Several shades of conflict☆

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Abstract

Recently, the measurement of conflict between belief functions has been given an axiomatic foundation by Destercke and Burger, resulting in conflict being measured as the inconsistency of their conjunctive combination and excluding metric distances as suitable candidates for conflict measures. The contribution of this paper is twofold. First, we define a parameterised family of consistency measures which encompasses three existing definitions of consistency of a belief function. An induced family of conflict measures between belief functions is then derived, and each of these conflict measures is shown to satisfy the previously proposed axiomatization. The family of conflict measures defines several shades of conflict as it encompasses the classical measure of conflict, associated to the weakest definition of consistency, as well as two other conflict measures associated, respectively, to a stronger definition of consistency by Yager and to the strongest definition of consistency by Destercke and Burger. The different measures are illustrated on a toy example of vessel destination estimation. Second, we provide a geometric view on consistency measures as well as on the associated conflict measures. In particular, we show that the consistency of a belief function (whatever the considered definition of consistency) is its distance to the belief function representing the state of total inconsistency. This geometric view is then transposed to conflict measures, shedding some new light on the relation

☆This paper is an extended and revised version of [1]. Section 4 is an extension of the results of [1] encompassing a parameterised family of conflict measures, itself presented in Section 3.

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between conflict and distances.

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1. Introduction

Intelligent systems need to be able to cope with large amount of information that often displays different dimensions of imperfection. In particular, inconsistent evidence is a challenging problem that may arise when an information source is partially reliable and provides inaccurate or aberrant information. When inconsistent information needs to be merged, a particular attention should be paid to characterising, measuring and understanding the conflict. The conflict can be used as an indicator of a lack of reliability of some source and further used to discount the corresponding piece of information in the overall fusion process or to choose the most appropriate combination rule (see, *e.g.*, [2]). Besides its importance in deriving relevant overall belief, the conflict can be used to support decision making. For instance in maritime security, inconsistency may reveal maritime anomalies such as vessels deviating from normalcy (*e.g.*, “off-route vessels”, “too fast vessels”) and those possibly spoofing the Automatic Identification System (AIS) signal to hide suspect behaviour [3, 4].

Investigating methods to characterise and measure conflict in information fusion has attracted particular attention in the setting of belief functions [5, 6]. To clarify the semantics of conflict measures for belief functions, axiomatic foundations have been recently provided in [7] and [8] and corresponding measures proposed: Martin’s measure combines a degree of inclusion with a distance, whereas Destercke and Burger measure conflict between belief functions as the inconsistency yielded by their conjunctive combination. The uniqueness of these measures with respect to their set of axioms was not addressed though. Besides, although the two sets of axioms of [7] and [8] highly overlap, they differ notably by the identity axiom present in [8] and not in [7]. Whether a conflict measure between two identical belief functions should be null or not has been discussed in several work [7, 8, 9, 10, 11], questioning the appropriateness of the classical conflict measure in belief function theory [6] to quantify conflict in all situations, and led to alternative considerations of conflict. Liu [9] proposes to complement the classical conflict measure with a distance measure, Daniel [10, 12] distinguishes between internal, external and total conflict, Destercke and Burger [7] consider knowledge about source dependence, and Burger [11] investigates the suitability of distance measures and other geometrical objects for conflict measurement.

In this paper, we follow the axiomatization proposed by Destercke and Burger [7] for conflict measures, which encompasses the classical measure of conflict, and revisit and extend some of their results. Specifically, we tackle two research questions: (1) the uniqueness of the proposed measures, and (2) whether geometrical objects may be relevant for this kind of measures of conflict.

Our starting point are the two different definitions (one being stronger than the other one) of the notion of nonconflicting belief functions proposed in [7]. We define a new parameterised family of conflict measures that captures gradual notions ("shades") of conflict. The proposed family satisfies Destercke and Burger's axiomatic approach, and interestingly, subsumes their definitions of nonconflict as extreme cases. In particular, we show that so-called strong nonconflict can be captured by a sound measure which is an alternative to the original contour-based measure proposed in [7]. We also show that this family is compatible with a geometric view and shed some new light on the relation between conflict and distances.

The study in [7] is quite general and considers the whole spectrum from unknown dependence to known dependence between the sources providing the belief functions. We do not tackle these more general situations in this paper, and consider only the case where it can be safely assumed that the sources are independent, although we note that our results can readily be extended to the more general case of known dependence.

This paper is organised as follows. Necessary concepts of belief function theory, as well as Destercke and Burger's axiomatic approach to conflict measurement, are recalled in Section 2. In Section 3, a parameterised family of conflict measures is unveiled and its special cases and properties are discussed. A geometric perspective on the proposed measures is presented in Section 4, before concluding in Section 5.

2. Preliminaries

In this section, necessary concepts of belief function theory are first recalled. Then, the axiomatic approach to conflict measurement of Destercke and Burger [7] is presented.

2.1. Belief function theory

The theory of belief functions is a framework for uncertainty modeling and reasoning. It was originally introduced by Dempster [13, 5] in the context of statistical inference, as a theory of imprecise probabilities. It was extended by Shafer [6] and then by Smets and Kennes [14] to handle subjective uncertainty related to fixed quantities. In this latter interpretation of this framework, called Transferable Belief Model (TBM) and considered in this paper, the beliefs held by an agent about the actual value $\boldsymbol{x}$ taken by a variable defined on a finite domain $\mathcal{X} = \{x_1, \ldots, x_K\}$ (called *frame*), are modeled by a so-called *mass function* defined as a mapping $m : 2^{\mathcal{X}} \rightarrow [0, 1]$ verifying $\sum_{A \subseteq \mathcal{X}} m(A) = 1$ with $m(\emptyset) \geq 0$. The mass $m(A)$ represents the amount of belief allocated to the fact of knowing only that $\boldsymbol{x} \in A$. The set of all mass functions on $\mathcal{X}$ is denoted by $\mathcal{M}$.

Subsets A of $\mathcal{X}$ such that $m(A) > 0$ are called *focal sets* of m, and the set of focal sets of m is denoted by $\mathcal{F}$. In order to simplify some expressions, the number $|\mathcal{F}|$ of focal sets of m is denoted by $\mathfrak{F}$. Several specific cases of mass functions are often distinguished. A mass function m is called:

- *categorical* if $m(A) = 1$ for some $A \subseteq \mathcal{X}$, in which case it defines a classical set and will be denoted by m_A in the following;
- *vacuous* if $m(\mathcal{X}) = 1$ and denoted by $m_{\mathcal{X}}$. It represents total ignorance;
- *empty* if $m(\emptyset) = 1$ and denoted by $m_{\emptyset}$. It represents total inconsistency in the agent's beliefs about the set of values that are conceivable for $\boldsymbol{x}$ [15];
- *normalised* if $m(\emptyset) = 0$.

Equivalent representations of a mass function m are the *plausibility function* and the *belief function*, defined respectively as, for all $A \subseteq \mathcal{X}$,

$$pl(A) = \sum_{B \cap A \neq \emptyset} m(B), \tag{1}$$

and

$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B). \tag{2}$$

That is, $pl(A)$ is the amount of belief consistent with $\boldsymbol{x} \in A$, and $bel(A)$ is the amount of belief implying $\boldsymbol{x} \in A$. The plausibility function restricted to the singletons of $\mathcal{X}$ is the *contour function* $\pi : \mathcal{X} \rightarrow [0, 1]$ such that $\pi(x) = pl(\{x\})$, for all $x \in \mathcal{X}$. Due to the one-to-one correspondence between functions m, bel and pl, any of these functions may be loosely referred to as "belief function" for simplicity – it should nonetheless always be clear from the context what is meant from a technical point of view.

We note that in the imprecise probabilistic interpretation of belief function theory, the belief and plausibility measures bel and pl represent boundaries on an ill-known probability measure P on $\mathcal{X}$ and m is associated to the set $\mathcal{P}(m)$ of compatible probability measures defined by $\mathcal{P}(m) = \{P | \forall A \subseteq \mathcal{X}, bel(A) < P(A)\}$.

The informative contents of two mass functions can be compared using the notion of specialisation [16]: a mass function m_1 defined on $\mathcal{X}$ is said to be a *specialisation* of another mass function m_2 defined on $\mathcal{X}$, which is denoted by $m_1 \sqsubseteq m_2$, if and only if there exists a non-negative square matrix $S = [S(A,B)]$, $A, B \in 2^{\mathcal{X}}$, verifying

$$\sum_{A \subseteq \mathcal{X}} S(A,B) = 1, \quad \forall B \subseteq \mathcal{X},$$
$$S(A,B) > 0 \Rightarrow A \subseteq B, \quad A, B \subseteq \mathcal{X},$$
$$m_1(A) = \sum_{B \subseteq \mathcal{X}} S(A,B) m_2(B), \quad \forall A \subseteq \mathcal{X}.$$

The term $S(A,B)$ may be seen as the proportion of the mass $m_2(B)$ which "flows down" to A. The specialisation relation extends the relation of inclusion between classical sets. Let us also recall that we have [16]

$$m_1 \sqsubseteq m_2 \Rightarrow pl_1(A) \leq pl_2(A), \quad \forall A \subseteq \mathcal{X}. \tag{3}$$

Another useful notion is that of *refined* mass function. Recall that a refinement of a space $\mathcal{X}$ to a space $\mathcal{Y}$ is formally defined as a function $\rho : 2^{\mathcal{X}} \to 2^{\mathcal{Y}}$ such that the set $\{\rho(\{x\}) | x \in \mathcal{X}\}$ is a partition of $\mathcal{Y}$, and $\rho(A) = \bigcup_{x \in A} \rho(\{x\}), \forall A \subseteq \mathcal{X}$. If ρ is a refinement function from $\mathcal{X}$ into $\mathcal{Y}$, the refined mass function $\rho(m_i)$, denoted by $m_{i\rho}$ for short, of some mass function m_i is defined such that for any $A \in \mathcal{F}_i$, with $\mathcal{F}_i$ the set of focal sets of m_i, we have $m_i(A) = m_{i\rho}(\rho(A))$.

The TBM is appealing because it makes it possible to combine multiple pieces of information about a variable. One of the most classical combination rule of the theory is Dempster's unnormalised rule [13], also known as *conjunctive rule*. Let m_1 and m_2 be two mass functions representing pieces of information about $\boldsymbol{x}$. Their combination by the conjunctive rule, denoted by $\textcircled{\cap}$, results in the mass function $m_{1\textcircled{\cap}2}$ defined by, for all $A \subseteq \mathcal{X}$,

$$m_{1\textcircled{\cap}2}(A) \quad = \sum_{B \cap C = A} m_1(B) m_2(C). \tag{4}$$

This combination is appropriate when m_1 and m_2 have been provided by two independent and reliable sources. Furthermore, the rule $\textcircled{\cap}$ is commutative and associative, and admits the vacuous mass function $m_{\mathcal{X}}$ as neutral element.

In the TBM, conditioning of a mass function m by some $B \subseteq \mathcal{X}$ is equivalent to conjunctive combination of m with the categorical mass function m_B. The result is denoted by $m[B]$, with $m[B] = m \textcircled{\cap} m_B$. The conjunctive rule admits a simple expression using conditioning:

$$m_{1\textcircled{\cap}2}(A) = \sum_{B \subseteq \mathcal{X}} m_1(B) m_2[B](A), \forall A \subseteq \mathcal{X}. \quad (5)$$

2.2. Axiomatic approach to conflict measurement

The axiomatic approach to conflict measurement of [7] relies on the notion of consistency of a mass function, which is recalled first.

2.2.1. Consistency of a mass function

Although a *totally inconsistent* information state is uniquely represented as the empty mass function $m_\emptyset$ [7, 15], *total consistency* of an information state represented by a mass function can be understood differently. Specifically, two different definitions of total consistency are considered in [7]:

Definition 1 (Logical consistency [7]). *A mass function m is* logically consistent *iff* $\bigcap_{A \in \mathcal{F}} A \neq \emptyset$.

Definition 2 (Probabilistic consistency [7]). *A mass function m is* probabilistically consistent *iff* $m(\emptyset) = 0$ *(*i.e.*, m is normalised).*

Logical consistency of m corresponds to logical consistency between the focal sets of m, hence its name. Probabilistic consistency takes its name from the fact that $m(\emptyset) = 0$ is equivalent to $\mathcal{P}(m) \neq \emptyset$. We note that a logically consistent mass function is normalised and thus is also probabilistically consistent. Hence, logical consistency is a stronger form of consistency than probabilistic consistency.

Based on these notions of total inconsistency and total consistency, two properties that a measure of consistency ϕ of a mass function should obey have been defined in [7]:

Property 1 (Bounded [7]). *A measure of consistency should be bounded,* i.e.*, possess minimal and maximal values.*

Property 2 (Extreme consistent values [7]). *A measure of consistency should reach its maximal value if and only if information is totally consistent (according to the considered definition), and its minimal value if and only if information is totally inconsistent.*

Property 2 depends thus on the definition of total consistency considered.

Remarking that $\bigcap_{A\in\mathcal{F}} A \neq \emptyset \Leftrightarrow \exists x \in \mathcal{X}$ s.t. $\pi(x) = 1$ [7, Lemma 1], two consistency measures ϕ_π and ϕ_m from $\mathcal{M}$ to $[0,1]$ are proposed in [7]:

$$\begin{aligned} \phi_\pi(m) &= \max_{x\in\mathcal{X}} \pi(x), \\ \phi_m(m) &= 1 - m(\emptyset). \end{aligned}$$

The measure $\phi_\pi(m)$ has been proposed originally in [10] to quantify the internal conflict of a belief function. As detailed in [7], measure ϕ_π satisfies Property 1 and Property 2, in the case where total consistency is understood according to Definition 1. Measure ϕ_m on the other hand satisfies Property 1 and Property 2, in the case where total consistency is understood according to Definition 2. Furthermore, we have $\phi_m(m) \geq \phi_\pi(m)$ for any $m \in \mathcal{M}$ [7, Lemma 2].

Measure ϕ_π agrees with the TBM interpretation of belief functions since a mass function is totally logically consistent if and only if it considers that at least one value in $\mathcal{X}$ is totally plausible [7]. Furthermore, it is argued in [7] that ϕ_m is in accordance with the imprecise probabilistic interpretation of belief functions since a mass function is totally probabilistically consistent if and only if its associated set $\mathcal{P}(m)$ is not empty. Yet, as reported in [7, Section VII.A], the correlation between ϕ_m and ϕ_π is high enough in some cases, to consider ϕ_m as a good approximation of ϕ_π.

Despite Destercke and Burger [7] position about measure ϕ_m, we remark that this measure has been justified by Smets [15] as a measure of consistency *in the TBM interpretation* of belief functions, using an argument based on the notion of belief updating. Smets concludes in [15] that ϕ_m quantifies the amount of consistency present in the mass function representing the agent's beliefs about the set of values $\mathcal{X}$ that are conceivable for $\boldsymbol{x}$ by the agent. This can simply be seen by remarking that $pl(\mathcal{X}) = 1 - m(\emptyset) = \phi_m(m)$, that is $\phi_m(m)$ is the amount of belief consistent with the proposition that the actual value $\boldsymbol{x}$ is in $\mathcal{X}$ (we also have $bel(\mathcal{X}) = \phi_m(m)$). In particular, we have $m(\emptyset) = 0 \Leftrightarrow pl(\mathcal{X}) = 1$ so that we could also name the notion of probabilistic consistency as *frame consistency* (in contrast, $\phi_\pi(m)$ is the maximum amount of belief consistent with $\boldsymbol{x} = x$ for some $x \in \mathcal{X}$, and logical consistency corresponds thus to consistency with at least one value in $\mathcal{X}$). This view on ϕ_m is totally in line with the so-called open-world assumption of the TBM, where $m(\emptyset)$ quantifies the belief that $\boldsymbol{x}$ does not lie in $\mathcal{X}$. Overall, ϕ_m and ϕ_π seem thus both relevant in the TBM interpretation of belief functions (ϕ_m being in addition relevant in the imprecise probabilistic interpretation).

Let us finally remark that besides ϕ_π and ϕ_m, another measure of consistency of a mass function has been proposed by Yager in [17]:

$$\phi_Y(m) = \sum_{A\cap B\neq\emptyset} m(A)m(B). \tag{6}$$

This measure reaches its maximum, that is a mass function m is considered totally consistent according to Yager, if and only if the focal sets of m have a pairwise non-empty intersection. This alternative view on total consistency will be called in this paper *pairwise consistency*:

Definition 3 (Pairwise consistency)**.** *A mass function m is* pairwise consistent *iff* $\forall(A,B)\in\mathcal{F}^2, A\cap B\neq\emptyset$.

It is easy to check that measure ϕ_Y satisfies Property 1 and Property 2, in the case where total consistency is understood according to Definition 3.

2.2.2. Conflict between mass functions

As rightfully remarked in [7], two mass functions can be considered as totally conflicting if none of their focal sets intersect. Formally:

Definition 4 (Total conflict [7])**.** *Let m_1 and m_2 be two mass functions with sets of focal sets $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. Let $\mathcal{D}_i = \cup_{A\in\mathcal{F}_i} A$ denote the disjunction*[1] *of all focal sets of m_i. m_1 and m_2 are* totally conflicting *when $\mathcal{D}_1\cap\mathcal{D}_2=\emptyset$.*

However, there does not seem to be a unique way to define totally nonconflicting mass functions. As a matter of fact, two[2] definitions of nonconflicting mass functions are considered in [7]:

Definition 5 (Strong nonconflict [7])**.** *Two mass functions m_1 and m_2 are said to be* strongly nonconflicting *if and only if*

$$\bigcap_{A\in\{\mathcal{F}_1\cup\mathcal{F}_2\}} A\neq\emptyset.$$

Definition 6 (nonconflict [7])**.** *m_1 and m_2 are said to be* nonconflicting *if and only if $\forall(A,B)$ such that $A\in\mathcal{F}_1, B\in\mathcal{F}_2$, we have $A\cap B\neq\emptyset$.*

[1] The disjunction of all focal sets is called the *core* of the belief function by Shafer [6].

[2] In [7], three definitions of nonconflicting mass functions are considered (Definitions 4, 5 and 6 in [7]). However, in the case of independent sources, which is assumed in this paper, Definitions 5 and 6 in [7] are equivalent.

Definition 5 requires that all focal sets of m_1 and m_2 have a non-empty intersection, which is stronger than requiring each focal set of m_1 to have a non-empty intersection with each focal set of m_2 as required by Definition 6.

Based on these notions of totally conflicting and totally nonconflicting mass functions, five properties that a measure of conflict κ between two mass functions defined on $\mathcal{X}$ and provided by two independent sources, should satisfy are provided in [7]. The five properties are the following (further details on the motivations of these requirements can be found in [7]):

Property 3 (Extreme conflict values [7]). *$\kappa(m_1, m_2) = 0$ if and only if m_1 and m_2 are nonconflicting (according to the considered definition) and $\kappa(m_1, m_2) = 1$ if and only if m_1 and m_2 are totally conflicting.*

Property 4 (Symmetry [7]). *$\kappa(m_1, m_2) = \kappa(m_2, m_1)$.*

Property 5 (Imprecision monotonicity [7]). *If $m_1 \sqsubseteq m_{1'}$, then $\kappa(m_1, m_2) \geq \kappa(m_{1'}, m_2)$.*

Property 6 (Ignorance is bliss [7]). *If m_2 is vacuous, then $\kappa(m_2, m_1) = 1 - \phi(m_1)$.*

Property 7 (Insensitivity to refinement [7]). *If ρ is a refinement function from $\mathcal{X}$ into $\mathcal{Y}$, then $\kappa(m_1, m_2) = \kappa(m_{1\rho}, m_{2\rho})$.*

As Property 2 for consistency measures, which depends on the definition of total consistency considered, Property 3 for conflict measures depends on the definition of nonconflict considered (*i.e.*, either Definition 5 or Definition 6). Property 5 states that the conflict should not increase as the imprecision (defined in terms of specialisation) of a mass function increases. Property 6 states that a state of ignorance should not conflict with any other state of information represented by a mass function m_1, whilst accounting for the fact that this state of information m_1 may be partially inconsistent itself – we note however that this property does not specify which kind of consistency (*e.g.*, logical or probabilistic) should consistency measure ϕ satisfy. Property 7 states that the refinement of a mass function should not change its conflict value with any other mass function refined in the same way.

In [7], the authors propose to evaluate the conflict between two mass functions as the inconsistency of their conjunctive combination, where inconsistency is the "inverse" of consistency (*i.e.*, $1 - \phi(\cdot)$. More precisely, they propose two measures of conflict satisfying Properties 3-7, induced by consistency measures ϕ_π and ϕ_m:

$$\begin{aligned}
\kappa_\pi(m_1, m_2) &= 1 - \phi_\pi(m_{1\textcircled{\cap}2}) = 1 - \max_{x \in \mathcal{X}} \pi_{1\textcircled{\cap}2}(x), \\
\kappa_m(m_1, m_2) &= 1 - \phi_m(m_{1\textcircled{\cap}2}) = m_{1\textcircled{\cap}2}(\emptyset).
\end{aligned}$$

Measure κ_π satisfies Properties 3-7, when nonconflict in Prop. 3 is understood in terms of Definition 5, whereas κ_m satisfies these properties when nonconflict is understood in terms of Definition 6. We note that with κ_π, Prop. 6 is satisfied for $\phi = \phi_\pi$, whereas with κ_m this property is satisfied for $\phi = \phi_m$. Let us finally stress that $\kappa_m(m_1, m_2)$ is nothing but the classical measure of conflict $m_{1 \textcircled{\cap} 2}(\emptyset)$ in the TBM.

3. *N*-consistency and induced conflict

The previous section has reviewed the axiomatic approach to conflict measurement proposed in [7]. It was recalled that the measures κ_π and κ_m satisfy this approach. In particular, measure κ_π, which relies on the measure ϕ_π of logical consistency, is suitable as a measure for strong nonconflict. In this section, it is shown that there exists an alternative measure of logical consistency (*i.e.*, a measure ϕ satisfying Property 2 for Definition 1) and that its induced conflict measure (defined as the inconsistency of the conjunctive combination) is an alternative measure for strong nonconflict (*i.e.*, a measure κ satisfying Property 3 for Definition 5). More generally, a parameterised family of consistency measures capturing different forms of consistency including logical, pairwise and probabilistic consistency is introduced and is shown to induce a family of conflict measures satisfying the axiomatic approach of [7] for a family of definitions of the notion of nonconflicting mass functions subsuming strong nonconflict (Definition 5) and nonconflict (Definition 6).

3.1. N-consistency

Let us start by introducing the following definition:

Definition 7 (N-consistency)**.** *A mass function m is said to be* consistent of order N *(*N-consistent *for short), with $1 \leq N \leq \mathfrak{F}$, iff its focal sets are N-wise consistent,* i.e., *$\forall \mathcal{F}' \subseteq \mathcal{F}$ s.t. $|\mathcal{F}'| = N$, we have*

$$\bigcap_{A \in \mathcal{F}'} A \neq \emptyset.$$

In addition, let ϕ_N denote the measure from $\mathcal{M}$ to $[0, 1]$ such that, for $1 \leq N \leq \mathfrak{F}$ and all $m \in \mathcal{M}$,

$$\phi_N(m) \quad := \quad 1 - m^N(\emptyset), \tag{7}$$

where m^N denotes the mass function resulting from the combination of m by itself N times, *i.e.*

$$m^N \quad := \quad m^{N-1} \textcircled{\cap} m,$$

with $m^0 := m_{\mathcal{X}}$. Hence, we have $m^1 = m$, $m^2 = m \textcircled{\cap} m$ and more generally $m^N = \textcircled{\cap}_1^N m$. Note that $m^N(\emptyset)$, *i.e.*, the mass associated to the empty set after N combinations of m by itself, is called *auto-conflict of order* N of mass function m in [18].

We will show in the following that the family ϕ_N measures gradual notions ("shades") of consistency of m as N varies and in particular encompasses the three forms of consistency already defined in the literature and recalled in Section 2.2.1.

3.2. 1-consistency

Let us first remark that probabilistic consistency (Definition 2) of a mass function m is nothing but 1-consistency, since we have

$$m(\emptyset) = 0 \quad \Leftrightarrow \quad A \neq \emptyset, \quad \forall A \in \mathcal{F}.$$

In addition, we have $\phi_m(m) = \phi_1(m)$, for all $m \in \mathcal{M}$, since $\phi_1(m) = 1 - m(\emptyset)$, and thus $\phi_1(m)$ is a measure of probabilistic consistency (or 1-consistency), in the sense that it satisfies Properties 1 and 2 in the case where total consistency is understood according to Definition 2.

Let $\kappa_1(m_1, m_2) := 1 - \phi_1(m_{1\textcircled{\cap}2})$, for all $m_1, m_2 \in \mathcal{M}$. We have $\kappa_1(m_1, m_2) = \kappa_m(m_1, m_2)$, for all $m_1, m_2 \in \mathcal{M}$, and κ_1 is thus a measure for nonconflict (Definition 6) in the sense that it verifies Properties 3-7, when nonconflict in Prop. 3 is understood in terms of Definition 6,.

It will be useful to remark that the notion of nonconflicting mass functions (Definition 6) can be equivalently presented as follows. Let m_1 and m_2 be any two mass functions and let $\mathcal{F}_{12} := \{A \cap B | A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$. It is clear that: m_1 and m_2 are nonconflicting (Definition 6) $\Leftrightarrow A \neq \emptyset, \forall A \in \mathcal{F}_{12}$. In other words, nonconflict is equivalent to each set in $\mathcal{F}_{12}$ being non-empty, *i.e.*, to the sets in $\mathcal{F}_{12}$ being "1-wise" consistent. Let us note that $\mathcal{F}_{12} = \mathcal{F}_{1\textcircled{\cap}2}$, with $\mathcal{F}_{1\textcircled{\cap}2}$ the set of focal sets of $m_{1\textcircled{\cap}2}$.

3.3. $\mathfrak{F}$-consistency

Lemma 1 below shows that the notion of logical consistency of a mass function (Definition 1) is a particular case of that of N-consistency, when N is the number of focal sets of m.

Lemma 1. *m is logically consistent if and only if m is $\mathfrak{F}$-consistent.*

Proof. m is $\mathfrak{F}$-consistent $\Leftrightarrow \forall \mathcal{F}' \subseteq \mathcal{F}$ s.t. $|\mathcal{F}'| = \mathfrak{F}$, we have $\bigcap_{A \in \mathcal{F}'} A \neq \emptyset$. There is only one set $\mathcal{F}'$ s.t. $|\mathcal{F}'| = \mathfrak{F}$, which is $\mathcal{F}' = \mathcal{F}$. Hence m is $\mathfrak{F}$-consistent $\Leftrightarrow \bigcap_{A \in \mathcal{F}} A \neq \emptyset$. □

Moreover, the following results hold for any mass function $m \in \mathcal{M}$:

Lemma 2. *m is $\mathfrak{F}$-consistent if and only if $m^{\mathfrak{F}}(\emptyset) = 0$.*

Proof. Follows from $m^{\mathfrak{F}}(\emptyset) = \sum_{\cap_{i=1}^{\mathfrak{F}} A_i = \emptyset} \prod_{i=1}^{\mathfrak{F}} m(A_i)$. □

Lemma 3. *m is totally inconsistent if and only if $m^{\mathfrak{F}}(\emptyset) = 1$.*

Proof. $\Rightarrow$: $m(\emptyset) = 1$, *i.e.*, m is totally inconsistent, implies clearly $m^{\mathfrak{F}}(\emptyset) = 1$.

$\Leftarrow$: assume this is not true, *i.e.*, $m^{\mathfrak{F}}(\emptyset) = 1$ and $m(\emptyset) \neq 1$. We reach a contradiction since $m(\emptyset) \neq 1$ implies that $\exists A \subseteq \mathcal{X}, A \neq \emptyset$, s.t. $m(A) > 0$, in which case we have $m^{\mathfrak{F}}(A) \geq (m(A))^{\mathfrak{F}} > 0$, and thus $m^{\mathfrak{F}}(\emptyset) \neq 1$. □

Lemmas 1, 2 and 3 suggest to use $\phi_{\mathfrak{F}}(m) = 1 - m^{\mathfrak{F}}(\emptyset)$ as an alternative measure of logical consistency of a mass function m. Indeed, these results show that similarly to ϕ_π, $\phi_{\mathfrak{F}}$ verifies Property 1 and Property 2, in the case where total consistency is understood according to Definition 1 (logical consistency). This measure appears thus as justified as ϕ_π to evaluate logical consistency of a mass function.

Remark 1. *We note that the measures ϕ_π and $\phi_{\mathfrak{F}}$ are not equal. For instance, denoting by*

$$m : (A_1, m(A_1); \ldots ; A_{\mathfrak{F}}, m(A_{\mathfrak{F}}))$$

a mass function m with $\mathfrak{F}$ focal elements A_i with associated masses $m(A_i)$, then if m is defined on $\mathcal{X} = \{d_1, d_2, d_3\}$ by

$$m : (\{d_1, d_2\}, 0.8; \{d_3\}, 0.2)\,,$$

we have $\phi_\pi(m) = \max_{x \in \mathcal{X}} \pi(x)$ =0.8 $\neq \phi_{\mathfrak{F}}(m) = 1 - m^2(\emptyset)$ =0.68 (since $\mathfrak{F} = 2$). Further details about the relationships between these measures will be given in Section 3.6.

The measure $\kappa_\pi(m_1, m_2)$ associated to the notion of strong nonconflict (Definition 5) is derived from $\phi_\pi(m_{1 \circledcap 2})$, which is a measure of logical consistency of $m_{1 \circledcap 2}$. Let $\mathfrak{F}_{12}$ denote the cardinality of $\mathcal{F}_{12}$, *i.e.*, $\mathfrak{F}_{12} = |\mathcal{F}_{12}|$. Then $\phi_{\mathfrak{F}_{12}}(m_{1 \circledcap 2})$ is also a measure of logical consistency of $m_{1 \circledcap 2}$. This prompts us to consider the following conflict measure from $\mathcal{M} \times \mathcal{M}$ to $[0, 1]$:

$$\kappa_{\mathfrak{F}_{12}}(m_1, m_2) := 1 - \phi_{\mathfrak{F}_{12}}(m_{1 \circledcap 2}). \tag{8}$$

Proposition 1. *Measure $\kappa_{\mathfrak{F}_{12}}$ satisfies Properties 3-7, when nonconflict in Property 3 is understood in terms of Definition 5 (strong nonconflict).*

Proof. We show each property in turn:

- Property 3 (Extreme conflict values): $\kappa_{\mathfrak{F}_{12}}(m_1, m_2) = 0 \Leftrightarrow \phi_{\mathfrak{F}_{12}}(m_{1 \textcircled{\cap} 2}) = 1 \Leftrightarrow m^{\mathfrak{F}_{12}}_{1 \textcircled{\cap} 2}(\emptyset) = 0$, which is equivalent using Lemma 2 to $m_{1 \textcircled{\cap} 2}$ is $\mathfrak{F}_{12}$-consistent, which in turn is equivalent using Lemma 1 to $m_{1 \textcircled{\cap} 2}$ is logically consistent, *i.e.*, $\cap_{A \in \mathcal{F}_{12}} A \neq \emptyset$ or equivalently $\exists x \in \mathcal{X}$ s.t. $x \in A$, $\forall A \in \mathcal{F}_{12}$. From the definition of $\textcircled{\cap}$, we have $\exists x \in \mathcal{X}$ s.t. $x \in A$, $\forall A \in \mathcal{F}_{12}$ iff m_1 and m_2 are strongly nonconflicting.

  $\kappa_{\mathfrak{F}_{12}}(m_1, m_2) = 1 \Leftrightarrow \phi_{\mathfrak{F}_{12}}(m_{1 \textcircled{\cap} 2}) = 0 \Leftrightarrow m^{\mathfrak{F}_{12}}_{1 \textcircled{\cap} 2}(\emptyset) = 1$, which using Lemma 3 is equivalent to $m_{1 \textcircled{\cap} 2}$ is totally inconsistent, *i.e.*, $m_{1 \textcircled{\cap} 2}(\emptyset) = 1$. From the definition of $\textcircled{\cap}$, $m_{1 \textcircled{\cap} 2}(\emptyset) = 1$ iff m_1 and m_2 are totally conflicting.

- Property 4 (Symmetry): We have

$$\begin{aligned}
\kappa_{\mathfrak{F}_{12}}(m_1, m_2) &= m^{\mathfrak{F}_{12}}_{1 \textcircled{\cap} 2}(\emptyset) \\
&= (\textcircled{\cap}^{\mathfrak{F}_{12}}_{i=1} m_{1 \textcircled{\cap} 2})(\emptyset) \\
&= (\textcircled{\cap}^{\mathfrak{F}_{12}}_{i=1} (m_1 \textcircled{\cap} m_2))(\emptyset) \\
&= (\textcircled{\cap}^{\mathfrak{F}_{12}}_{i=1} (m_2 \textcircled{\cap} m_1))(\emptyset) \\
&= (\textcircled{\cap}^{\mathfrak{F}_{12}}_{i=1} m_{2 \textcircled{\cap} 1})(\emptyset) \\
&= m^{\mathfrak{F}_{12}}_{2 \textcircled{\cap} 1}(\emptyset) \\
&= \kappa_{\mathfrak{F}_{12}}(m_2, m_1).
\end{aligned}$$

- Property 5 (Imprecision monotonicity): $\textcircled{\cap}$ is monotonic with respect to $\sqsubseteq$ [19, Proposition 2] and thus $m_1 \sqsubseteq m_{1'} \Rightarrow m_1 \textcircled{\cap} m_2 \sqsubseteq m_{1'} \textcircled{\cap} m_2, \forall m_2 \in \mathcal{M}$. Using [2, Lemma 3], we obtain

$$\textcircled{\cap}^{\mathfrak{F}_{12}}_{i=1} (m_1 \textcircled{\cap} m_2) \sqsubseteq \textcircled{\cap}^{\mathfrak{F}_{12}}_{i=1} (m_{1'} \textcircled{\cap} m_2).$$

  That is $m^{\mathfrak{F}_{12}}_{1 \textcircled{\cap} 2} \sqsubseteq m^{\mathfrak{F}_{12}}_{1' \textcircled{\cap} 2}$, which implies using (3) that

$$\begin{aligned}
pl^{\mathfrak{F}_{12}}_{1 \textcircled{\cap} 2}(\mathcal{X}) &\leq pl^{\mathfrak{F}_{12}}_{1' \textcircled{\cap} 2}(\mathcal{X}) \\
\Leftrightarrow 1 - m^{\mathfrak{F}_{12}}_{1 \textcircled{\cap} 2}(\emptyset) &\leq 1 - m^{\mathfrak{F}_{12}}_{1' \textcircled{\cap} 2}(\emptyset) \\
\Leftrightarrow \kappa_{\mathfrak{F}_{12}}(m_1, m_2) &\geq \kappa_{\mathfrak{F}_{12}}(m_{1'}, m_2).
\end{aligned}$$

- Property 6 (Ignorance is bliss): $m_2(\mathcal{X}) = 1$ means that $m_2 = m_{\mathcal{X}}$. It is thus the neutral element of $\textcircled{\cap}$ and we have $\mathfrak{F}_{12} = \mathfrak{F}_1$, from which we obtain

$$
\begin{aligned}
\kappa_{\mathfrak{F}_{12}}(m_1, m_2) &= m_{1 \bigcirc\!\!\!\!\cap\, 2}^{\mathfrak{F}_{12}}(\emptyset) \\
&= (\bigcirc\!\!\!\!\cap_{i=1}^{\mathfrak{F}_{12}} m_{1 \bigcirc\!\!\!\!\cap\, 2})(\emptyset) \\
&= (\bigcirc\!\!\!\!\cap_{i=1}^{\mathfrak{F}_{12}} m_1)(\emptyset) \\
&= m_1^{\mathfrak{F}_{12}}(\emptyset) \\
&= m_1^{\mathfrak{F}_1}(\emptyset) \\
&= 1 - \phi_{\mathfrak{F}_1}(m_1).
\end{aligned}
$$

- Property 7 (Insensitivity to refinement): We have

$$
\begin{aligned}
\kappa_{\mathfrak{F}_{12}}(m_1, m_2) &= m_{1 \bigcirc\!\!\!\!\cap\, 2}^{\mathfrak{F}_{12}}(\emptyset) \\
&= \sum_{\cap_{i=1}^{\mathfrak{F}_{12}}(A_i \cap B_i)=\emptyset} \prod_{i=1}^{\mathfrak{F}_{12}} m_1(A_i) m_2(B_i),
\end{aligned}
$$

  and

$$
\begin{aligned}
\kappa_{\mathfrak{F}_{12}}(m_{1\rho}, m_{2\rho}) &= m_{1\rho \bigcirc\!\!\!\!\cap\, 2\rho}^{\mathfrak{F}_{12}}(\emptyset) \\
&= \sum_{\cap_{i=1}^{\mathfrak{F}_{12}}(\rho(A_i) \cap \rho(B_i))=\emptyset} \prod_{i=1}^{\mathfrak{F}_{12}} m_{1\rho}(\rho(A_i)) m_{2\rho}(\rho(B_i)).
\end{aligned}
$$

  $\forall (A_1, B_1, \ldots, A_{\mathfrak{F}_{12}}, B_{\mathfrak{F}_{12}}) \in \times_{i=1}^{\mathfrak{F}_{12}} \mathcal{F}_1 \times \mathcal{F}_2$, we have:

  - either $\cap_{i=1}^{\mathfrak{F}_{12}}(\rho(A_i) \cap \rho(B_i)) \neq \emptyset$, in which case since $\{\rho(\{x\}) | x \in \mathcal{X}\}$ is a partition of $\mathcal{Y}$, $\exists x \in \mathcal{X}$ s.t. $x \in A_i$ and $x \in B_i$, $i = 1, \ldots, \mathfrak{F}_{12}$, *i.e.*, $\cap_{i=1}^{\mathfrak{F}_{12}}(A_i \cap B_i) \neq \emptyset$;
  - or $\cap_{i=1}^{\mathfrak{F}_{12}}(\rho(A_i) \cap \rho(B_i)) = \emptyset$, in which case $\nexists x \in \mathcal{X}$ s.t. $x \in A_i$ and $x \in B_i$, $i = 1, \ldots, \mathfrak{F}_{12}$, *i.e.*, $\cap_{i=1}^{\mathfrak{F}_{12}}(A_i \cap B_i) = \emptyset$.

  Hence, $\forall (A_1, B_1, \ldots, A_{\mathfrak{F}_{12}}, B_{\mathfrak{F}_{12}})$, when the mass $\prod_{i=1}^{\mathfrak{F}_{12}} m_{1\rho}(\rho(A_i)) m_{2\rho}(\rho(B_i))$ is allocated to $\emptyset$, so does the mass $\prod_{i=1}^{\mathfrak{F}_{12}} m_1(A_i) m_2(B_i)$, and when the mass $\prod_{i=1}^{\mathfrak{F}_{12}} m_{1\rho}(\rho(A_i)) m_{2\rho}(\rho(B_i))$ is not allocated to $\emptyset$, so does the mass $\prod_{i=1}^{\mathfrak{F}_{12}} m_1(A_i) m_2(B_i)$. Property 7 is then obtained since

$$
\prod_{i=1}^{\mathfrak{F}_{12}} m_{1\rho}(\rho(A_i)) m_{2\rho}(\rho(B_i)) = \prod_{i=1}^{\mathfrak{F}_{12}} m_1(A_i) m_2(B_i),
$$

  $\forall (A_1, B_1, \ldots, A_{\mathfrak{F}_{12}}, B_{\mathfrak{F}_{12}})$.

□

In the proof of Property 6 of Proposition 1, we remark that if m_2 is vacuous then $\kappa_{\mathfrak{F}_{12}}(m_1, m_2) = 1 - \phi_{\mathfrak{F}_1}(m_1)$, *i.e.*, the consistency of m_1 is evaluated using measure $\phi_{\mathfrak{F}_1}$, which is a measure of the logical consistency of m_1. In other words, we have a similar behaviour to that of κ_π, which satisfies Prop.6 for $\phi = \phi_\pi$ that is another measure of the logical consistency of m_1.

According to Proposition 1, measure $\kappa_{\mathfrak{F}_{12}}(m_1, m_2)$, which is nothing but the auto-conflict of order $\mathfrak{F}_{12}$ of the mass function $m_{1 \circledcap 2}$, constitutes thus a sound alternative to measure $\kappa_\pi(m_1, m_2)$ for evaluating the (strong) conflict between two mass functions m_1 and m_2.

Let us finally note that we have the following equivalence: m_1 and m_2 are strongly nonconflicting (Definition 5) $\Leftrightarrow \bigcap_{A \in \mathcal{F}_{12}} A \neq \emptyset$. In other words, strong nonconflict is equivalent to the intersection of all sets in $\mathcal{F}_{12}$ being non-empty, *i.e.*, to the sets in $\mathcal{F}_{12}$ being $\mathfrak{F}_{12}$-wise consistent.

3.4. 2-consistency

It is clear that pairwise consistency (Definition 3) of a mass function m is equivalent to m being 2-consistent.

Furthermore, the measure ϕ_Y of consistency proposed by [17] (see Eq. (6)) is obtained with $N = 2$ in (7):

$$\phi_Y(m) = \phi_2(m) = 1 - m^2(\emptyset), \quad \forall m \in \mathcal{M}.$$

Hence, ϕ_2 is a measure of pairwise consistency of m, since it satisfies Properties 1 and 2 in the case where total consistency is understood according to Definition 3.

We established that probabilistic consistency of a mass function m is equivalent to m being 1-consistent and that logical consistency of m is equivalent to m being $\mathfrak{F}$-consistent. Hence, pairwise consistency of m can be situated in between the two extremes that are probabilistic consistency and logical consistency:

- probabilistic consistency is the weakest form – it requires focal sets of m to be "onewise" consistent, *i.e.*, each focal set is non-empty;
- then comes pairwise consistency – focals sets need to be pairwise consistent, *i.e.*, the intersection of any two focal sets must be non-empty;
- and logical consistency is the strongest form – focal sets must be $\mathfrak{F}$-consistent, *i.e.*, the intersection of all focal sets must be non-empty.

Similarly, we have seen that nonconflict (Definition 6) is equivalent to each set in $\mathcal{F}_{12}$ being non-empty, whereas strong nonconflict is equivalent to the intersection of all sets in $\mathcal{F}_{12}$ being non-empty. Considering the above discussion about the three possible definitions of total consistency, nonconflict and strong nonconflict appear thus to be two extreme forms of nonconflict, and Yager's definition of consistency suggests then an alternative definition of nonconflict:

Definition 8. *m_1 and m_2 are said to be* nonconflicting of order 2 *(2-nonconflicting for short) if and only if $\forall A \in \mathcal{F}_{12}, B \in \mathcal{F}_{12}$, we have $A \cap B \neq \emptyset$.*

Let us denote by κ_2 the measure of conflict induced by ϕ_2, *i.e.*, the measure $\kappa_2(m_1, m_2) : \mathcal{M} \times \mathcal{M} \rightarrow [0,1]$ defined by

$$\kappa_2(m_1, m_2) \ := \ 1 - \phi_2(m_{1 \circledcap 2}).$$

Proposition 2. *Measure κ_2 satisfies Properties 3-7, when nonconflict in Property 3 is understood in terms of Definition 8 (2-nonconflict).*

Proof. We only show Property 3 (the other properties can be shown using a similar proof to that of Proposition 1 – for Property 6, we obtain $\kappa_2(m_1, m_2) = 1 - \phi_2(m_2)$):

$\kappa_2(m_1, m_2) = 0 \Leftrightarrow \phi_2(m_{1 \circledcap 2}) = 1 \Leftrightarrow m^2_{1 \circledcap 2}(\emptyset) = 0 \Leftrightarrow m_{1 \circledcap 2}$ is 2-consistent $\Leftrightarrow \forall A \in \mathcal{F}_{12}, B \in \mathcal{F}_{12}$, we have $A \cap B \neq \emptyset$.

$\kappa_2(m_1, m_2) = 1 \Leftrightarrow m^2_{1 \circledcap 2}(\emptyset) = 1$, which, using a similar proof to that of Lemma 3, is equivalent to $m_{1 \circledcap 2}$ is totally inconsistent, *i.e.*, $m_{1 \circledcap 2}(\emptyset) = 1$. From the definition of $\circledcap$, $m_{1 \circledcap 2}(\emptyset) = 1$ iff m_1 and m_2 are totally conflicting. □

According to Proposition 2, $\kappa_2(m_1, m_2)$ is thus a measure for 2-nonconflict.

3.5. A family of conflict measures

The three definitions of nonconflict encountered so far suggest the following generalisation, obtained by applying the notion of N-consistency to the sets in $\mathcal{F}_{12}$:

Definition 9. *m_1 and m_2 are said to be* nonconflicting of order N *(N-nonconflicting for short), with $1 \leq N \leq \mathfrak{F}_{12}$, if and only if $\forall A_i \in \mathcal{F}_{12}, i = 1, \ldots, N$, we have $\bigcap_{i=1}^{N} A_i \neq \emptyset$.*

N-nonconflict subsumes obviously 2-nonconflict (Definition 8), which is recovered for $N = 2$, but also nonconflict (Definition 6, recovered for N=1) and strong nonconflict (Definition 5, recovered for $N = \mathfrak{F}_{12}$).

$\phi(m)$				$\kappa(m_1,m_2)=1-\phi(m_{1\circledcap 2})$			
Total inconsistency				**Total conflict**			
$m(\emptyset)=1$				$\mathcal{D}_1\cap\mathcal{D}_2=\emptyset$			
Total consistency				**Total nonconflict**			
Probabilistic consistency	$\forall A\in\mathcal{F}$, $A\neq\emptyset$	$\phi_1=\phi_m$ [7]	1-consistency	Nonconflict	$\forall A\in\mathcal{F}_{12}$, $A\neq\emptyset$	$\kappa_1=\kappa_m$ [7]	1-nonconflict
Pairwise consistency	$\forall(A,B)\in\mathcal{F}^2$, $A\cap B\neq\emptyset$	ϕ_2 [17]	2-consistency	Pairwise nonconflict	$\forall(A,B)\in\mathcal{F}_{12}^2$, $A\cap B\neq\emptyset$	κ_2	2-nonconflict
Logical consistency	$\bigcap_{A\in\mathcal{F}} A\neq\emptyset$	$\phi_{\mathfrak{F}}$ ϕ_π [7]	$\mathfrak{F}$-consistency	Strong nonconflict	$\bigcap_{A\in\mathcal{F}_{12}} A\neq\emptyset$	$\kappa_{\mathfrak{F}_{12}}$ κ_π [7]	$\mathfrak{F}_{12}$-nonconflict

Table 1: Notions of consistency and associated nonconflict.

Let $\kappa_N(m_1,m_2):\mathcal{M}\times\mathcal{M}\to[0,1]$ be the measure defined, for $1\leq N\leq\mathfrak{F}_{12}$, by

$$\kappa_N(m_1,m_2) \;:=\; 1-\phi_N(m_{1\circledcap 2}). \tag{9}$$

Proposition 3. *Measure κ_N satisfies Properties 3-7, when nonconflict in Property 3 is understood in terms of Definition 9 (N-nonconflict).*

Proof. The proof is similar to that of Proposition 2. □

Proposition 3 is a generalisation of Propositions 1 and 2. The parameterised family of conflict measures κ_N includes obviously measures $\kappa_{\mathfrak{F}_{12}}$ and κ_2, but also the classical measure of conflict in the TBM, *i.e.*, κ_m, which is equivalent to κ_1.

Remark 2. *Let $m_1=m_A$ and $m_2=m_B$, for some $A,B\subseteq\mathcal{X}$. In this case, we have $\mathfrak{F}_{12}=1$ and thus the family of conflict measures κ_N reduces to the measure κ_1. Besides, if A and B are consistent,* i.e., *$A\cap B\neq\emptyset$, then $\kappa_1(m_1,m_2)=0$, and if A and B are inconsistent,* i.e., *$A\cap B=\emptyset$, then $\kappa_1(m_1,m_2)=1$. In other words, if m_1 and m_2 are categorical, then the family of conflict measures κ_N reduces to classical inconsistency assessment between sets.*

Table 1 summarises the notions of consistency and associated conflict together with the associated definitions and measures. The measures defined so far in the literature are mentioned with the proper reference, while the ones proposed in this paper are not assigned any specific reference. Only three values of N are considered for consistency (1, 2 and $\mathfrak{F}$) and for conflict (1, 2 and $\mathfrak{F}_{12}$) in Table 1.

Lemma 4. $\forall m_1 \in \mathcal{M}, \forall m_2 \in \mathcal{M}$, *we have* $\kappa_{N-1}(m_1, m_2) \leq \kappa_N(m_1, m_2)$.

Proof. Follows from the fact that auto-conflict verifies $m^{N-1}(\emptyset) \leq m^N(\emptyset)$ [18]. □

Lemma 4 shows that as N increases, the stronger the conflict measure becomes. In particular, $\kappa_{\mathfrak{F}_{12}}$ is the strongest conflict measure, while κ_1 is the weakest. An equivalent relationship can straightforwardly be derived between consistency measures ϕ_N.

The following Section 3.6 further studies the relative behaviours of the measures considered so far.

3.6. Correlation analysis

In this section, we conduct some correlation analysis of the different measures, the new ϕ_N family for $N = 1, \ldots, \mathfrak{F}$ (including $\phi_m = \phi_1$ and $\phi_Y = \phi_2$), the existing logical consistency measure ϕ_π and their counter-part conflict measures, to illustrate and better grasp their relative behaviours. In particular, the experiments show that the measures capture different shades of consistency (and conflict). We use the Spearman coefficient which is a rank correlation coefficient quantifying how much two measures are monotonically correlated. It reaches the value 1 if the two measures are linked through some increasing monotonic function.

3.6.1. Conflict measures κ_N vs κ_π

We first study how the family of conflict measures κ_N behaves relatively to the existing (strong) conflict measure κ_π, using a similar experiment to that in [7, Section VII.A].

For a given size of the frame of discernment $\mathcal{X}$, we have drawn randomly (following Algorithm 3 of [20]) 5000 couples of normalised mass functions (m_1, m_2) having $2^{|\mathcal{X}|} - 1$ focal sets . Table 2 shows the Spearman correlation between κ_π and κ_N for different values of N and of $|\mathcal{X}|$. The first line of Table 2 corresponds to the first line of Table II of [7, Section VII.A] with very similar values, while the other lines differ in the sense that our N does not correspond to the number of sources combined as in [7]. We can observe that for $|\mathcal{X}| = 2$ and $|\mathcal{X}| = 3$, for which we have $\mathfrak{F}_{12} = 4$ and $\mathfrak{F}_{12} = 8$ respectively, the correlation between $\kappa_{\mathfrak{F}_{12}}$ and κ_π is really high (0.9895 and 0.9941, respectively).

In the experiment of [7, Section VII.A], it was observed that the correlation between κ_1 and κ_π is relatively high, and that this correlation increases as the cardinality of the frame decreases and as the number of sources to be

		Size of the frame of discernment $|\mathcal{X}|$					
		2	3	4	5	6	7
N	1	0.8467	0.7747	0.7268	0.6624	0.6116	0.5551
	2	0.9535	0.8960	0.8451	0.7772	0.7187	0.6616
	3	0.9796	0.9452	0.9024	0.8395	0.7713	0.7064
	4	0.9895	0.9689	0.9373	0.8853	0.8168	0.7477
	5	-	0.9811	0.9585	0.9175	0.8546	0.7851
	6	-	0.9877	0.9716	0.9395	0.8847	0.8178
	7	-	0.9916	0.9798	0.9548	0.9081	0.8457
	8	-	0.9941	0.9852	0.9655	0.9260	0.8694

Table 2: Spearman correlation between κ_π and κ_N according to $|\mathcal{X}|$ and $N \leq 8$.

combined increases. In our experiment, a similar observation can be made: the correlation between κ_N and κ_π increases as $|\mathcal{X}|$ decreases and as N increases, for the values of $|\mathcal{X}|$ and N considered here. This latter result can be better observed in Figure 1, which displays the scatter plots for the specific case of $|\mathcal{X}| = 3$, whose corresponding correlation coefficients can be found in column "$|\mathcal{X}| = 3$" of Table 2.

Our experiment shows that while κ_1 and κ_π are quite correlated as both are measures of conflict, the family of measures κ_N captures different shades of conflict and the correlation with κ_π increases with N. However, it shows that the measures $\kappa_{\mathfrak{F}_{12}}$ and κ_π for strong nonconflict do not have a maximal correlation and hence are different (a fact already known thanks to Remark 1).

3.6.2. Consistency measures ϕ_N

Here we study the correlation between ϕ_N and ϕ_π on the one hand and within the family ϕ_N as N varies on the other hand, by analysing the impact of the number of focal elements $\mathfrak{F}$ as well as the size of the frame of discernment $|\mathcal{X}|$. We thus prefer Algorithm 7 to Algorithm 3, both of [20], which allows to control the number of focal elements of the mass function generated.

As one can see from Figure 2, the correlation between $\phi_\mathfrak{F}$ and ϕ_π, which are two measures of logical consistency, is high (> 0.93) whatever the size of the frame of discernment and the number of focal elements considered here. Furthermore, if we consider a particular value of $\mathfrak{F}$, for instance $\mathfrak{F} = 5$, it appears that ϕ_π is more correlated to $\phi_\mathfrak{F}$ than to any ϕ_N, for all $N < \mathfrak{F}$, as shown in Figure 3. Figure 3(a) displays how the correlation between ϕ_π and ϕ_N increases with N for different sizes of $\mathcal{X}$, while Figure 3(b) is the

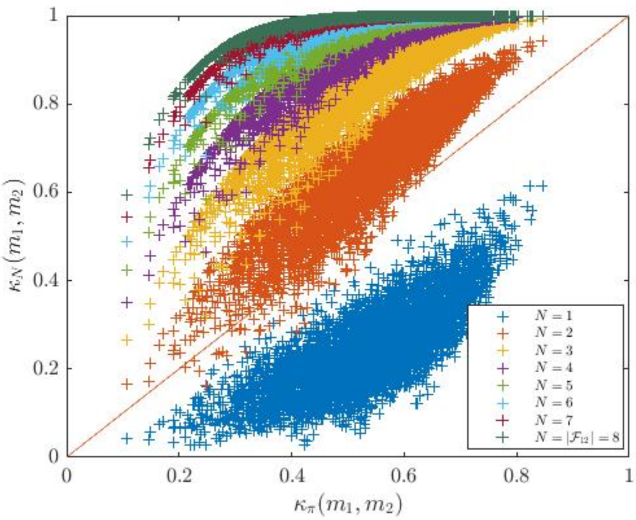


Figure 1: Scatter plots of measures κ_π and κ_N obtained from 5000 couples of randomly generated mass functions over $\mathcal{X}$ such that $|\mathcal{X}| = 3$.

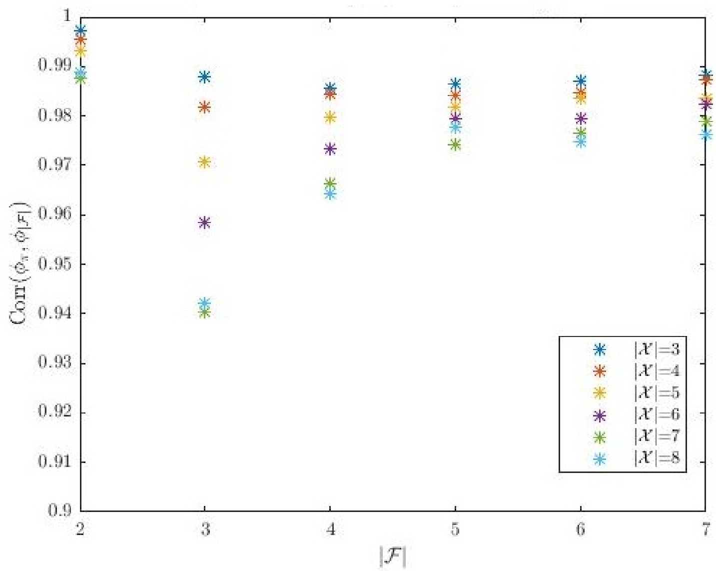
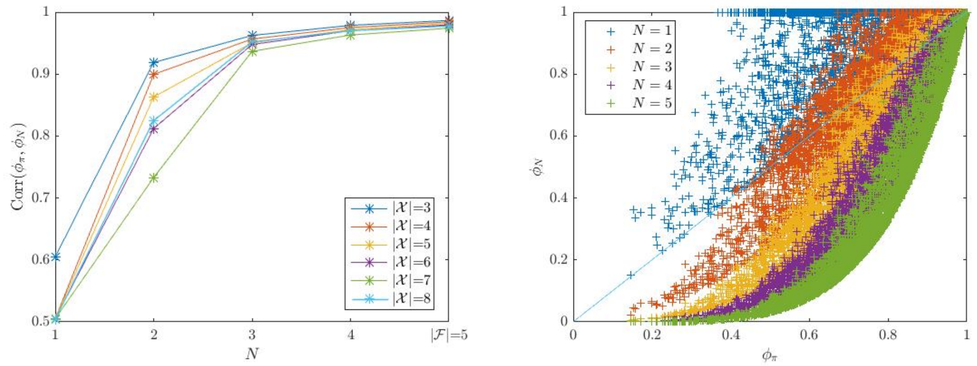


Figure 2: Spearman correlation between ϕ_π and $\phi_{\mathfrak{F}}$ for 5000 randomly generated mass functions.

corresponding scatter plot for the specific case of $|\mathcal{X}| = 4$.



(a) Spearman correlation between ϕ_π and ϕ_N, N varying, for several $|\mathcal{X}|$.

(b) Scatter plots of ϕ_π and ϕ_N, $|\mathcal{X}| = 4$.

Figure 3: Correlation between ϕ_π and ϕ_N, 5000 randomly generated mass functions over $\mathcal{X}$ with $\mathfrak{F} = 5$ focal sets.

We conclude this correlation analysis section with the behaviour of the ϕ_N measures between themselves. Table 3 reports the pairwise correlation coefficients between ϕ_N and ϕ_M, for $1 \leq N, M \leq \mathfrak{F} = 5$ and $|\mathcal{X}| = 4$. We

		N				
		1	2	3	4	5
M	1	1.0000	0.6459	0.5869	0.5591	0.5438
	2	-	1.0000	0.9829	0.9653	0.9520
	3	-	-	1.0000	0.9965	0.9910
	4	-	-	-	1.0000	0.9987
	5	-	-	-	-	1.0000

Table 3: Correlation between ϕ_N and ϕ_M for $1 \leq N, M \leq \mathfrak{F} = 5$ and $|\mathcal{X}| = 4$.

observe that the correlation between successive pairs of measures (ϕ_N, ϕ_{N+1}) increases as N increases. Additionally, we observe that ϕ_N is less correlated to ϕ_{N+2} than to ϕ_{N+1}.

The scatter plots of Figure 4 illustrate the correlation coefficients corresponding to the second diagonal of Table 3. For a given m, the measures ϕ_N are less correlated for low values of N.

We illustrated that the family of consistency measures ϕ_N allows to capture different shades of consistency, and thus that the derived family of conflict measures κ_N allows to capture different shades of conflict between two mass functions m_1 and m_2. In practice, the family of measures κ_N offers

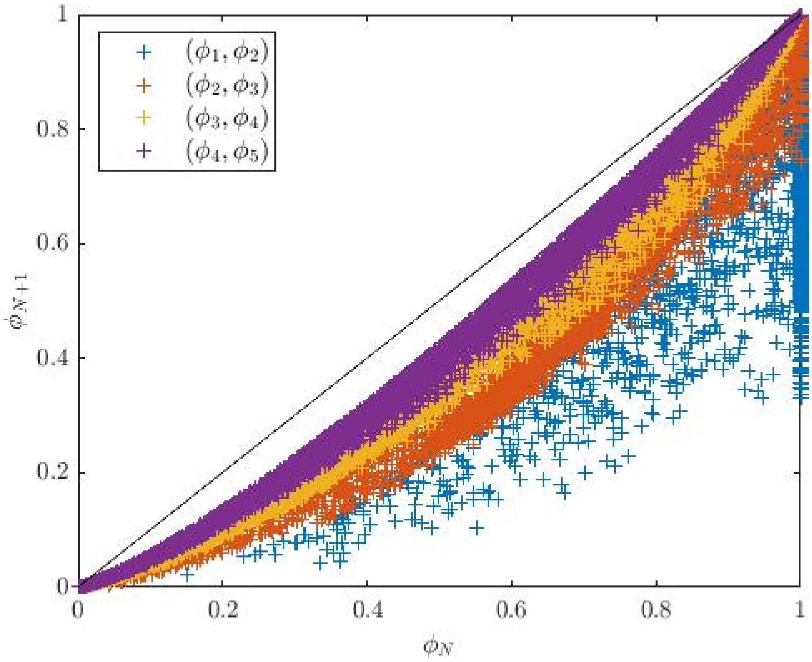


Figure 4: Scatter plots of ϕ_N and ϕ_{N+1} for 5000 random mass functions such that $|\mathcal{X}| = 4$ and $\mathfrak{F} = 5$.

alternatives to evaluate the validity of the result of the combination by the conjunctive rule $\textcircled{\cap}$, as it will be illustrated in Section 3.7.

3.7. Toy example

We illustrate some of the consistency and conflict measures, as well as some of their relationships, described previously, on a problem of vessel destination estimation and associated anomaly detection, in which we would like to estimate a vessel's destination while detecting any inconsistency given different sources of information.

Let $\mathcal{X} = \{d_1, d_2, d_3, d_4\} = \{\text{Savona}, \text{Genova}, \text{La Spezia}, \text{Livorno}\}$ denote the set of possible destinations for the vessel.

We consider a source S_1, as an algorithm analysing the kinematic features of the vessel, compared to some pre-extracted patterns-of-life as maritime routes [21]. Once computed, the routes provide contextual information describing some normalcy against which the current positions of vessels can be compared. In particular, assigning a vessel to a route provides information about its final destination [3]. However, because the routes share some portions of the sea, some indeterminacy may occur and a subset of possible destinations may rather be deduced. Let m_1 be the mass function that encodes the information provided by the source S_1:

$$m_1 : (\{d_1, d_2, d_3\}, 0.6; \{d_1, d_2, d_4\}, 0.3; \{d_3, d_4\}, 0.1).$$

We have:

$$\begin{aligned} m_1^2 : \quad & (\{d_1, d_2, d_3\}, 0.36; \{d_1, d_2, d_4\}, 0.09; \{d_3, d_4\}, 0.01; \\ & \{d_1, d_2\}, 0.36; \{d_4\}, 0.06; \{d_3\}, 0.12). \end{aligned}$$

and

$$\begin{aligned} m_1^3 : \quad & (\{d_1, d_2, d_3\}, 0.216; \{d_1, d_2, d_4\}, 0.027; \{d_3, d_4\}, 0.001; \\ & \{d_1, d_2\}, 0.486; \{d_4\}, 0.036; \{d_3\}, 0.126; \emptyset, 0.108). \end{aligned}$$

The mass function m_1 is 1-consistent (probabilistically consistent) and 2-consistent (pairwise consistent), but not 3-consistent (*i.e.*, not logical consistent since $\mathfrak{F}_1 = 3$). Indeed:

$$\begin{cases} \phi_1(m_1) = \phi_2(m_1) = 1, \\ \phi_3(m_1) = 1 - m_1^3(\emptyset) = 0.892. \end{cases} \tag{10}$$

Consider now a second source S_2 which provides a piece of evidence encoded by the following mass function, corresponding to the subjective assessment of a human operator, knowing the maritime traffic of the area and excluding Genova (d_2) as a possible destination for that vessel, but considering as more probable the destination of Savona (d_1) or Livorno (d_4) than that of La Spezia (d_3):

$$m_2 : (\{d_1, d_3, d_4\}, 0.7; \{d_1, d_4\}, 0.3)$$

The conjunctive combination of m_1 and m_2 has $\mathfrak{F}_{12} = 5$ focal sets:

$$m_{1 \text{\textcircled{$\cap$}} 2} : (\{d_1, d_3\}, 0.42; \{d_1, d_4\}, 0.3; \{d_3, d_4\}, 0.07; \{d_1\}, 0.18; \{d_4\}, 0.03)$$

The mass functions provided by the sources S_1 and S_2 are 1-nonconflicting since $\kappa_1(m_1, m_2) = 0$. However, they are not nonconflicting of order 2 (and consequently higher orders) since some focal sets of $m_{1 \text{\textcircled{$\cap$}} 2}$ are pairwise disjoint (*e.g*, $\{d_3, d_4\} \cap \{d_1\} = \emptyset$):

$$\begin{cases} \kappa_1(m_1, m_2) = 1 - \phi_1(m_{1 \text{\textcircled{$\cap$}} 2}) = m^1_{1 \text{\textcircled{$\cap$}} 2}(\emptyset) = m_{1 \text{\textcircled{$\cap$}} 2}(\emptyset) = 0, \\ \kappa_2(m_1, m_2) = 1 - \phi_2(m_{1 \text{\textcircled{$\cap$}} 2}) = m^2_{1 \text{\textcircled{$\cap$}} 2}(\emptyset) = 0.0612, \\ \kappa_3(m_1, m_2) = 1 - \phi_3(m_{1 \text{\textcircled{$\cap$}} 2}) = m^3_{1 \text{\textcircled{$\cap$}} 2}(\emptyset) = 0.1908, \\ \kappa_4(m_1, m_2) = 1 - \phi_4(m_{1 \text{\textcircled{$\cap$}} 2}) = m^4_{1 \text{\textcircled{$\cap$}} 2}(\emptyset) = 0.2999, \\ \kappa_5(m_1, m_2) = \kappa_{\mathfrak{F}_{12}}(m_1, m_2) = 1 - \phi_{\mathfrak{F}_{12}}(m_{1 \text{\textcircled{$\cap$}} 2}) = m^{\mathfrak{F}_{12}}_{1 \text{\textcircled{$\cap$}} 2}(\emptyset) = 0.3865. \end{cases} \tag{11}$$

Note that, as stated in Lemma 4, the conflict does not decrease with N.

The (1-)nonconflict of the mass functions is a valid justification for considering the result of their conjunctive combination for further reasoning [22] and based on some standard decision procedure (such as the pignistic transform [23]), d_1=Savona would be a sensible estimated destination.

We note also that while κ_1 is null, κ_2 is very small (0.0612), and $\kappa_{\mathfrak{F}_{12}}$, the measure for strong nonconflict, is quite higher (0.3865). A criterion for combination based on $\kappa_{\mathfrak{F}_{12}}$ instead of κ_1 would lead then possibly to the decision to not combine the sources.

4. A norm-based view on conflict

Section 3 has brought to light the existence of a family of conflict measures, respecting Destercke and Burger's axiomatization of conflict quantification and encompassing Yager's definition of consistency.

As argued in [10, 7] and further developed in [11], distances between belief functions [24] are questionable to measure their conflict; for instance, they do not satisfy Properties 5 (imprecision monotonicity) and 7 (insensitivity to refinement). Still, this does not mean that geometrical objects are not relevant to conflict quantification as will be shown in this section.

After recalling in Section 4.1 necessary material on norms and distances, we lay bare in Section 4.2 a pseudo-norm based view on consistency measures, which leads us to investigate the relationship between the consistency of a mass function and its distance to the state of total inconsistency. Then, this geometric view with respect to consistency measures is carried over to conflict measures in Section 4.3.

4.1. Norms and distances

Any vector $\boldsymbol{v}$ of the Cartesian space $\mathbb{R}^N$ spanned by the set of vectors $\{\boldsymbol{e}_i, 1 \leq i \leq N\}$, can be written as $\boldsymbol{v} = \sum_{1 \leq i \leq N} v_i \boldsymbol{e}_i$, with $v_i \in \mathbb{R}$ the coordinate of $\boldsymbol{v}$ along dimension $\boldsymbol{e}_i$.

By definition, a *norm* is a function $n : \mathbb{R}^N \to [0, \infty)$ that satisfies the following properties:

(n.1) Definiteness: $n(\boldsymbol{v}) = 0 \Leftrightarrow \boldsymbol{v} = \boldsymbol{0}$ (*i.e.*, $v_i = 0$, $i = 1, \ldots, N$),
(n.2) Homogeneity: $n(b\boldsymbol{v}) = |b| n(\boldsymbol{v})$, $\forall b \in \mathbb{R}$, $\forall \boldsymbol{v} \in \mathbb{R}^N$,
(n.3) Subadditivity (triangle inequality): $n(\boldsymbol{v}_1 + \boldsymbol{v}_2) \leq n(\boldsymbol{v}_1) + n(\boldsymbol{v}_2)$, $\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^N$.

A *pseudo-norm* corresponds to changing condition (n.1) in the definition of the norm to $n(\boldsymbol{v}) = 0 \Leftarrow \boldsymbol{v} = \boldsymbol{0}$ [25].

The function

$$n_{\boldsymbol{w}}^{p}(\boldsymbol{v}) := (\sum_i w_i |v_i|^p)^{1/p}, \tag{12}$$

with $p \geq 1$ and where, for $i = 1, \ldots, N$, finite $w_i > 0$, is an example of a norm; if we allow some w_i to equal zero, then $n_{\boldsymbol{w}}^{p}$ is a pseudo-norm [25]. The weights w_i actually distort the space of reference by increasing or reducing the importance of some dimensions. The (pseudo-)norm $n_{\boldsymbol{w}}^{1}$ will be more simply denoted by $n_{\boldsymbol{w}}$ and we have $n_{\boldsymbol{w}}(\boldsymbol{v}) = \sum_i w_i |v_i|$. Note also that $n_{\boldsymbol{w}}^{\infty}(\boldsymbol{v}) = \max \boldsymbol{v}$, $\forall \boldsymbol{w}$ such that $\sum_i w_i = 1$ [26]. As a consequence n^{∞} will denote any norm $n_{\boldsymbol{w}}^{\infty}$ with $\boldsymbol{w}$ such that $\sum_i w_i = 1$.

A metric or distance function (called simply *distance*) is a function $d : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ that satisfies the following properties:

(d.1) Definiteness: $d(\boldsymbol{v}_1, \boldsymbol{v}_2) = 0 \Leftrightarrow \boldsymbol{v}_1 = \boldsymbol{v}_2$,
(d.2) Symmetry: $d(\boldsymbol{v}_1, \boldsymbol{v}_2) = d(\boldsymbol{v}_2, \boldsymbol{v}_1)$, $\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^N$,
(d.3) Triangle inequality: $d(\boldsymbol{v}_1, \boldsymbol{v}_2) \leq d(\boldsymbol{v}_1, \boldsymbol{v}_3) + d(\boldsymbol{v}_2, \boldsymbol{v}_3)$, $\forall \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^N$.

A *pseudo-distance* corresponds to changing condition (d.1) in the definition of the distance to $d(\boldsymbol{v}_1, \boldsymbol{v}_2) = 0 \Leftarrow \boldsymbol{v}_1 = \boldsymbol{v}_2$ [25]. If n is a norm (resp. pseudo-norm) then $d_n(\boldsymbol{v}_1, \boldsymbol{v}_2) := n(\boldsymbol{v}_1 - \boldsymbol{v}_2)$ is a distance (resp. pseudo-distance) and is called the induced distance (resp. pseudo-distance) from n. The distance (resp. pseudo-distance) induced by the norm (resp. pseudo-norm) $n_{\boldsymbol{w}}^{p}$ will be denoted by $d_{\boldsymbol{w}}^{p}$; $d_{\boldsymbol{w}}^{1}$ will be more simply denoted by $d_{\boldsymbol{w}}$ and the distance induced by n^{∞} will be denoted by d^{∞}.

4.2. Consistency as distance to total inconsistency

Let $\mathcal{E}_{\mathcal{X}}$ denote the Cartesian space $\mathbb{R}^{2^K}$ spanned by the set of vectors $\{\boldsymbol{e}_A, A \subseteq \mathcal{X}\}$. Any vector $\boldsymbol{v}$ of $\mathcal{E}_{\mathcal{X}}$ can then be written as $\boldsymbol{v} = \sum_{A \subseteq \mathcal{X}} v_A \boldsymbol{e}_A$, with $v_A \in \mathbb{R}$ the coordinate of $\boldsymbol{v}$ along dimension $\boldsymbol{e}_A$. A mass function m may then be represented as the vector $\boldsymbol{m}$ of $\mathcal{E}_{\mathcal{X}}$ such that $v_A = m(A)$. Similarly, a plausibility function pl may be represented by the vector $\boldsymbol{pl} = \sum_{A \subseteq \mathcal{X}} pl(A) \boldsymbol{e}_A$, with its plausibility values $pl(A)$ as coordinates of $\boldsymbol{pl}$. The vector associated to m_A is denoted by $\boldsymbol{m}_A$. Special cases are the empty mass function $\boldsymbol{m}_{\emptyset}$ and the vacuous mass function $\boldsymbol{m}_{\mathcal{X}}$.

Let $k : 2^{\mathcal{X}} \to [0, 1]$ be the function defined by $k(A) := 1 - pl(A)$, for all $A \subseteq \mathcal{X}$. Function k may be represented by the vector $\boldsymbol{k} = \sum_{A \subseteq \mathcal{X}} k(A) \boldsymbol{e}_A$ of $\mathcal{E}_{\mathcal{X}}$. Lemma 5 below links function k with other well-known quantities.

Lemma 5. *For all $A \subseteq \mathcal{X}$, we have*

$$\begin{aligned} k(A) &= \kappa_1(m, m_A) && (13) \\ &= m[A](\emptyset). && (14) \end{aligned}$$

Proof. For all $A \subseteq \mathcal{X}$, we have

$$\begin{aligned} 1 - pl(A) &= 1 - \sum_{B \cap A \neq \emptyset} m(B) \\ &= \sum_{B \cap A = \emptyset} m(B) \qquad (15) \\ &= (m_A \bigcirc\!\!\!\!\cap\, m)(\emptyset). \end{aligned}$$

Besides, $\kappa_1(m, m_A) = (m_A \bigcirc\!\!\!\!\cap\, m)(\emptyset)$ and $m[A](\emptyset) = (m_A \bigcirc\!\!\!\!\cap\, m)(\emptyset)$ by definition. □

From (13), $k(A)$ is the amount of conflict, according to κ_1, between m and $\boldsymbol{x} \in A$. From (15), $k(A)$ can also be interpreted as the amount of belief inconsistent with $\boldsymbol{x} \in A$. Hereafter, we will refer to k as the *inconsistency* function associated to m. k is in one-to-one correspondence with m.

We are ready to show the first result of this section:

Proposition 4. *For all $m \in \mathcal{M}$ and $1 \leq N \leq \mathfrak{F}$, we have*

$$\phi_N(m) = n_{\boldsymbol{m}^{N-1}}(\boldsymbol{pl}), \qquad (16)$$

$$m^N(\emptyset) = n_{\boldsymbol{m}^{N-1}}(\boldsymbol{k}). \qquad (17)$$

Proof. We have $m^N = m^{N-1} \bigcirc\!\!\!\!\cap\, m$, which using (5) yields

$$\begin{aligned} m^N(\emptyset) &= \sum_{A \subseteq \mathcal{X}} m^{N-1}(A) m[A](\emptyset) \\ &= \sum_{A \subseteq \mathcal{X}} m^{N-1}(A) k(A) \\ &= n_{\boldsymbol{m}^{N-1}}(\boldsymbol{k}), \end{aligned}$$

and

$$\begin{aligned} \phi_N(m) &= 1 - m^N(\emptyset) \\ &= 1 - \sum_{A \subseteq \mathcal{X}} m^{N-1}(A) k(A) \\ &= 1 - \sum_{A \subseteq \mathcal{X}} m^{N-1}(A)(1 - pl(A)) \\ &= \sum_{A \subseteq \mathcal{X}} m^{N-1}(A) pl(A) \\ &= n_{\boldsymbol{m}^{N-1}}(\boldsymbol{pl}). \end{aligned}$$

□

Proposition 4 shows that the N-consistency of a mass function amounts to the $n_{\boldsymbol{m}^{N-1}}$ pseudo-norm of the vector $\boldsymbol{pl}$ and its N-inconsistency amounts to the same pseudo-norm but of the vector $\boldsymbol{k}$. Note that $n_{\boldsymbol{m}^{N-1}}$ is in general a pseudo-norm, and not a norm, since it is possible that $m^{N-1}(A) = 0$ for some $A \subseteq \mathcal{X}$.

Corollary 1. *From $\boldsymbol{k} = \boldsymbol{1} - \boldsymbol{pl}$ and $\phi_N(m) = 1 - m^N(\emptyset)$, we also have the following equalities:*

$$\begin{aligned} \phi_N(m) &= 1 - n_{\boldsymbol{m}^{N-1}}(\boldsymbol{k}) \\ &= n_{\boldsymbol{m}^{N-1}}(\boldsymbol{1} - \boldsymbol{k}), \end{aligned}$$

and

$$\begin{aligned} m^N(\emptyset) &= 1 - n_{\boldsymbol{m}^{N-1}}(\boldsymbol{pl}) \\ &= n_{\boldsymbol{m}^{N-1}}(\boldsymbol{1} - \boldsymbol{pl}). \end{aligned} \tag{18}$$

Corollary 1 shows that it is equivalent to compute the pseudo-norm of the inverse of the inconsistency function (resp. plausibility function), and to compute the inverse of the pseudo-norm of the inconsistency function (resp. plausibility function). Briefly, the pseudo-norm and inverse commute.

Proposition 4 leads to the main result of this section, which expresses the consistency measures ϕ_N in terms of pseudo-distances to the state of total inconsistency:

Proposition 5. *For all $m \in \mathcal{M}$ and $1 \leq N \leq \mathfrak{F}$, we have*

$$\phi_N(m) = d_{\boldsymbol{m}^{N-1}}(\boldsymbol{pl}, \boldsymbol{pl}_{\emptyset}). \tag{19}$$

Proof. Eq. (19) follows from (16) and $\boldsymbol{pl}_{\emptyset} = \boldsymbol{0}$. □

In short, Proposition (5) shows that the N-consistency of a mass function is nothing but its pseudo-distance, induced from $n_{\boldsymbol{m}^{N-1}}$, to the totally inconsistent knowledge state. Accordingly, the farther a mass function from total inconsistency, the more consistent it is, which makes sense. It may be worth noting that this proposition holds because $\boldsymbol{pl}_{\emptyset}$ is a null vector in space $\mathcal{E}_{\mathcal{X}}$, and coincides thus with the origin of this space.

Particular cases $N = 1, 2, \mathfrak{F}$, of Proposition (5) yield

$$\phi_1(m) = d_{\boldsymbol{m}^0}(\boldsymbol{pl}, \boldsymbol{pl}_{\emptyset}) = d_{\boldsymbol{m}_{\mathcal{X}}}(\boldsymbol{pl}, \boldsymbol{pl}_{\emptyset}), \tag{20}$$

$$\phi_2(m) = d_{\boldsymbol{m}^1}(\boldsymbol{pl}, \boldsymbol{pl}_{\emptyset}) = d_{\boldsymbol{m}}(\boldsymbol{pl}, \boldsymbol{pl}_{\emptyset}), \tag{21}$$

$$\phi_{\mathfrak{F}}(m) = d_{\boldsymbol{m}^{\mathfrak{F}-1}}(\boldsymbol{pl}, \boldsymbol{pl}_{\emptyset}). \tag{22}$$

Example 1. *Let us illustrate Eqs.* (20)-(22) *with mass function* m_1 *from Section 3.7,* i.e.,

$$m_1 : (\{d_1, d_2, d_3\}, 0.6; \{d_1, d_2, d_4\}, 0.3; \{d_3, d_4\}, 0.1).$$

This mass function has $\mathfrak{F}_1 = 3$ *focal sets and we have seen (Eq.* (10)*) that* $\phi_1(m_1) = \phi_2(m_1) = 1$ *and* $\phi_{\mathfrak{F}_1}(m_1) = \phi_3(m_1) = 0.892$.

Using (20)-(22)*, we find, respectively:*

$$\begin{aligned}
\phi_1(m_1) &= d_{\boldsymbol{m}_{\mathcal{X}}}(\boldsymbol{pl}_1, \boldsymbol{pl}_\emptyset) \\
&= n_{\boldsymbol{m}_{\mathcal{X}}}(\boldsymbol{pl}_1) \\
&= \sum_{A \subseteq \mathcal{X}} m_{\mathcal{X}}(A) pl_1(A) = m_{\mathcal{X}}(\mathcal{X}) pl_1(\mathcal{X}) = 1,
\end{aligned}$$

$$\begin{aligned}
\phi_2(m_1) &= d_{\boldsymbol{m}_1}(\boldsymbol{pl}_1, \boldsymbol{pl}_\emptyset) \\
&= n_{\boldsymbol{m}_1}(\boldsymbol{pl}_1) \\
&= \sum_{A \subseteq \mathcal{X}} m_1(A) pl_1(A) \\
&= m_1(\{d_1, d_2, d_3\}) pl_1(\{d_1, d_2, d_3\}) + m_1(\{d_1, d_2, d_4\}) pl_1(\{d_1, d_2, d_4\}) \\
&\quad + m_1(\{d_3, d_4\}) pl_1(\{d_3, d_4\}) \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\phi_{\mathfrak{F}_1}(m_1) &= d_{\boldsymbol{m}_1^{\mathfrak{F}_1 - 1}}(\boldsymbol{pl}_1, \boldsymbol{pl}_\emptyset) \\
&= d_{\boldsymbol{m}_1^2}(\boldsymbol{pl}_1, \boldsymbol{pl}_\emptyset) \\
&= n_{\boldsymbol{m}_1^2}(\boldsymbol{pl}_1) \\
&= \sum_{A \subseteq \mathcal{X}} m_1^2(A) pl_1(A) \\
&= m_1^2(\{d_1, d_2, d_3\}) pl_1(\{d_1, d_2, d_3\}) + m_1^2(\{d_1, d_2, d_4\}) pl_1(\{d_1, d_2, d_4\}) \\
&\quad + m_1^2(\{d_3, d_4\}) pl_1(\{d_3, d_4\}) + m_1^2(\{d_1, d_2\}) pl_1(\{d_1, d_2\}) \\
&\quad + m_1^2(\{d_4\}) pl_1(\{d_4\}) + m_1^2(\{d_3\}) pl_1(\{d_3\}) \\
&= 0.892.
\end{aligned}$$

Remark 3. *Similar results to Propositions 4 and 5 exist for* ϕ_π*. Indeed, let us denote by* $\mathcal{E}_x$ *the* K*-dimensional subspace of* $\mathcal{E}_{\mathcal{X}}$ *spanned by the set* $\{\boldsymbol{e}_x, x \in \mathcal{X}\}$ *corresponding to singletons. Then, the contour function* π *associated to a* *mass function m is represented by the vector* $\boldsymbol{\pi} = \sum_{x\in\mathcal{X}} \pi(x)\boldsymbol{e}_x$ *of* $\mathcal{E}_x$. *Since* $\phi_\pi(m) = \max_{x\in\mathcal{X}} \pi(x)$ *and* $\boldsymbol{\pi}_\emptyset$ *is a null vector in space* $\mathcal{E}_x$, *we have*

$$\begin{aligned}\phi_\pi(m) &= n^\infty(\boldsymbol{\pi}) \\ &= d^\infty(\boldsymbol{\pi}, \boldsymbol{\pi}_\emptyset). \end{aligned} \tag{23}$$

Note also that $\phi_1(m) = 1 - m(\emptyset) = pl(\mathcal{X}) = \max_{A\subseteq\mathcal{X}} pl(A)$ *and thus we have, besides the equality* (20),

$$\begin{aligned}\phi_1(m) &= n^\infty(\boldsymbol{pl}) \\ &= d^\infty(\boldsymbol{pl}, \boldsymbol{pl}_\emptyset). \end{aligned} \tag{24}$$

Hence, $\phi_\pi(m)$ *and* $\phi_1(m)$ *amount to the* n^∞ *norms of the contour and plausibility functions, respectively, as well as to the* d^∞ *distances of these functions to total inconsistency.*

Let us denote by d_ϕ the pseudo-distance associated to consistency measure ϕ, via the equalities (19) and (23) (*i.e.*, $d_{\phi_N} = d_{\boldsymbol{m}^{N-1}}$ and $d_{\phi_\pi} = d^\infty$). This section has thus shown that the consistency of a mass function amounts, for any consistency measure ϕ considered in this paper, to its pseudo-distance d_ϕ to total inconsistency.

Note that the above results suggest that measure $\phi_N^p(m) := n^p_{\boldsymbol{m}^{N-1}}(\boldsymbol{pl})$, with $p \neq 1, \infty$, *i.e.*, p different from the values considered above, might be interesting to investigate as a candidate consistency measure. This is left for further research.

4.3. Geometric perspective on conflict measures

As a natural extension of the preceding results, based on the fact that a given conflict measure is induced by a consistency measure, it is shown below how conflict measures κ_N can be expressed in terms of distances to the total inconsistency state.

Proposition 6. *For any* $m_1, m_2 \in \mathcal{M}$, *we have*

$$\begin{aligned}\kappa_N(m_1, m_2) &= 1 - n_{\boldsymbol{m}^{N-1}_{1\bigcirc\!\!\!\!\cap 2}}(\boldsymbol{pl}_{1\bigcirc\!\!\!\!\cap 2}) \\ &= n_{\boldsymbol{m}^{N-1}_{1\bigcirc\!\!\!\!\cap 2}}(\boldsymbol{k}_{1\bigcirc\!\!\!\!\cap 2})\end{aligned}$$

Proof. Follows from Eq. (9) and Proposition 4. □

Proposition 6 shows that the κ_N conflict between mass functions amounts to the $n_{\boldsymbol{m}^{N-1}_{1\bigcirc\!\!\!\!\cap 2}}$ pseudo-norm of the inconsistency function of their combination. Equivalently, it is equal to one minus the $n_{\boldsymbol{m}^{N-1}_{1\bigcirc\!\!\!\!\cap 2}}$ pseudo-norm of the plausibility function of their combination.

As the norm is simply the distance to the origin, Proposition 5 leads to the following relation between conflict and pseudo-distance:

Proposition 7. *For any* $m_1, m_2 \in \mathcal{M}$ *and* $1 \leq N \leq \mathfrak{F}_{12}$*, we have*

$$\kappa_N(m_1, m_2) \;=\; 1 - d_{\boldsymbol{m}_{1\textcircled{\cap}2}^{N-1}}(\boldsymbol{pl}_{1\textcircled{\cap}2}, \boldsymbol{pl}_{\emptyset}).$$

Special cases $N = 1, 2, \mathfrak{F}_{12}$, yield

$$\begin{aligned}\kappa_1(m_1, m_2) &= 1 - d_{\boldsymbol{m}_{\mathcal{X}}}(\boldsymbol{pl}_{1\textcircled{\cap}2}, \boldsymbol{pl}_{\emptyset}),\\ \kappa_2(m_1, m_2) &= 1 - d_{\boldsymbol{m}_{1\textcircled{\cap}2}}(\boldsymbol{pl}_{1\textcircled{\cap}2}, \boldsymbol{pl}_{\emptyset}),\\ \kappa_{\mathfrak{F}_{12}}(m_1, m_2) &= 1 - d_{\boldsymbol{m}^{\mathfrak{F}_{12}-1}}(\boldsymbol{pl}_{1\textcircled{\cap}2}, \boldsymbol{pl}_{\emptyset}).\end{aligned}$$

In particular, we note that the classical conflict measure $m_{1\textcircled{\cap}2}(\emptyset)$ between mass functions in the TBM is equal to one minus the pseudo-distance $d_{\boldsymbol{m}_{\mathcal{X}}}$ between the plausibility function of their combination and the plausibility function of the empty mass function.

Informally, Proposition 7 shows that the conflict between m_1 and m_2 amounts to one minus the pseudo-distance between their conjunctive combination and the totally inconsistent knowledge state (the counterpart to Remark 3 for conflict measure κ_π yields a similar conclusion: $\kappa_\pi(m_1, m_2) = 1 - d^\infty(\boldsymbol{\pi}_{1\textcircled{\cap}2}, \boldsymbol{\pi}_{\emptyset})$). This shows that while a distance between m_1 and m_2 is not an appropriate measure of their conflict [7, 11], distances can be used to express it.

Conversely, conflict measures can be used to express the distance between m_1 and m_2, as detailed in Remark 4 for a special case. Whether such relationships exist for other distances and conflict measures is an open question.

Remark 4. *Let the Euclidean distance between the plausibility functions be denoted* d^2 *and defined as*

$$d^2(pl_1, pl_2) := \sqrt{\sum_{A \subseteq \mathcal{X}} (pl_1(A) - pl_2(A))^2}.$$

By Lemma 5, we obtain for all $m_1, m_2 \in \mathcal{M}$*:*

$$d^2(pl_1, pl_2) \;=\; \sqrt{\sum_{A \subseteq \mathcal{X}} (\kappa_1(m_1, m_A) - \kappa_1(m_2, m_A))^2}. \tag{25}$$

As can be seen with (25)*,* $d^2(pl_1, pl_2)$ *does* not *evaluate how much* m_1 *and* m_2 *are in conflict with each other, but rather quantifies how much they are* *in conflict (according to the classical conflict measure κ_1) with the same categorical mass functions m_A (sets A), $A \subseteq \mathcal{X}$. This gives also some intuition behind the property that $d^2(pl_1, pl_2) = 0$ iff $m_1 = m_2$ whereas we can have $\kappa_N(m_1, m_2) \neq 0$ for $m_1 = m_2$: if $m_1 = m_2 = m$ then m_1 and m_2 have the same conflict with the same sets,* i.e., $\kappa_1(m_1, m_A) - \kappa_1(m_2, m_A) = 0$ *$\forall A \subseteq \mathcal{X}$, whatever m may be, whereas $\kappa_N(m_1, m_2) = 0$ iff m_1 and m_2 are N-nonconflicting, that is m^2 is N-consistent.*

5. Conclusions

Conflict between belief functions has been defined by Destercke and Burger as the inconsistency of the belief function resulting from their conjunctive combination. The three existing definitions of the consistency of a belief function were shown in this paper to be specific cases, or *shades*, of a parameterised family of consistency definitions, which we called consistency of order N. We introduced a corresponding family of consistency measures and derived an associated family of conflict measures between belief functions. Each of these conflict measures is shown to verify a list of desirable properties for quantifying conflict. The family of conflict measures encompasses the classical measure of conflict in belief function theory, associated to the weakest ("brightest") definition of consistency, as well as two other conflict measures associated, respectively, to a stronger ("darker") definition of consistency by Yager and to the strongest ("darkest") definition of consistency by Destercke and Burger (called logical consistency). In addition, a geometric view on consistency measures as well as on the associated conflict measures was provided. In particular, we showed that measuring the consistency of a belief function amounts to measuring its distance to the totally inconsistent knowledge state, whatever the definition of consistency considered. A similar result was also obtained for conflict measures.

Our conflict measures are applicable in the case where it can be safely assumed that the sources are independent. However, the measures can readily be extended to the case where the dependence between sources is known and is captured by a joint mass function as in [7]. Handling the case where the dependence structure cannot be uniquely identified would need some further work, but it seems possible (in particular, measure κ_1 for nonconflict has already been adapted to this case in [7]). Besides, similarly to what has been done in [7] for measure κ_π, it would be interesting to find a decomposition of the conflict κ_N of two belief functions, in terms of some internal conflict (inconsistency) of each of the belief functions and of some external conflict between them, such that this decomposition satisfies some sensible relations originally introduced in [10].

Other interesting perspectives include finding theoretical or practical considerations that could help in selecting in a given application, one specific conflict measure among the κ_N measures, or in choosing between the two measures for strong nonconflict $\kappa_{\mathfrak{F}_{12}}$ and κ_π. In addition, it would be interesting to know whether there exist other pseudo-norms of the plausibility vector, besides the family of pseudo-norms we have introduced and presented, which could serve as consistency measures.

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