}$. Besides, we also have $E_{pl}^{CD} < E_{pl}^{CR}$ if $(pl_S(\{b\})^2 + (pl_S(\{c\})^2 < (1 - pl_S(\{a\})^2$; an example of a source output satisfying this latter inequality is: $m_S(\{a\}) = 0.6$ and $m_S(\{b\}) = m_S(\{c\}) = 0.2$.

Table 6: Results for the minimization of E_{pl} with the data in Table 4 for each contextual correction mechanism for both sensors 1 and 2.

Contextual correction	Sensor 1	Sensor 2
CD	$\boldsymbol{\beta} = (0.76, 1.00, 1.00)$ $E_{pl}(\boldsymbol{\beta}) = 3.39$	$\boldsymbol{\beta} = (0.74, 1.00, 1.00)$ $E_{pl}(\boldsymbol{\beta}) = 4.81$
CR	$\boldsymbol{\beta} = (0.94, 0.66, 0.38)$ $E_{pl}(\boldsymbol{\beta}) = 2.33$	$\boldsymbol{\beta} = (0.65, 0.22, 0.55)$ $E_{pl}(\boldsymbol{\beta}) = 2.39$
CN	$\boldsymbol{\beta} = (0.33, 1.00, 0.45)$ $E_{pl}(\boldsymbol{\beta}) = 2.59$	$\boldsymbol{\beta} = (0.63, 0.06, 0.86)$ $E_{pl}(\boldsymbol{\beta}) = 2.25$

8.5.2 An illustrative example

Let us consider the data given in Table 4.

Results of the minimization of E_{pl} for CD, CR and CN are summarized in Table 6 for both sensors 1 and 2. Let us recall that $\boldsymbol{\beta} = (\beta_{\{a\}}, \beta_{\{h\}}, \beta_{\{r\}})$ for CD, and $\boldsymbol{\beta} = (\beta_{\overline{\{a\}}}, \beta_{\overline{\{b\}}}, \beta_{\overline{\{c\}}})$ for CR and CN, with different meanings and associated transformations for each vector.

In order to analyze the results presented in Table 6, one may look at the mass transfers associated with these learnt vectors, that is the transformations of the sensor outputs induced by CD, CR and CN. For instance, the CD learnt vector $\boldsymbol{\beta} = (0.76, 1.00, 1.00)$ for sensor 1 indicates that for each $B \subseteq \mathcal{X}$, a portion $(1 - 0.76 = 0.24)$ of mass $m_S(B)$ should be transferred to $B \cup \{a\}$. One may also use the interpretation given to the parameters β_A; for instance having $\beta_{\{a\}} = 0.76$ indicates that sensor 1 negatively lies for airplanes with mass 0.24. Yet, we find it more instructive as well as more appropriate considering the discrepancy measure used, to analyze those results in light of the changes they suggest on the plausibilities given to the singletons by the sensors. We focus on such an analysis hereafter.

For CD it can be observed that $\beta_{\{h\}} = \beta_{\{r\}} = 1$ for both sensors, which means from the previous Remark 14 that $pl_S(\{r\})$ and $pl_S(\{h\})$ do not have to be increased to improve E_{pl}, which is not the case for $pl_S(\{a\})$ as it has to be increased since $\beta_{\{h\}} < 1$ ($pl_S(\{a\})$ will be increased slightly more for sensor 2 than for sensor 1 since $0.76 > 0.74$). This means that the sensors may be considered cautious enough concerning the plausibilities they allocate to objects of type h and r, but may be a bit too bold concerning objects of type a. In particular, if for any object to be classified they give a low plausibility to h or to r then it should be left as is, but if they give a low plausibility to a then it should be increased.

Based on Remark 15, the CR learnt vector for sensor 1 indicate that $pl_S(\{a\})$ should (almost) not be decreased, whereas $pl_S(\{h\})$ and even more $pl_S(\{r\})$ have to be decreased for this sensor. This means that sensor 1 is bold enough for a, but too cautious for h and especially too cautious for r. So if sensor 1 gives a high plausibility to h or to r, then it should be decreased (even more so for r), but a high plausibility on a should remain roughly as is. Conclusions are different for sensor 2: all plausibilities should be substantially decreased. Precisely, only 65% of $pl_S(\{a\})$ should be left on a, 22% of $pl_S(\{h\})$ should be left on h and 55% of $pl_S(\{r\})$ should be left on r.

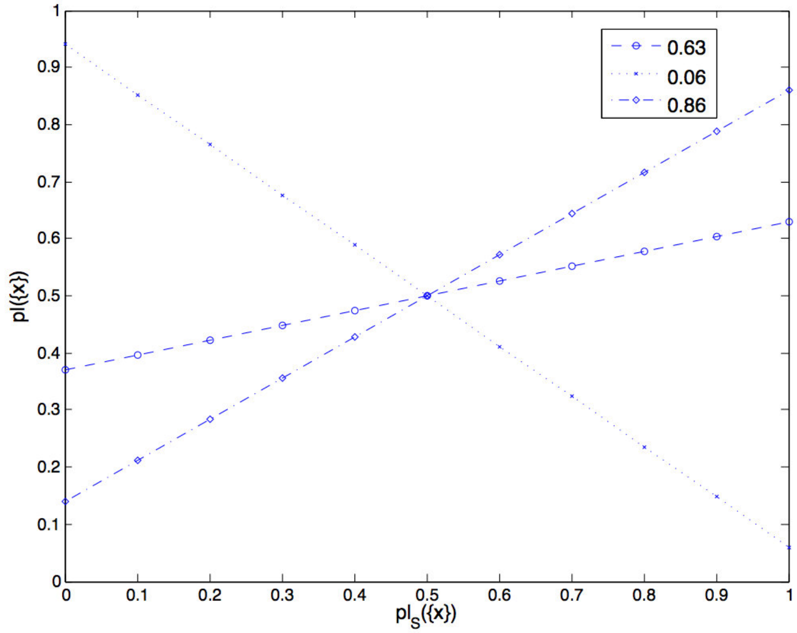


Figure 3: Plausibility on a singleton $pl(\{x\})$ after CN correction of $pl_S(\{x\})$, for three different values of $\beta_{\overline{\{x\}}}$.

For CN, we know from Remark 16 that we have $pl(\{x\}) = 0.5 + (pl_S(\{x\}) - 0.5)(2\beta_{\overline{\{x\}}} - 1)$ for each $x \in \mathcal{X}$. Figure 3 shows this latter equation for the three values of $\beta_{\overline{\{x\}}}$ learnt for sensor 2. As it can be seen on this figure, if we have $pl_S(\{h\}) < 0.5$ for sensor 2, then this plausibility on h should be increased, for instance if $pl_S(\{h\}) = 0.2$ then this value should be increased to $pl(\{h\}) = 0.5 + (pl_S(\{h\}) - 0.5)(2 \cdot \beta_{\overline{\{h\}}} - 1) = 0.5 + (0.2 - 0.5)(2 \cdot 0.06 - 1) = 0.764$. Conversely, if $pl_S(\{h\}) > 0.5$, then this plausibility on h should be decreased. For a and r, the situation is similar: if the plausibilities on these singletons are lower than 0.5, then they should be increased, and if they are greater than 0.5, they should be decreased, albeit with different rates. Of note for sensor 1, is the fact that $pl_S(\{h\})$ should not be altered at all since $\beta_{\overline{\{h\}}} = 1$ for this sensor, which is quite different from the situation observed for sensor 2.

This small illustrative example shows that CD, CR and CN yield different tunings of a source. Perhaps most importantly, on a practical side, Table 6 shows that CR and CN also permit to obtain lower values for E_{pl} than those reached with CD, for both sensors 1 and 2, which confirms the advantages in some cases of CR and CN over CD exposed in Example 19 concerning the minimization of E_{pl}. This demonstrates the interest of investigating alternative correction mechanisms distinct from a discounting process. A second observation is that a lower value for E_{pl} using CN has been obtained for sensor 2 compared to the value obtained with CR, and conversely a lower value for E_{pl} using CR has been obtained for sensor 1 compared to the value obtained with CN, which shows the potential utility of both mechanisms in terms of performance improvement.

Finally, let us note that even if CR and CD are related (CR amounts to the negation of the CD of the negation of the information provided by the source [20]), CR and CD parameters minimizing E_{pl} (42) cannot be deduced analytically from each other, as illustrated by Example 20, which shows that knowing the vector $\boldsymbol{\beta}$ minimizing E_{pl} for CD does not imply knowing the vector $\boldsymbol{\beta}$ minimizing E_{pl} for CR.

Example 20. *Let us modify in Table 4, MF* $m_{S_1}\{o_1\}$ *by*

$$\begin{aligned} m_{S_1}\{o_1\}(\{r\}) &= 0.5282, \\ m_{S_1}\{o_1\}(\{h,r\}) &= 0.3000, \\ m_{S_1}\{o_1\}(\mathcal{X}) &= 0.1718, \end{aligned}$$

i.e.*, information coming from sensor* 1 *is slightly deteriorated, the truth being* a*. Then, learnings of CD parameters for sensors* 1 *and* 2 *yield the same vector* $\boldsymbol{\beta} = (0.74, 1.00, 1.00)$*, while learnings of CR parameters yield* $\boldsymbol{\beta} = (0.92, 0.68, 0.38)$ *for sensor* 1 *and* $\boldsymbol{\beta} = (0.65, 0.22, 0.55)$ *for sensor* 2*.*

8.6 Application in classification

Despite the formal elegance of CD, CR and CN, their usefulness in belief function based applications might be challenged. Therefore, an experiment is conducted in this section to demonstrate their ability to improve the performances of an evidential classifier.

The chosen evidential classifier is the evidential k-nearest neighbour classifier (*ev-knn*) introduced by Denœux in [5]. It is used on a 5-class classification problem with data generated from 5 bivariate normal distributions with respective means $\mu_{x_1} = (0,0)$, $\mu_{x_2} = (2,0)$, $\mu_{x_3} = (0,2)$, $\mu_{x_4} = (2,2)$, $\mu_{x_5} = (1,1)$ and common variance matrix

$$\Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}.$$

For each class $x \in \mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$, 1000 instances have been generated. The total amount of data is then composed of 5000 instances and is illustrated in Figure 4.

The principle of ev-knn [5] is to consider the k-nearest neighbours n_i (according to a distance measure d, *e.g.*, the Euclidean distance), $i \in \{1, \ldots, k\}$, of a new instance a to be classified, and to build a MF regarding the class of a, which results from the combination by Dempster's rule of k MFs m_i, $i \in \{1, \ldots, k\}$, each one of these MFs being associated with a neighbour n_i of a, and reflecting a piece of evidence regarding the class of the instance a to be classified. With $x^i \in \mathcal{X}$ the class of n_i, each m_i, $i \in \{1, \ldots, k\}$, is defined in the following manner:

$$m_i(A) = \begin{cases} \tau \cdot e^{-\gamma_{x^i}\,(d(a,n_i))^2} & \text{if } A = \{x^i\}, \\ 1 - \tau \cdot e^{-\gamma_{x^i}\,(d(a,n_i))^2} & \text{if } A = \mathcal{X}, \\ 0 & \text{otherwise,} \end{cases} \tag{55}$$

with τ a constant in $[0,1]$, γ_x a positive constant depending on class $x \in \mathcal{X}$, and d a distance. For each neighbour n_i of a, the knowledge of the class x^i of n_i is a piece of evidence increasing the belief that the class of a is x^i depending on the distance $d(a, n_i)$ between a and n_i.

Following the simple heuristic used in [5], τ has been fixed to 0.95 and γ_x has been chosen equal to the inverse of the mean distance between each sample of class x, which belongs to the training set of the classifier ev-knn. The number k of neighbours has been chosen equal to 3.

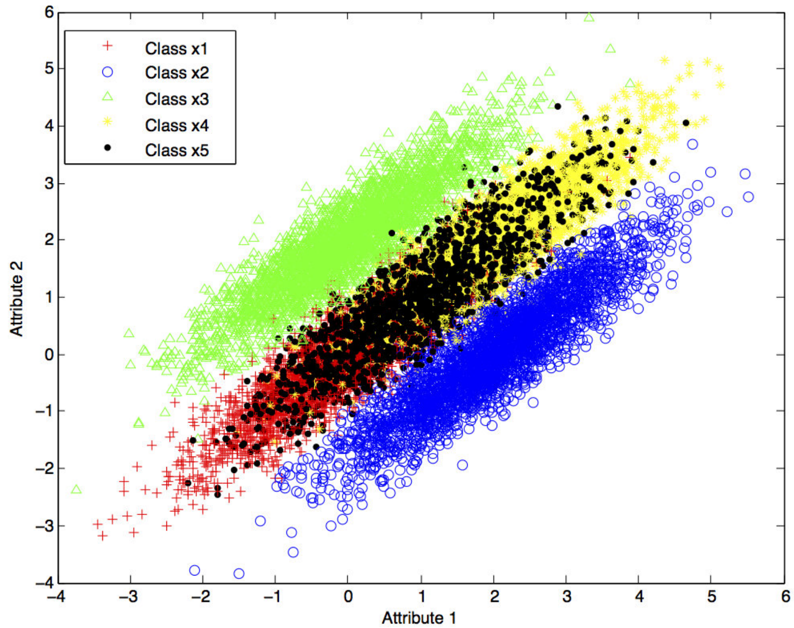


Figure 4: Illustration of the data generated for a 5-class classification problem with 2 features.

Let us note that these latter choices for the parameters of the ev-knn classifier may not be the most optimal, *i.e.*, the ones leading to the best possible performances with respect to the experiment described in this section; in particular, an approach has been proposed in [41] to optimize these parameters. Similarly, ev-knn may not be the (evidential) classifier, with the best possible performances on this experiment. The idea in this section is not to show that correction mechanisms yield the best possible classifier on a given classification problem. Rather, the aim is to illustrate the fact that given a real-life situation where one has access to an information source, such as a sensor, which can not really be controlled (*i.e.*, is like a black box) and which is not necessarily the best possible sensor out there, it may be possible to improve this source using correction mechanisms. Hence, the role played by ev-knn in our experiment is simply to simulate such a source.

The 5000 instances displayed in Figure 4, have been divided into three parts:

1. The first third of the data constitutes the training set of ev-knn, *i.e.*, the set used to learn the parameters γ_x for each class $x \in \mathcal{X}$, as well as the set of neighbours used to classify samples;

2. The second third of the data constitutes the training set for the correction mechanisms, *i.e.*, it is dedicated to learn the best CD, CR and CN of ev-knn outputs;

3. The last third of the data constitutes the test set, *i.e.*, it is used to test these learnt corrections of ev-knn, as well as ev-knn.

The performances of ev-knn without correction (ev-knn), ev-knn with CD (CD), ev-knn with CR (CR), and ev-knn with CN (CN), are reported in Figure 5 using ROC curves (a higher curve corresponds to higher performance); the ROC curves were obtained using the plausibility transformation.

As may be expected, samples of classes x_2 and x_3, which are clearly disjoint from the samples of the other classes (cf Figure 4), are very well classified by ev-knn (cf Figures 5b and 5c). Besides, CD, CR and CN neither improve nor deteriorate ev-knn outputs for these classes.

Samples of classes x_1, x_4 and x_5, overlap (cf Figure 4) and are not so well classified by ev-knn (cf Figures 5a, 5d and 5e). Most interestingly, CD, CR and CN succeed in improving ev-knn outputs for these more difficult classes, which is an experimental evidence of the interest of these correction mechanisms to improve the performance of a source.

9 Conclusion

The aim of this study was twofold: (1) to enlarge the set of tools available to deal with contextual knowledge about the quality of a source, and (2) to provide a practical means to obtain such kind of knowledge.

Several conclusions may be drawn from our works and with respect to this aim.

First, it is indeed possible to find useful complements to contextual discounting based on a coarsening and its recent extension to an arbitrary set of contexts: as illustrated through numerous examples, CD, CR, CN, CdD, CdR, CD+ and CR+ may be used to account for various situations with respect to meta-knowledge about a source, and some of them have even been shown to be also interesting to improve a source performance.

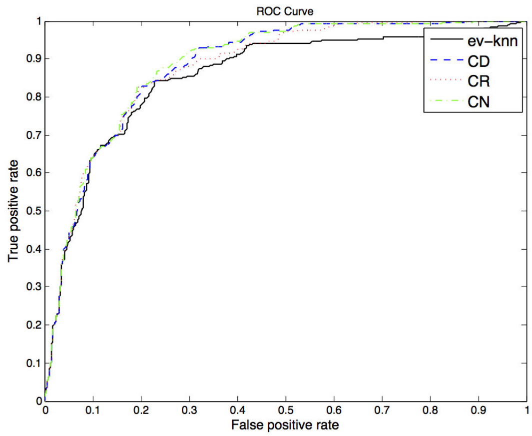
The pitfall is that these contextual correction mechanisms correspond to quite specific meta-knowledge about a source, and accordingly to rather precise and different interpretations of a given piece of information provided by the source, and thus one must be careful in choosing one or the other mechanism and in setting its associated parameters $\mathcal{A}$ and $\boldsymbol{\beta}$.

When not much is known about the quality of a source, this variety and associated difficulty may seem daunting. Fortunately, if labelled data are available, we have shown that it is possible to learn which mechanism is the best (in terms of performance), and its associated parameters. In this case, the subtlety resides in using an error criterion that is meaningful, that leads itself to easy analysis and that also leads to an optimization problem, which can be solved efficiently.

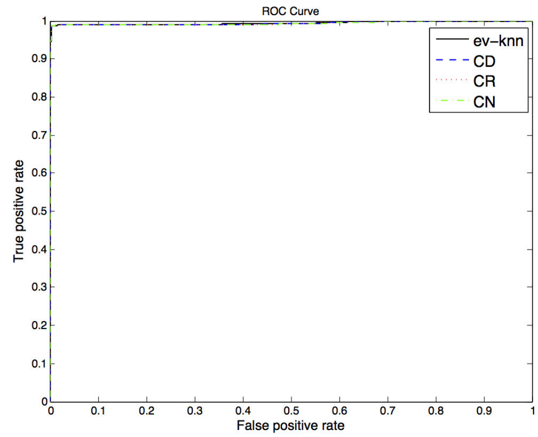
This study may be continued in various directions, which are left for further research.

In [28], it was envisioned to derive a contextual version of negating by considering a situation where the agent holds beliefs concerning the truthfulness of the source conditionally on different subsets of $\mathcal{X}$, much as contextual discounting based on a coarsening was originally derived in [23] and is extended to an arbitrary set of subsets in [21]. It would be interesting to study this alternative path to derive a contextual version of negating and to compare the resulting correction mechanism to CN.

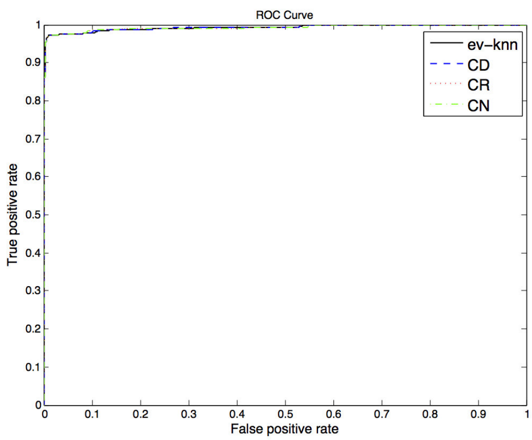
Compared to CD and CR, some formal results remain to be obtained for CN. In particular, since the inverse of the rule $\textcircled{\scriptsize$\cap$}$, and precisely the 0-commonality function, is at the base of what would be contextual de-negating and thus of contextual negating based on the canonical decomposition, it would be interesting to study further this function in order to know under which conditions it does not equal to 0, similarly as we know that commonality $q(A) \neq 0$ for all $A \subseteq \mathcal{X}$ if the mass function is non dogmatic.



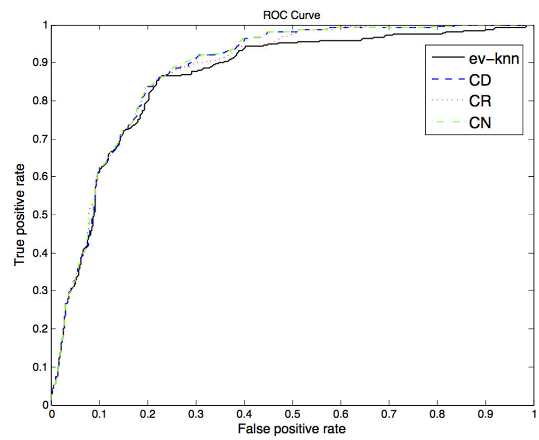
(a) Positive class: class x_1.



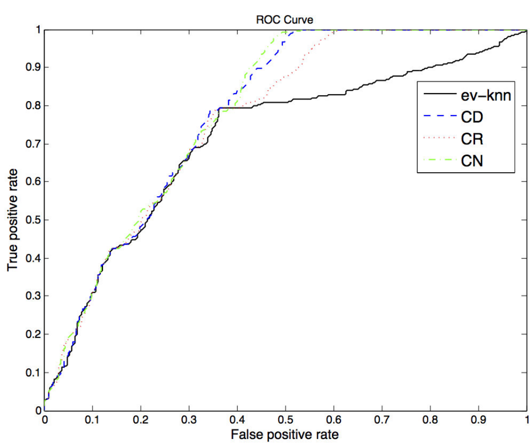
(b) Positive class: class x_2.



(c) Positive class: class x_3.



(d) Positive class: class x_4.



(e) Positive class: class x_5.

Figure 5: ROC curves for each of the 5 classes.

The conjunctive and equivalence rules, which are at the base of CR and CN respectively, are actually two extreme members of a family of rules known as the α-conjunctions [35, 25] depending on a parameter $\alpha \in [0,1]$. Hence, formally, CR and CN may easily be presented as two members of a family of contextual correction mechanisms based on the α-conjunctions. While it might be possible to find an interpretation for all the members in this family, their main interest might reside in applications, where they could be used as flexible tools to improve a source. This would require extending to this family of correction mechanisms, the efficient means proposed in this paper to learn CR and CN.

Concerning the learning of contextual correction parameters, other discrepancy measures (in particular distances [15, 18]) than the one based on the plausibility function may be studied. Other approaches for obtaining these parameters may also be investigated, such as expert elicitation procedures (see, *e.g.*, [29] for an expert elicitation procedure of the knowledge associated with discounting) or methods using confusion matrices, which may be appropriate for discovering truthfulness [17]. Finally, these approaches should also be looked at to obtain the parameters associated with the extension of contextual discounting based on a coarsening uncovered recently in [21].

Acknowledgments

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A A general model of source truthfulness

In this appendix, a general model of source truthfulness is presented. It includes as particular cases the three states ℓ_A, p_A, and n_A introduced in Sections 3 and 4.

A.1 Elementary-level truthfulness

Assume that a source S provides a piece of information on the value taken by $\mathbf{x}$ of the form $\mathbf{x} \in B$, for some $B \subseteq \mathcal{X}$.

Let us now consider a particular value $x \in \mathcal{X}$. In Sections 3.1 and 4.1, the notions of (non) truthful, *positively* (non) truthful and *negatively* (non) truthful for a value $x \in \mathcal{X}$ have been defined (Definitions 1, 3 and 4, respectively). Let us denote by $\mathbf{t}_x$ a variable with associated frame $\mathcal{T}_x = \{t_x, \neg t_x, \neg t_x^p, \neg t_x^n\}$ allowing us to model the global truthfulness of the source with respect to the value x, where:

- t_x corresponds to the case where the source tells the truth whatever it says about the value x, *i.e.*, to a (positively and negatively) truthful source for x;
- $\neg t_x$ corresponds to the case where the source lies whatever it says about the value x, *i.e.*, to a (positively and negatively) non truthful source for x;
- $\neg t_x^p$ corresponds to the case of a source that lies only when it says that x is a possibility for $\mathbf{x}$, *i.e.*, to a positively non truthful and negatively truthful source for x;

- $\neg t_x^n$ corresponds to the case of a source that lies only when it tells that x is not a possibility for $\mathbf{x}$, *i.e.*, to a positively truthful and negatively non truthful source for x.

Thus, there are four possible cases:

1. Suppose the source tells x is possibly the actual value of $\mathbf{x}$, *i.e.*, the information $\mathbf{x} \in B$ provided by the source is such that $x \in B$.
   (a) If the source is assumed to be in state t_x or $\neg t_x^n$, then one must conclude that x is possibly the actual value of $\mathbf{x}$;
   (b) If the source is assumed to be in state $\neg t_x^p$ or $\neg t_x$, then one must conclude that x is not a possibility for the actual value of $\mathbf{x}$;
2. Suppose the source tells x is not a possibility for the actual value of $\mathbf{x}$, *i.e.*, $x \notin B$.
   (a) If the source is assumed to be in state t_x or $\neg t_x^p$, then one must conclude that x is not a possibility for the actual value of $\mathbf{x}$;
   (b) If the source is assumed to be in state $\neg t_x^n$ or $\neg t_x$, then one must conclude that x is possibly the actual value of $\mathbf{x}$;

A.2 Two truthfulness assumptions

Let $\mathcal{T}$ denote the possible states of S with respect to its truthfulness for all $x \in X$. By definition, $\mathcal{T} = \times_{x \in \mathcal{X}} \mathcal{T}_x$.

Let $h_A^{t_1,t_2} \in \mathcal{T}$, $A \subseteq \mathcal{X}$, denote the state where the source is in state $t_1 \in \mathcal{T}_x$ for all $x \in A$, and in state $t_2 \in \mathcal{T}_x$ for all $x \notin A$. For instance, let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$, $A = \{x_3, x_4\}$, $t_1 = t_x$ and $t_2 = \neg t_x^p$, then

$$h_A^{t_1,t_2} = h_{\{x_3,x_4\}}^{t_x,\neg t_x^p} = \left(\neg t_{x_1}^p, \neg t_{x_2}^p, t_{x_3}, t_{x_4}\right),$$

i.e., the source is truthful for x_3 and x_4, and is positively non truthful and negatively truthful for x_1 and x_2.

Consider now the following question: what must one conclude about $\mathbf{x}$ when the source tells $\mathbf{x} \in B$ and is assumed to be in some state $h_A^{t_1,t_2}$? To answer this question, one merely needs to look in turn at each $x \in \mathcal{X}$ and:

1. to find for each of those $x \in \mathcal{X}$, whether $x \notin B$ or $x \in B$;
2. to find for each of those $x \in \mathcal{X}$, whether $x \notin A$ or $x \in A$, and, accordingly, which of t_1 or t_2 applies for each of those $x \in \mathcal{X}$;
3. and then, using information obtained at steps 1 and 2, to find out for each of those $x \in \mathcal{X}$, which one of the four cases 1.a), 1.b), 2.a) or 2.b) described at the very end of Section A.1 applies.

Table 7 lists exhaustively, *i.e.*, for all possible cases with respect to the membership of a given value x to the sets B and A, and for all possible couples (t_1, t_2), whether one should deduce that a given value $x \in \mathcal{X}$ is possibly the actual value of $\mathbf{x}$ or not – the former is indicated by a 1 and the latter by a 0 in columns (t_1, t_2). According to Table

Table 7: Interpretations of the source testimony according to couples (t_1, t_2).

$x \in B$	$x \in A$	$\neg t_x^p, \neg t_x^p$	$t_x, \neg t_x^p$	$\neg t_x^p, t_x$	t_x, t_x	$\neg t_x, \neg t_x^p$	$\neg t_x^n, \neg t_x^p$	$\neg t_x, t_x$	$\neg t_x^n, t_x$	$\neg t_x^p, \neg t_x$	$t_x, \neg t_x$	$\neg t_x^p, \neg t_x^n$	$t_x, \neg t_x^n$	$\neg t_x, \neg t_x$	$\neg t_x^n, \neg t_x$	$\neg t_x, \neg t_x^n$	$\neg t_x^n, \neg t_x^n$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

7, when the source is assumed to be in, *e.g.*, state $h_A^{t_x,\neg t_x^p}$, then one should deduce that $x \in \mathcal{X}$ is a possible value for $\mathbf{x}$ iff $x \in B$ and $x \in A$, and therefore, since this holds for all $x \in \mathcal{X}$, one should deduce that $\mathbf{x} \in B \cap A$. We may then remark that state $h_A^{t_x,\neg t_x^p}$ is actually nothing but state p_A discussed in Section 4.2. Similarly, state $h_A^{\neg t_x^n,t_x}$ corresponds to state n_A of Section 4.3, and state $h_A^{t_x,\neg t_x}$ correspond to state ℓ_A of Section 3.2.

In addition, let us remark that states ℓ_A, p_A, n_A, induce that a testimony $\mathbf{x} \in B$ should be combined with subset A using operators $\underline{\cap}$, $\cap$, $\cup$, respectively. This comes from the fact that these states are associated respectively to logical equality, conjunction, and disjunction, as can be seen from Table 7. More generally, this latter table shows that the couples $(t_1, t_2) \in \mathcal{T}_x^2$ actually yield all possible binary Boolean connectives; a formally interesting result pertaining to information correction which has some similarity with what was done recently in information fusion in [28] and in [27], where the conjunctive rule and the notion of conflict, respectively, are extended to other binary Boolean connectives than the conjunction.

Remark 17. *In this paper, we are only interested by the states based on the couples* $(t_1, t_2) \in \{(t_x, \neg t_x^p), (\neg t_x^n, t_x), (t_x, \neg t_x)\}$ *since they are the only ones needed for our developments. Yet, let us note that the other couples may be useful. For instance, the state* $h_{x_i}^{\neg t_x^n,\neg t_x^p}$*, which is such that* $\Gamma_B(h_{x_i}^{\neg t_x^n,\neg t_x^p}) = x_i$*, for all* $B \subseteq \mathcal{X}$*, allows one to recover the correction mechanism used in [1] to favor a given element* $x_i \in \mathcal{X}$*.*

B Single correction perspective

Interestingly, the BBCs corresponding to simple contextual biases put forward by Proposition 7, are equivalent to a single BBC corresponding to some particular knowledge on the truthfulness of the information source as shown by Proposition 17.

Proposition 17.

$$(\circ_{A\in\mathcal{A}}\, f_{m^{\mathcal{H}}_{A,\cup}})(m_S) \quad = \quad f_{m^{\mathcal{H}}_{\mathcal{A},\cup}}(m_S), \tag{56}$$

with $m^{\mathcal{H}}_{\mathcal{A},\cup} = \textcircled{\cup}_{A\in\mathcal{A}} m^{\mathcal{H}}_{A,\cup}$.

Proof. Let us first consider the left side of Equation (56). From Proposition 7, we have

$$(\circ_{A\in\mathcal{A}}\, f_{m^{\mathcal{H}}_{A,\cup}})(m_S) \quad = \quad m_S \textcircled{\cup}_{A\in\mathcal{A}} A_{\beta_A}. \tag{57}$$

From the definition of $\textcircled{\cup}$, we have that Equation (57) allocates, for all $\mathcal{C} \subseteq \mathcal{A}$ and all $B \subseteq \mathcal{X}$, the quantity

$$m_S(B) \cdot \prod_{A\in\mathcal{C}} (1-\beta_A) \cdot \prod_{D\in\mathcal{A}\setminus\mathcal{C}} \beta_D$$

to $B\cup(\cup_{A\in\mathcal{C}}A)\subseteq\mathcal{X}$, and a null mass to all $D\subseteq\mathcal{X}, D\neq B\cup(\cup_{A\in\mathcal{C}}A), \forall\mathcal{C}\subseteq\mathcal{A}, \forall B\subseteq\mathcal{X}$.

Let us now consider the right side of Equation (56). Let $\mathcal{H}_{\mathcal{A},n}=\{n_A, A\in\mathcal{A}\}$. The mass function $m^{\mathcal{H}}_{\mathcal{A},\cup}$ clearly has $2^{|\mathcal{H}_{\mathcal{A},n}|}$ focal sets, which are $\mathcal{H}_{\mathcal{A},n}$ and the sets $\{t\cup\{n_A, A\in\mathcal{C}\}\}$, for all $\mathcal{C}\subset\mathcal{A}$. Besides, from the definition of $\textcircled{\cup}$, the mass function $m^{\mathcal{H}}_{\mathcal{A},\cup}$ is such that

$$
\begin{aligned}
m^{\mathcal{H}}_{\mathcal{A},\cup}(\mathcal{H}_{\mathcal{A},n}) &= \prod_{A\in\mathcal{A}}(1-\beta_A),\\
m^{\mathcal{H}}_{\mathcal{A},\cup}(\{t\cup\{n_A, A\in\mathcal{C}\}\}) &= \prod_{A\in\mathcal{C}}(1-\beta_A)\cdot\prod_{D\in\mathcal{A}\setminus\mathcal{C}}\beta_D, \quad \forall\mathcal{C}\subset\mathcal{A}.
\end{aligned}
$$

Furthermore, from the definition of the BBC procedure, we have that $f_{m^{\mathcal{H}}_{\mathcal{A},\cup}}(m_S)$ allocates,

- for all $\mathcal{C}\subset\mathcal{A}$ and all $B\subseteq\mathcal{X}$, the quantity

$$m_S(B)\cdot\prod_{A\in\mathcal{C}}(1-\beta_A)\cdot\prod_{D\in\mathcal{A}\setminus\mathcal{C}}\beta_D$$

  to

$$
\begin{aligned}
\Gamma_B(\{t\cup\{n_A, A\in\mathcal{C}\}\}) &= \Gamma_B(t)\cup(\cup_{A\in\mathcal{C}}\Gamma_B(n_A))\\
&= B\cup(\cup_{A\in\mathcal{C}}(B\cup A))\\
&= B\cup(\cup_{A\in\mathcal{C}}A)
\end{aligned}
$$

- and for $\mathcal{C}=\mathcal{A}$ and all $B\subseteq\mathcal{X}$, the quantity

$$m_S(B)\cdot\prod_{A\in\mathcal{A}}(1-\beta_A)=m_S(B)\cdot\prod_{A\in\mathcal{C}}(1-\beta_A)\cdot\prod_{D\in\mathcal{A}\setminus\mathcal{C}}\beta_D$$

  to

$$
\begin{aligned}
\Gamma_B(\mathcal{H}_{\mathcal{A},n}) &= \Gamma_B(\{n_A, A\in\mathcal{C}\})\\
&= \cup_{A\in\mathcal{C}}\Gamma_B(n_A)\\
&= \cup_{A\in\mathcal{C}}(B\cup A)\\
&= B\cup(\cup_{A\in\mathcal{C}}A),
\end{aligned}
$$

  which completes the proof.

□

In other words, CD can also be viewed as a single correction under meta-knowledge $m^{\mathcal{H}}_{\mathcal{A},\cup}$ obtained by combining disjunctively the pieces of meta-knowledge $m^{\mathcal{H}}_{A,\cup}, A\in\mathcal{A}$.

A similar proposition (Proposition 18) to Proposition 17 exists for CR (although it is not its strict counterpart). It relies on the following technical lemma, which puts forward a particular subset of $\mathcal{H}$, denoted by P_A, that induces the same transformation to a testimony $\mathbf{x}\in B$ as contextual lie p_A.

Lemma 1.

$$\Gamma_B(p_A) = \Gamma_B(P_A), \quad \forall A \subseteq \mathcal{X}, \quad \forall B \subseteq \mathcal{X},$$

with P_A defined by $P_A = \{p_x | x \in A\} \cup p_\emptyset$.

Proof. We have

$$\begin{aligned}
\Gamma_B(P_A) &= \Gamma_B(p_\emptyset) \bigcup_{x\in A} \Gamma_B(p_x) = (B \cap \emptyset) \bigcup_{x \in A} (B \cap x) \\
&= (B \cap \emptyset) \bigcup_{x\in A} (B \cap x) = \bigcup_{x \in A} (B \cap x) \\
&= B \cap (\bigcup_{x\in A} x) = B \cap A = \Gamma_B(p_A).
\end{aligned}$$

□

Proposition 18.

$$m_S \circledcap_{A\in\mathcal{A}} A^{\beta_A} \quad = \quad f_{m^{\mathcal{H}}_{\mathcal{A},\cap}}(m_S) \tag{58}$$

with $m^{\mathcal{H}}_{\mathcal{A},\cap} = \circledcap_{A\in\mathcal{A}} m^{\mathcal{H}}_{A,P,\cap}$, where

$$m^{\mathcal{H}}_{A,P,\cap}(P_{\mathcal{X}}) \quad = \quad \beta_A, \quad m^{\mathcal{H}}_{A,P,\cap}(P_A) = 1 - \beta_A,$$

and $P_A = \{p_x | x \in A\} \cup p_\emptyset$, for all $A \subseteq \mathcal{X}$.

Proof. First, let us study the left side of (58). From the definition of $\circledcap$, we have that, for all $B \subseteq \mathcal{X}$, the quantity $m_S(B) \cdot (\circledcap_{A\in\mathcal{A}} A^{\beta_A})(C)$, with C a focal set of $\circledcap_{A\in\mathcal{A}} A^{\beta_A}$, is allocated to set $B \cap C$.

Moreover, from the definitions of A^{β_A} and $m^{\mathcal{H}}_{A,P,\cap}$, it is straightforward to see that

$$(\circledcap_{A\in\mathcal{A}} A^{\beta_A})(C) = (\circledcap_{A\in\mathcal{A}} m^{\mathcal{H}}_{A,P,\cap})(P_C)$$

holds for any focal set C of $\circledcap_{A\in\mathcal{A}} A^{\beta_A}$.

Studying now the right side of (58). From the definition of the BBC procedure and of $m^{\mathcal{H}}_{\mathcal{A},\cap}$, the quantity

$$m_S(B) \cdot (\circledcap_{A\in\mathcal{A}} m^{\mathcal{H}}_{A,P,\cap})(P_C) \quad = \quad m_S(B) \cdot (\circledcap_{A\in\mathcal{A}} A^{\beta_A})(C)$$

is allocated to set $\Gamma_B(P_C) = \Gamma_B(p_C)$ (using Lemma 1) which is in turn equal to $B \cap C$. □

In other words, CR can be viewed as a single correction under meta-knowledge $m^{\mathcal{H}}_{\mathcal{A},\cap}$ obtained by combining conjunctively the pieces of meta-knowledge $m^{\mathcal{H}}_{A,P,\cap}$, each of them inducing the same correction as the pieces of meta-knowledge $m^{\mathcal{H}}_{A,\cap}$ (24) underlying CR[10].

[10]This statement holds because $\Gamma_B(P_X) = \Gamma_B(p_X)$ (using Lemma 1), and state p_X corresponds to state t (Remark 3).

C Proof of Proposition 15

A proof for Proposition 15 is provided in Appendix C.2. It uses the matrix notation for belief functions and in particular some technical results using this notation and concerning rule $\textcircled{$\underline{\cap}$}$, which are first recalled in Appendix C.1.

C.1 Matrix notation

Matrix calculus can be applied to belief functions in order to simplify their mathematics [37]. A MF m (and its associated functions, *e.g.*, q) can be seen as a column vector of size $2^{|\mathcal{X}|}$, whose elements are ordered according to the so-called binary order: the ith element of the vector $\mathbf{m}$ corresponds to the set with elements indicated by 1 in the binary representation of $i-1$. For instance, let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$. The first element ($i=1$) of the vector $\mathbf{m}$ corresponds to $\emptyset$ since the binary representation of $1-1$ is 0000. The twelfth element ($i=12$) corresponds to $\{x_1, x_2, x_4\}$ since the binary representation of $12-1$ is 1011.

Let us denote by $\mathbf{Kron}(\mathbf{A}, \mathbf{B})$ the $mp \times nq$ matrix resulting from the Kronecker product of a $m \times n$ matrix $\mathbf{A}$ with a $p \times q$ matrix $\mathbf{B}$. The matrix $\mathbf{Kron}(\mathbf{A}, \mathbf{B})$ is defined by:

$$\mathbf{Kron}(\mathbf{A}, \mathbf{B}) = \begin{bmatrix} A(1,1)\mathbf{B} & \cdots & A(1,n)\mathbf{B} \\ \vdots & \ddots & \vdots \\ A(m,1)\mathbf{B} & \cdots & A(m,n)\mathbf{B} \end{bmatrix}.$$

The transformation of a MF m into its associated commonality function q admits a simple expression using matrix notation. We have:

$$\mathbf{q} = \mathbf{Q} \cdot \mathbf{m},$$

with $\mathbf{Q}$ a matrix that can be obtained in a simple way using Kronecker multiplication, from the building block $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$:

$$\mathbf{Q}^{i+1} = \mathbf{Kron}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Q}^{i} \right), \mathbf{Q}^1 = 1,$$

where $\mathbf{Q}^{i+1}$ denotes the matrix $\mathbf{Q}$ when $|\mathcal{X}| = i$. On the other hand, m can be recovered from q as follows:

$$\mathbf{m} = \mathbf{Q}^{-1} \cdot \mathbf{q}.$$

As mentioned in Section 2.1.2, combination by the rule $\textcircled{$\underline{\cap}$}$ can be expressed similarly as combination by $\textcircled{$\cap$}$ and $\textcircled{$\cup$}$, that is by a simple pointwise product expression, as shown by Smets [35, 37]. The counterpart of the commonality and implicability functions on which the pointwise product expression of $\textcircled{$\underline{\cap}$}$ is based, is called 0-commonality in [26, 24]. Let $\underline{q}$ denote the 0-commonality function associated to a MF m. We have, for any two MFs m_1 and m_2 [35, 37]:

$$\underline{q}_{1 \textcircled{$\underline{\cap}$} 2}(A) = \underline{q}_1(A) \cdot \underline{q}_2(A), \quad \forall A \subseteq \mathcal{X}.$$

Smets [35, 37] showed that function $\underline{q}$ can be obtained as follows:

$$\underline{\mathbf{q}} \;=\; \underline{\mathbf{Q}} \cdot \mathbf{m},$$

with $\underline{\mathbf{Q}}$ a matrix, which as shown by Pichon and Denœux [26, 24] can easily be obtained by Kronecker multiplication, similarly as $\mathbf{Q}$, but using building block $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

In the sequel, we will also denote by $\underline{\mathbf{B}}$ the matrix obtained by Kronecker multiplication, similarly as $\mathbf{Q}$, but using building block $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. This matrix plays only a technical role in the proof that follows.

C.2 Plausibility on singletons after CN

The proof of Proposition 15 requires the following technical Lemmas 2, 3, 4, 5 and 6.

Lemma 2.

$$\underline{Q}(B, A) \;=\; (-1)^{|\overline{A} \cap B|}, \quad \forall A, B \subseteq \mathcal{X}. \tag{59}$$

Proof. From [37, page 26 ($\underline{\mathbf{Q}}$ corresponding to $\mathbf{G}$ with $\alpha = 0$)], column A of matrix $\underline{\mathbf{Q}}$ is $\mathbf{V}_A \cdot \mathbf{1}$ ($\mathbf{1}$ denotes the column vector which components are 1), with $\mathbf{V}_A$ a matrix defined by $\mathbf{V}_A = \prod_{x \notin A} \mathbf{V}_{\overline{x}}$, where $\mathbf{V}_{\overline{x}} = [V_{\overline{x}}(A, B)]$, $\forall x \in \mathcal{X}$, $\forall A, B \subseteq \mathcal{X}$, with:

$$V_{\overline{x}}(A, B) = \begin{cases} 1 & \text{if } x \notin A, \quad A = B, \\ -1 & \text{if } x \in A, \quad A = B, \\ 0 & \text{if } A \neq B. \end{cases} \tag{60}$$

Matrices $\mathbf{V}_{\overline{x}}$ are diagonal, hence we have for all $A, B \subseteq \mathcal{X}$:

$$\begin{aligned} \underline{Q}(B, A) &= (\mathbf{V}_A \cdot \mathbf{1})(B) = V_A(B, B) = \prod_{x \notin A} V_{\overline{x}}(B, B) \\ &= \left(\prod_{x \notin A, x \in B} V_{\overline{x}}(B, B) \right) \cdot \left(\prod_{x \notin A, x \notin B} V_{\overline{x}}(B, B) \right) \\ &= \prod_{x \notin A, x \in B} V_{\overline{x}}(B, B) = \prod_{x \in \overline{A} \cap B} V_{\overline{x}}(B, B) = (-1)^{|\overline{A} \cap B|}. \end{aligned}$$

□

Lemma 3. *For all $A \subset \mathcal{X}$, the 0-commonality function $\underline{q}_A$ associated to the simple MF A^{β_A}, $\beta_A \in [0, 1]$, is defined for all $B \subseteq \mathcal{X}$ by:*

$$\underline{q}_A(B) \;=\; \begin{cases} 1 & \textit{if } |\overline{A} \cap B| \textit{ is even,} \\ 2 \cdot \beta_A - 1 & \textit{otherwise.} \end{cases} \tag{61}$$

Proof.

$$\begin{aligned} \underline{q}_A(B) &= \sum_{C \subseteq \mathcal{X}} \underline{Q}(B, C) \cdot (A^{\beta_A})(C) = \underline{Q}(B, A) \cdot (1 - \beta_A) + \underline{Q}(B, \mathcal{X}) \cdot \beta_A. \\ &= (-1)^{|\overline{A} \cap B|} \cdot (1 - \beta_A) + \beta_A. \text{ (Using Lemma 2)} \end{aligned}$$

□

Lemma 4.

$$\underline{\mathbf{Q}}^{-1} \quad = \quad 0.5^{|\mathcal{X}|} \cdot \underline{\mathbf{B}}. \qquad (62)$$

Proof. From [24, Corollary 6.1], $\underline{\mathbf{Q}}^{-1}$ may be obtained by Kronecker multiplication using the building block $0.5 \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The lemma follows from property $\mathbf{Kron}(k \cdot \mathbf{A}, \mathbf{B}) = \mathbf{Kron}(\mathbf{A}, k \cdot \mathbf{B}) = k \cdot \mathbf{Kron}(\mathbf{A}, \mathbf{B})$, k scalar, of Kronecker multiplication and the fact that $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is the building block of $\underline{\mathbf{B}}$. □

Lemma 5.

$$\underline{B}(B, A) \quad = \quad (-1)^{|A \cap \overline{B}|}, \quad \forall A, B \subseteq \mathcal{X}. \qquad (63)$$

Proof. Using [24, Proposition 6.7], we have $\underline{\mathbf{Q}} = \mathbf{J} \cdot \underline{\mathbf{B}} \cdot \mathbf{J}$ with $\mathbf{J}$ the square matrix, the elements of which are zeros except those on the secondary diagonal which are ones [37]. Placed before a matrix $\mathbf{M}$, matrix $\mathbf{J}$ inverses the rows of $\mathbf{M}$, which implies that $(\mathbf{J} \cdot \mathbf{M})(\overline{B}, A) = \mathbf{M}(B, A)$, $\forall A, B \subseteq \mathcal{X}$. Placed after a matrix $\mathbf{M}$, $\mathbf{J}$ inverses the columns of $\mathbf{M}$ which yields to $(\mathbf{M} \cdot \mathbf{J})(B, \overline{A}) = \mathbf{M}(B, A)$, $\forall A, B \subseteq \mathcal{X}$. Thus: $\underline{Q}(\overline{B}, \overline{A}) = \underline{B}(B, A)$, $\forall A, B \subseteq \mathcal{X}$, which, using Lemma 2, gives (63). □

Lemma 6. *For all $x \in \mathcal{X}$, for all $C \subseteq \mathcal{X}$ such that $C \neq \emptyset$ and $C \neq \{x\}$:*

$$\sum_{B \subseteq \overline{\{x\}}} (-1)^{|C \cap B|} \quad = \quad 0. \qquad (64)$$

Proof. Let $x \in \mathcal{X}$, $\mathcal{X}^\star = \mathcal{X} \setminus \{x\} = \overline{\{x\}}$ and $C^\star = C \setminus \{x\}$, $\forall C \subseteq \mathcal{X}$. We have $B \cap C = B \cap C^\star \subseteq \mathcal{X}^\star$, $\forall C \subseteq \mathcal{X}$, $\forall B \subseteq \mathcal{X}^\star$. Thus, $\forall C \subseteq \mathcal{X}$ s.t. $C \neq \emptyset$ and $C \neq \{x\}$, $\sum_{B \subseteq \overline{\{x\}}} (-1)^{|C \cap B|}$ is equal to $\sum_{B \subseteq \mathcal{X}^\star} (-1)^{|C^\star \cap B|}$ with $C^\star \subseteq \mathcal{X}^\star$ and $C^\star \neq \emptyset$. Let $m = |C^\star|$, $n = |\overline{C^\star}|$, $\mathcal{P}_{even} = \{B \subseteq \mathcal{X}^\star, |C^\star \cap B| \text{ is even}\}$, $\mathcal{P}_{odd} = \{B \subseteq \mathcal{X}^\star, |C^\star \cap B| \text{ is odd}\}$. Let us recall that there are $\binom{m}{k} 2^n$ subsets of $\mathcal{X}^\star$ with k elements in $C^\star$. We then have: $\sum_{B \subseteq \mathcal{X}^\star} (-1)^{|C^\star \cap B|} = |\mathcal{P}_{even}| - |\mathcal{P}_{odd}| = \sum_{k\, even} \binom{m}{k} 2^n - \sum_{k\, odd} \binom{m}{k} 2^n = 2^n \sum_{k=0}^{m} \binom{m}{k} (-1)^k = 0$ (Binomial theorem). □

Proposition 15 can then be proved as follows.

Proof. Let pl and $\underline{q}$ be, respectively, the plausibility and 0-commonality functions associated to MF m defined by $m = m_S \circledcap_{A \in \mathcal{A}} A^{\beta_A}$ with $\beta_A \in [0, 1]$, for all $A \in \mathcal{A}$.

For all $x \in \mathcal{X}$, we then have:

$$\begin{aligned}
pl(\{x\}) &= \sum_{A\cap\{x\}\neq\emptyset} m(A) = \sum_{x\in A}(\underline{\mathbf{Q}}^{-1}\cdot\underline{\mathbf{q}})(A) = \sum_{x\in A}(0.5^{|\mathcal{X}|}\cdot\underline{\mathbf{B}}\cdot\underline{\mathbf{q}})(A) \text{ (Using Lemma 4)}\\
&= 0.5^{|\mathcal{X}|}\cdot\sum_{x\in A}(\underline{\mathbf{B}}\cdot\underline{\mathbf{q}})(A) = 0.5^{|\mathcal{X}|}\cdot\sum_{x\in A}\sum_{C}\underline{B}(A,C)\cdot\underline{q}(C)\\
&= 0.5^{|\mathcal{X}|}\cdot\sum_{C}\underline{q}(C)\sum_{x\in A}\underline{B}(A,C) = 0.5^{|\mathcal{X}|}\cdot\sum_{C}\underline{q}(C)\sum_{x\in A}(-1)^{|C\cap\overline{A}|} \text{ (Using Lemma 5)}\\
&= 0.5^{|\mathcal{X}|}\cdot\sum_{C}\underline{q}(C)\sum_{B\subseteq\overline{\{x\}}}(-1)^{|C\cap B|}\\
&= 0.5^{|\mathcal{X}|}\cdot\left(\underline{q}(\emptyset)\sum_{B\subseteq\overline{\{x\}}}(-1)^{|\emptyset\cap B|}+\underline{q}(\{x\})\sum_{B\subseteq\overline{\{x\}}}(-1)^{|\{x\}\cap B|}+\sum_{C\neq\emptyset,\{x\}}\underline{q}(C)\sum_{B\subseteq\overline{\{x\}}}(-1)^{|C\cap B|}\right).
\end{aligned}$$

Since there are $2^{|\mathcal{X}|-1}$ subsets of $\overline{\{x\}}$, this last equation becomes:

$$\begin{aligned}
&0.5^{|\mathcal{X}|}\cdot\left(2^{|\mathcal{X}|-1}\cdot\underline{q}(\emptyset)+2^{|\mathcal{X}|-1}\cdot\underline{q}(\{x\})+\sum_{C\neq\emptyset,\{x\}}\underline{q}(C)\sum_{B\subseteq\overline{\{x\}}}(-1)^{|C\cap B|}\right)\\
=\;&0.5\cdot\underline{q}(\emptyset)+0.5\cdot\underline{q}(\{x\})+0.5^{|\mathcal{X}|}\cdot\sum_{C\neq\emptyset,\{x\}}\underline{q}(C)\sum_{B\subseteq\overline{\{x\}}}(-1)^{|C\cap B|}. \qquad (65)
\end{aligned}$$

Using [24, Proposition 6.5], which tells us that $\underline{q}(\emptyset) = 1$, and Lemma 6, Equation (65) reduces to:

$$pl(\{x\}) = 0.5 + 0.5\cdot\underline{q}(\{x\}).$$

Besides, using Lemma 3 and the definition of $\underline{q}$:

$$\underline{q}(B) = \underline{q}_S(B)\cdot\prod_{A\in\mathcal{A},|\overline{A}\cap B| \text{ is odd}}(2\cdot\beta_A-1), \text{ for all } B\subseteq\mathcal{X}.$$

Thus, for all $x \in \mathcal{X}$:

$$\underline{q}(\{x\}) = \underline{q}_S(\{x\})\cdot\prod_{A\in\mathcal{A},|\overline{A}\cap\{x\}| \text{ is odd}}(2\cdot\beta_A-1) = \underline{q}_S(\{x\})\cdot\prod_{A\in\mathcal{A},A\subseteq\overline{\{x\}}}(2\cdot\beta_A-1).$$

At last:

$$\begin{aligned}
\underline{q}_S(\{x\}) &= (\underline{\mathbf{Q}}\cdot\mathbf{m}_S)(\{x\}) = \sum_{A\subseteq\mathcal{X}}\underline{Q}(x,A)\cdot m_S(A)\\
&= \sum_{A\cap\{x\}\neq\emptyset}\underline{Q}(x,A)\cdot m_S(A)+\sum_{A\subseteq\overline{\{x\}}}\underline{Q}(x,A)\cdot m_S(A)\\
&= \sum_{A\cap\{x\}\neq\emptyset}(-1)^{|\overline{A}\cap\{x\}|}\cdot m_S(A)+\sum_{A\subseteq\overline{\{x\}}}(-1)^{|\overline{A}\cap\{x\}|}\cdot m_S(A) \text{ (Using Lemma 2)}\\
&= \sum_{A\cap\{x\}\neq\emptyset}m_S(A)-\sum_{A\subseteq\overline{\{x\}}}m_S(A)\\
&= pl_S(\{x\})-b_S(\overline{\{x\}})\\
&= 2\cdot pl_S(\{x\})-1,
\end{aligned}$$

which completes the proof. □

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