Optimization Problems with Evidential Linear Objective

Tuan-Anh Vu[a], Sohaib Afifi[a], Eric Lefèvre[a], Frédéric Pichon[a]

[a] *Univ. Artois, UR 3926, Laboratoire de Genie Informatique et d'Automatique de l'Artois (LGI2A), F-62400 Bethune, France*

Abstract

We investigate a general optimization problem with a linear objective in which the coefficients are uncertain and the uncertainty is represented by a belief function. We consider five common criteria to compare solutions in this setting: generalized Hurwicz, strong dominance, weak dominance, maximality and E-admissibility. We provide characterizations for the non-dominated solutions with respect to these criteria when the focal sets of the belief function are Cartesian products of compact sets. These characterizations correspond to established concepts in optimization. They make it possible to find non-dominated solutions by solving known variants of the deterministic version of the optimization problem or even, in some cases, simply by solving the deterministic version.

Keywords: Belief function, Robust optimization, Combinatorial optimization, Linear programming.

1. Introduction

Our paper focuses on a very general class of optimization problems where the objective function is linear (LOP). LOP covers a broad range of practical problems in diverse areas such as transportation, scheduling, network design, and profit planning, to name only a few important domains. In many realistic situations, one often encounters uncertainty on the coefficients of the objective function. Various approaches have been developed to model the uncertainty on coefficients, including robust optimization frameworks that represent uncertainty using discrete scenario sets [19, 10, 12] and intervals [16, 17, 10, 19, 5]. In the former representation, all possible realizations or scenarios of coefficients are explicitly listed to obtain the so-called scenario set. In the interval representation, each coefficient is constrained to lie within a given closed interval, and the scenario set is the Cartesian product of these intervals.

Email addresses: `tanh.vu@univ-artois.fr` (Tuan-Anh Vu), `sohaib.afifi@univ-artois.fr` (Sohaib Afifi), `eric.lefevre@univ-artois.fr` (Eric Lefèvre), `frederic.pichon@univ-artois.fr` (Frédéric Pichon)

In this paper, we investigate the case where the uncertainty on the coefficients is *evidential*, *i.e.*, modelled by a belief function [21]. More specifically, we assume that each so-called focal set of the considered belief function is a Cartesian product of compact sets, with each compact set describing possible values of each coefficient. Such a belief function is a direct and natural generalization of the interval representation, which arises when intervals are extended to compact sets and probabilities are assigned to scenario sets. It can be illustrated as follows: in a network with three cities A, B, and C, under good weather conditions, it may take 20 to 30 minutes to travel from A to B, and 10 to 20 minutes to travel from B to C; however under bad weather conditions, the travel times from A to B (resp. B to C) takes 30 to 40 minutes (resp. 15 to 25 minutes) and the forecast tells us that the probability of good weather (resp. bad weather) is 0.8 (resp. 0.2).

In the presence of evidential uncertainty on coefficients, the notion of best, *i.e.*, optimal, solutions becomes ill-defined. In our preliminary work[1] [25], which considered the shortest path problem (SPP) where each path has an evidential weight, we drew inspiration from [10] and utilized decision theory under evidential uncertainty [7], to define the best paths as those that are non-dominated with respect to some preference relation over paths built on the notions of their lower and upper expected weights. Specifically, we studied the cases of the preference relations obtained from three common criteria for decision-making, namely generalized Hurwicz, strong dominance, and weak dominance.

Besides [25], optimization problems under evidential uncertainty were explored recently in [15, 22, 12]. The authors of [15, 22] considered various variants of the vehicle routing problem with different uncertainty factors. In the resulting optimization problems, solutions had evidential costs and were compared according to their upper expected costs, i.e., using a particular case of the generalized Hurwicz criterion. Guillaume *et al.* [12] considered the LOP problem with evidential coefficients, where each focal set of the belief function on the coefficients can be any discrete scenario set. They defined best solutions as the non-dominated ones according to the generalized Hurwicz criterion and they provided complexity results regarding the problem of finding such solutions.

In this paper, we expand upon the work [25] by investigating a much broader class of problems, *i.e.,* LOP, and by incorporating two additional well-known criteria from the literature [2]: maximality and E-admissibility. More specifically, this paper's primary contributions are summarized as follows:

1. We propose models for LOPs in which the coefficients in the objective are subject to evidential uncertainty. Here, each feasible solution is regarded as an act, which is a fundamental concept in decision theory. These models are based on five common criteria from the literature for comparing acts, namely generalized Hurwicz, strong dominance, weak dominance, maximality, and E-admissibility. A key feature of these models is that they

[1] This paper is an extended and revised version of [25].

make use of the expressive nature of the belief function framework as they allow for incomparability of some solutions due to a lack of information.

2. We provide a characterization for the non-dominated solutions of each criterion, given our assumption about the focal sets. These characterizations correspond to established concepts of optimization. This makes it possible to find non-dominated solutions by solving known variants of the deterministic version of the LOP or even, in some cases (e.g., the case of the generalized Hurwicz criterion), simply by solving its deterministic version. For instance, we can use SPP-related algorithms to efficiently find non-dominated solutions for the five criteria in the case of the SPP. In our opinion, this is the main advantage of our works compared to [15, 22, 12], where finding non-dominated solutions with respect to the Hurwicz criterion was much harder in general than solving the deterministic version.

We note that the idea of using decision theory under uncertainty, and specifically maximality and a special case of the generalized Hurwicz criterion, to formalize optimization problems under (severe) uncertainty was first proposed in [20], where the very general theory of coherent lower previsions is used as the uncertainty representation framework. However, the resulting models were studied in detail and connected to their deterministic counterparts only in a few special uncertainty cases, such as the case of intervals (vacuous previsions); the case of the evidential representation of uncertainty was not investigated.

The rest of this paper is organized as follows. Sections 2 and 3 present necessary background material on the LOP and belief function theory, respectively. Sections 4 and 5 are devoted to the formalization and resolution of the LOP with evidential coefficients, respectively. The paper ends with a conclusion in Section 6.

2. Optimization problems with a linear objective (LOP)

Many real-world problems have variables that are either integers or a mixture of integers and real numbers. In this paper, we mainly focus on the following optimization problem:

$$\begin{aligned} \max / \min \ & c^T x \\ \text{s.t. } & x \in \mathcal{X} \subseteq \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n_2}_{\geq 0} \text{ with } n_1 + n_2 = n. \end{aligned} \tag{LOP}$$

where $\mathcal{X} \neq \emptyset$ is a set of feasible solutions and c is a vector of objective function coefficients $c_i \in \mathbb{R}$.

A very important class of Problem LOP is *linear mixed-integer programming* (MIP) problems:

$$\begin{aligned} \max / \min \ & c^T x \\ \text{s.t. } & Mx \leq b, x \in \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n_2}_{\geq 0}. \end{aligned} \tag{MIP}$$

where M is a $m \times n$ matrix and b is a m-vector. We require that M and b have rational entries [28]. A practical instance of Problem MIP is the uncapacitated lot sizing problem (Example 1).

Example 1 (Uncapacitated lot sizing)**.** *The problem is to decide on a production plan for an n-period horizon for a single product. The parameters of the problem are:*

- *f_t, which is the fixed cost of producing in period t;*
- *p_t, which is the production cost in period t;*
- *h_t, which is the unit storage cost in period t;*
- *d_t, which is the demand in period t.*

The problem can be modelled by the following optimization problem:

$$\begin{aligned} \min \sum_{t=1}^{n} p_t x_t + \sum_{t=1}^{n} h_t s_t + \sum_{t=1}^{n} f_t y_t& \\ s_{t-1} + x_t = d_t + s_t \quad &(t = 1, 2, \ldots, n) \\ x_t \leq M y_t \quad &(t = 1, 2, \ldots, n) \\ s_0 = 0, s_t, x_t \geq 0, y_t \in \{0,1\} \quad &(t = 1, 2, \ldots, n) \end{aligned} \tag{ULS}$$

where the decision variables are:

- *x_t, which is the amount produced in period t;*
- *s_t, which is the stock at the end of period t;*
- *$y_t = 1$ if production occurs in t and $y_t = 0$ otherwise;*

and where M is a big constant value.

Problem LOP is referred to as a 0-1 combinatorial optimization problem (01COP) when $\mathcal{X} \subseteq \{0,1\}^n$:

$$\begin{aligned} \max / \min \ & c^T x \\ \text{s.t.} \ & x \in \mathcal{X} \subseteq \{0,1\}^n. \end{aligned} \tag{01COP}$$

This class includes many important problems. Below, we provide two of the most notable examples.

Example 2 (The shortest path problem (SPP))**.** *Let $G = (V, A)$ be a directed graph with set of vertices V, set of arcs A and weight $c_{ij} \geq 0$ for each arc (i, j) in A. Let s and t be two vertices in V called the source and the destination, respectively.*

Finding a s-t shortest path, i.e., *a s-t path of lowest weight, can be modelled as the following optimization problem:*

$$\begin{aligned} \min \sum_{(i,j)\in A} c_{ij}x_{ij} \\ \sum_{(s,i)\in A} x_{si} - \sum_{(j,s)\in A} x_{js} = 1 \\ \sum_{(t,i)\in A} x_{ti} - \sum_{(j,t)\in A} x_{jt} = -1 \\ \sum_{(k,i)\in A} x_{ki} - \sum_{(j,k)\in A} x_{jk} = 0, \quad \forall k \in V\backslash\{s,t\} \\ x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A \end{aligned} \tag{SPP}$$

where each s-t path is identified with a set $x = \{x_{ij}|(i,j) \in A\}$ of which element $x_{ij} = 1$ if arc (i,j) is in the path and $x_{ij} = 0$ otherwise.

Example 3 (The 0-1 knapsack problem (01KP))**.** *Suppose a company has a budget of W and needs to choose which items to manufacture from a set of n possible items, each with a production cost of w_i and fixed profit of p_i (all values are numbers in unit €). The 01KP involves selecting a subset of items to manufacture that maximizes the total profit while keeping the total production costs below W. The 01KP can be formulated as*

$$\begin{aligned} &\max \sum_{i=1}^{n} p_i x_i \\ &s.t. \ \sum_{i=1}^{n} w_i x_i \leq W \\ &\quad x_i \in \{0,1\} \quad (i = 1, 2, \ldots, n). \end{aligned} \tag{01KP}$$

The sets of feasible solution in Examples 2 and 3 are described by linear constraints. However, it should be noted that Problem 01COP is not limited to problems with linear constraints as $\mathcal{X}$ can be any set.

When $\mathcal{X}$ is a convex subset of $\mathbb{R}^n_{\geq 0}$, Problem LOP becomes a convex optimization problem (CV):

$$\begin{aligned} &\max / \min \ c^T x \\ &\quad \text{s.t. } x \in \mathcal{X} \subseteq \mathbb{R}^n_{\geq 0} \text{ is convex.} \end{aligned} \tag{CV}$$

This class includes linear programming as a particular case.

3. Belief function theory

Let Ω be the set, called frame of discernment, of all possible values of a variable of interest ω. In belief function theory [21], adapting the presentation of [27], partial knowledge about the true (unknown) value of ω, when Ω is a closed subset of $\mathbb{R}^n$ as will be the case in this paper, is represented by a mapping $m : \mathcal{C} \mapsto [0,1]$ called mass function, where $\mathcal{C}$ is assumed here to be a finite collection of closed subsets of Ω, such that $\sum_{A\in\mathcal{C}} m(A) = 1$ and $m(\emptyset) = 0$. Mass $m(A)$ quantifies the amount of belief allocated to the fact of knowing only that $\omega \in A$. A subset $A \subseteq \Omega$ is called a focal set of m if $m(A) > 0$. The set of all focal sets of m is denoted by $\mathcal{F}$.

The mass function m induces a belief function Bel and a plausibility function Pl defined on $\mathcal{B}(\Omega)$ the Borel subsets of Ω:

$$Bel(A) = \sum_{B\in\mathcal{F}:B\subseteq A} m(B), \quad Pl(A) = \sum_{B\in\mathcal{F}:B\cap A\neq\emptyset} m(B). \tag{1}$$

A probability measure P on $\mathcal{B}(\Omega)$ is compatible with m if $Bel(A) \leq P(A)\ \forall A \in \mathcal{B}(\Omega)$. We denote by $\mathcal{P}(m)$ the set of all probability measures that are compatible with m. The *upper expected value* $\overline{E}_m(h)$ and *lower expected value* $\underline{E}_m(h)$ of a bounded, measurable function function $h : \Omega \to \mathbb{R}$, relative to m, are defined as

$$\overline{E}_m(h) := \sup_{P\in\mathcal{P}(m)} E_P(h), \qquad \underline{E}_m(h) := \inf_{P\in\mathcal{P}(m)} E_P(h). \tag{2}$$

A well-known result [27, Section 2.4] states that the upper and lower expected values of h can be computed as:

$$\overline{E}_m(h) = \sum_{A\in\mathcal{F}} m(A) \sup_{\omega_i\in A} h(\omega_i), \tag{3}$$

$$\underline{E}_m(h) = \sum_{A\in\mathcal{F}} m(A) \inf_{\omega_i\in A} h(\omega_i). \tag{4}$$

When mass function m is clear from the context, $\overline{E}_m(h)$ and $\underline{E}_m(h)$ may be simply written $\overline{E}(h)$ and $\underline{E}(h)$, respectively.

Assume Ω represents the state of nature and its true value is known in the form of some mass function m. Assume further that a decision maker (DM) needs to choose an act (decision) f from a finite set $\mathcal{Q}$. The outcome of each act can vary based on the prevailing state of nature. Denoting by $\mathcal{O}$ the set of possible outcomes, each act can thus be formalized as a mapping $f : \Omega \to \mathcal{O}$.

Depending on the context, outcomes induce either utilities or costs. Utilities (resp. costs) of outcomes can be quantified by an utility function $u : \mathcal{O} \to \mathbb{R}$ (resp. cost function $l : \mathcal{O} \to \mathbb{R}$). We assume that for any f, $u \circ f$ (resp. $l \circ f$) is a bounded real-valued map. In the following, to keep the discussion concise, we concentrate on presenting the treatment when the outcomes are associated with an utility function since a cost minimization can be turned in a utility maximization by taking the negative. Moreover, to enhance comprehension, we will use a specific problem, the SPP, to illustrate the results of the cost function case in Section 5.

In this framework, the DM's preference over acts is denoted by $\succeq$, where $f \succeq g$ means that act f is preferred to act g. The preference relation is typically assumed to satisfy the reflexivity property ($f \succeq f$ for any f) and the transitivity property (if $f \succeq g$ and $g \succeq k$, then $f \succeq k$ for any f, g, and k), making it a preorder. Furthermore, if the relation is antisymmetric ($f = g$ for any f and g such that $f \succeq g$ and $g \succeq f$), then it becomes an order. Relation $\succeq$ is complete if for any two acts f and g, $f \succeq g$ or $g \succeq f$, otherwise, it is partial. Additionally, f is strictly (resp. equally) preferred to g, which is denoted by $f \succ g$ (resp. $f \sim g$), if $f \succeq g$ but not $g \succeq f$ (resp. if $f \succeq g$ and $g \succeq f$).

Typically, the DM seeks solutions in the set Opt of non-dominated acts:

$$Opt = \{f \in \mathcal{Q} : \nexists g \text{ such that } g \succ f\}. \tag{5}$$

If relation $\succeq$ is complete, finding one solution in Opt is enough since solutions in Opt are preferred equally between each other and strictly preferred to the rest $\mathcal{Q} \setminus Opt$. In this case, solutions in Opt are also called optimal acts. On the other hand, if relation $\succeq$ is partial, the DM may need to identify all solutions in Opt.

Usually, the DM constructs his preference over acts based on some criterion. We denote by $\succeq_{cr}$ his preference according to some criterion cr and by Opt_{cr} its associated set of non-dominated (or best) acts. In this paper, we consider five common criteria defined as follows for any two acts f and g [7]:

1. Generalized Hurwicz criterion: $f \succeq_{hu}^{\alpha} g$ if

   $$\alpha \overline{E}_m(u \circ f) + (1-\alpha)\underline{E}_m(u \circ f) \geq \alpha \overline{E}_m(u \circ g) + (1-\alpha)\underline{E}_m(u \circ g) \tag{6}$$

   for some fixed parameter $\alpha \in [0,1]$, representing an optimism/pessimism degree, and where $\overline{E}_m(u \circ f)$ and $\underline{E}_m(u \circ f)$ denote, respectively, the upper and lower expected utilities of act f with respect to mass function m. Relation $\succeq_{hu}^{\alpha}$ is complete and we have $f \succ_{hu}^{\alpha} g$ if (6) is strict. The set of non-dominated acts with respect to $\succeq_{hu}^{\alpha}$ is denoted by Opt_{hu}^{α}.

2. Strong dominance criterion: $f \succeq_{str} g$ if

   $$\underline{E}_m(u \circ f) \geq \overline{E}_m(u \circ g). \tag{7}$$

   Relation $\succeq_{str}$ is partial and we have $f \succ_{str} g$ if (7) is strict. The set of non-dominated acts with respect to $\succeq_{str}$ is denoted by Opt_{str}.

3. Weak dominance criterion: $f \succeq_{weak} g$ if

   $$\overline{E}_m(u \circ f) \geq \overline{E}_m(u \circ g) \text{ and } \underline{E}_m(u \circ f) \geq \underline{E}_m(u \circ g). \tag{8}$$

   Relation $\succeq_{weak}$ is partial and we have $f \succ_{weak} g$ if at least one inequality in (8) is strict. The set of non-dominated acts with respect to $\succeq_{weak}$ is denoted by Opt_{weak}.

4. Maximality criterion: $f \succeq_{max} g$ if

   $$\underline{E}_m(u \circ f - u \circ g) \geq 0 \iff \forall P \in \mathcal{P}(m), E_P(u \circ f) \geq E_P(u \circ g), \tag{9}$$

   Relation $\succeq_{max}$ is partial and we have $f \succ_{max} g$ if $\underline{E}_m(u \circ f - u \circ g) > 0$. The set of non-dominated acts with respect to $\succeq_{max}$ is denoted by Opt_{max}.

5. E-admissibility criterion: Let Opt_{adm} be the set of non-dominated solutions with respect to E-admissibility criterion, then $f \in Opt_{adm}$ iff there exists $P \in \mathcal{P}(m)$ such that $E_P(u \circ f) \geq E_P(u \circ g)$ for any act g.

Note that $Opt_{adm} \subseteq Opt_{max}$ and $Opt_{weak} \subseteq Opt_{max} \subseteq Opt_{str}$ with usually strict inclusions (see [8]).

We can observe that E-admissibility differs from other decision criteria, as it directly defines a set of non-dominated acts (*choice set*), without the need for explicitly defining a preference relation. However, we can still construct a preference relation from the choice set (see [7]).

Given these criteria, a relevant question for the DM is which criterion should be chosen. The choice of the criterion depends on factors such as its properties or its associated computational cost of determining non-dominated acts. For instance, when comparing strong dominance and maximality, the computational cost associated with maximality is generally higher than that of strong dominance, but strong dominance is more conservative than maximality since $Opt_{max} \subseteq Opt_{str}$. However, dealing with this question is beyond the scope of our paper. We refer to the excellent review papers of Troffaes [23] and Denoeux [7] for comprehensive discussions of these criteria.

4. LOP with evidential coefficients: modelling

In this section, we formalize what we mean by best solutions of Problem LOP when coefficients in the objective function are evidential, *i.e.*, are known in the form of a mass function, and we also describe a particular assumption about the focal sets of this mass function.

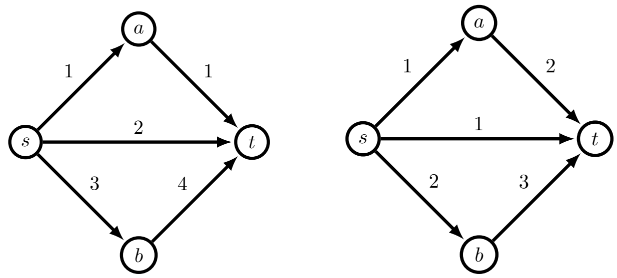
Let us assume that the coefficients c_i, for all $i \in 1, \ldots, n$, in the objective of Problem LOP are only partially known. More specifically, we consider the case where information about the coefficients is modelled by a mass function. Formally, let Ω_i be the frame of discernment for the variable c_i, *i.e.*, the set of possible values for the coefficients c_i and let $\Omega := \times_{i=1}^{n} \Omega_i$. Any $c \in \Omega$ will be called a scenario: it represents a possible assignment of values for all coefficients in the objective function. A mass function m on Ω, with set of focal sets denoted by $\mathcal{F} = \{F_1, \ldots, F_K\}$, represents uncertainty about the coefficients.

Example 4. *Consider the Problem SPP, let c^1 and c^2 be the two scenarios represented by Figures 1a and 1b, respectively. The mass function m such that $m(F_1) = 0.4$ and $m(F_2) = 0.6$, with $F_1 = \{c^1, c^2\}$ and $F_2 = \{c^1\}$, represents partial knowledge about arc weights.*

As will be seen, making a particular assumption about the nature of the focal sets of m is useful. This assumption relies on the following definition.

Definition 1. *Given a subset $A \subseteq \Omega$, we denote by $A^{\downarrow i}$ its projection on Ω_i. We say that A is a rectangle iff it can be expressed as the Cartesian product of its projections, that is: $A = \times_{i=1}^{n} A^{\downarrow i}$.*

The assumption about the focal sets of m is the following:



(a) Scenario c^1 (b) Scenario c^2

Figure 1: Two possible assignments of values, *i.e.,* two scenarios, for the arc weights.

Assumption 1 (Rectangular with Compact projections (RC))**.** *Each focal set of m is a rectangle where each of its projection is a compact subset of $\mathbb{R}$.*

Let m be a mass function satisfying the RC assumption and let F_r be a focal set of m. The minimum and maximum values of its projection $F_r^{\downarrow i}$ will be denoted hereafter l_i^r and u_i^r, respectively.

While assuming focal sets to be rectangular may seem restrictive, it has been argued in [1] that such focal sets arise in many practical situations, such as in the example given in the Introduction and, for instance, it results from the combination of marginal mass functions m^i defined on Ω_i under the assumption of independence [6]. The compactness assumption is also rather mild as it allows $F^{\downarrow i}$ to be, e.g., any closed (real) interval or any finite set of real numbers (and thus the practical situation of independent marginal mass functions m^i having closed intervals or finite sets as focal sets, fits the RC assumption). RC focal sets are further illustrated by Example 5 in a particular case where they are Cartesian products of intervals.

Example 5. *Consider the Problem SPP. Let m be the mass function such that $m(F_1) = 0.5$ and $m(F_2) = 0.5$ with focal sets F_1 and F_2, depicted in Figure 2, such that*

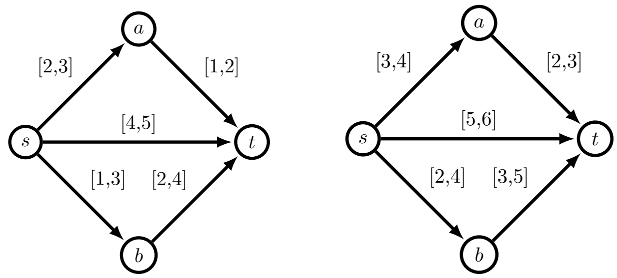
$$\begin{aligned} F_1 =& [l_{sa}, u_{sa}] \times [l_{sb}, u_{sb}] \times [l_{st}, u_{st}] \times [l_{at}, u_{at}] \times [l_{bt}, u_{bt}] \\ =& [2, 3] \times [1, 3] \times [4, 5] \times [1, 2] \times [2, 4]. \end{aligned}$$

and, similarly,

$$F_2 = [3, 4] \times [2, 4] \times [5, 6] \times [2, 3] \times [3, 5].$$

Each focal set is a subset of Ω. For instance, the scenario $c = \{c_{sa}, c_{sb}, c_{st}, c_{at}, c_{bt}\}$ with $c_{sa} = 2, c_{sb} = 3, c_{st} = 4, c_{at} = 1$ and $c_{bt} = 2$ is included in F_1.

When coefficients are evidential, *i.e.*, there is some uncertainty about them in the form of a mass function m on Ω, the preference over feasible solutions



(a) Focal set F_1 (b) Focal set F_2

Figure 2: Two focal sets which are Cartesian products of intervals.

with respect to the (uncertain) coefficients can be established using the decision-making framework recalled in Section 3. Specifically, the set Ω of scenarios represents the possible states of nature. The set of feasible solutions $\mathcal{X}$ represents the possible acts. By a slight abuse of notation, each solution x can be interpreted as a function $x : \Omega \to \mathcal{O}$ such that $x(c) = c^T x$, and the intended interpretation should be clear from the context.

If Problem LOP is a maximization problem (resp. minimization), the value $\sum_{i=1}^{n} c_i x_i$ of $x \in \mathcal{X}$ under scenario $c = \{c_i | i = 1, \ldots, n\} \in \Omega$ represents the utility $u \circ x(c)$ (resp. cost $l \circ x(c)$) of solution (act) x for the scenario (state of nature) c, with u (resp. l) being the identity function. From here on, we will use the notation x to represent $u \circ x$ and $l \circ x$ for convenience.

The preference over feasible solutions, and the associated best solutions, can then be defined using any of the five criteria recalled in Section 3. In the next section, we provide the main results of this paper, which concern best solutions with respect to these five criteria and under assumption RC.

Remark 1. *In [12], Problem LOP with evidential coefficients is also considered. The essential difference*[2] *between [12] and the present paper is the nature of the focal sets of the mass function m on the coefficients: in [12], they are assumed to be discrete scenario sets, whereas here we assume them to be RC. Hence, for instance, the mass function in Example 4 fits the setting of [12] but does not fit ours, whereas the mass function in Example 5 fits our setting but does not fit the one of [12].*

5. LOP with evidential coefficients: solving

In this section, we provide methods for finding best (non-dominated) solutions, with respect to the five criteria presented in Section 3, of Problem LOP

[2]Another important difference with [12] is that only the generalized Hurwicz criterion is considered in this latter paper, whereas we consider four additional criteria.

when coefficients in the objective function are evidential, *i.e.*, are known in the form of some mass function m on Ω with set of focal sets $\mathcal{F} = \{F_1, \dots, F_K\}$.

For $i \in \{1, \dots, n\}$, let π_i be the map from Ω to $\mathbb{R}$ such that $\pi_i(c) = c_i$, *i.e.*, $\pi_i(c)$ is nothing but coefficient c_i of scenario $c \in \Omega$. As will be seen, the upper $\overline{E}(\pi_i)$ and lower $\underline{E}(\pi_i)$ expected values of π_i with respect to m are central in our characterizations of the non-dominated solutions for the five criteria. These values can be computed easily under assumption RC:

Proposition 1. *Under assumption RC, we have*

$$\overline{E}(\pi_i) = \sum_{r=1}^{K} m(F_r) u_i^r, \tag{10}$$

$$\underline{E}(\pi_i) = \sum_{r=1}^{K} m(F_r) l_i^r. \tag{11}$$

Proof. We have

$$\overline{E}(\pi_i) = \sum_{r=1}^{K} m(F_r) \max_{c \in F_r} \pi_i(c) \tag{12}$$

$$= \sum_{r=1}^{K} m(F_r) \max_{c_i \in F_r^{\downarrow i}} c_i. \tag{13}$$

Similarly, we obtain $\underline{E}(\pi_i) = \sum_{r=1}^{K} m(F_r) \min_{c_i \in F_r^{\downarrow i}} c_i$. The proposition follows from the fact that under assumption RC, the projection $F_r^{\downarrow i}$ of focal set $F_r \in \mathcal{F}$, has maximum value u_i^r and minimum value l_i^r. □

To simplify the exposition of our results, $\overline{E}(\pi_i)$ and $\underline{E}(\pi_i)$ under assumption RC will be denoted hereafter by $\bar{u}_i$ and $\bar{l}_i$, respectively, *i.e.*, we have

$$\bar{u}_i := \sum_{r=1}^{K} m(F_r) u_i^r, \tag{14}$$

$$\bar{l}_i := \sum_{r=1}^{K} m(F_r) l_i^r. \tag{15}$$

Example 6 (Example 5 continued)**.** *Consider the Problem SPP and the mass function in Example 5, with evidential weighted graph in Figure 2. We have for instance for arc s-a:*

$$\bar{u}_{sa} = m(F_1) \cdot u_{sa}^1 + m(F_2) \cdot u_{sa}^2 \tag{16}$$

$$= 0.5 \cdot 3 + 0.5 \cdot 4 = 3.5, \tag{17}$$

$$\bar{l}_{sa} = 0.5 \cdot 2 + 0.5 \cdot 3 = 2.5. \tag{18}$$

We treat in this section the five criteria in the order that they were introduced in Section 3. Note that, as is the case for Proposition 1 above, all the following Propositions require assumption RC to hold, and thus, for conciseness, we will no longer explicitly state this assumption in the Propositions.

5.1. Generalized Hurwicz criterion

We give a characterization for non-dominated solutions with respect to the generalized Hurwicz criterion.

First, we can remark that this criterion relies on the notions of upper and lower expected utilities of acts, acts being here feasible solutions. The upper $\overline{E}(x)$ and lower $\underline{E}(x)$ expected utilities of a solution x can be computed easily under assumption RC:

Proposition 2. *(Under assumption RC) We have*

$$\overline{E}(x) = \sum_{i=1}^{n} \bar{u}_i x_i, \tag{19}$$

$$\underline{E}(x) = \sum_{i=1}^{n} \bar{l}_i x_i. \tag{20}$$

Proof. By definition and since each focal set is compact, the upper and lower expected utilities of x are

$$\overline{E}(x) = \sum_{r=1}^{K} m(F_r) \max_{c^r \in F_r} (\sum_{i=1}^{n} c_i^r x_i), \tag{21}$$

$$\underline{E}(x) = \sum_{r=1}^{K} m(F_r) \min_{c^r \in F_r} (\sum_{i=1}^{n} c_i^r x_i). \tag{22}$$

The inner maximum and minimum in (21) and (22) are obtained when each component c_i^r in c^r equals u_i^r and l_i^r, respectively. By regrouping terms we get the desired result. □

Since $\succeq_{hu}^{\alpha}$ is complete, it is sufficient to find one solution of the set Opt_{hu}^{α}, as explained in Section 3. To find one such solution, we need to solve the optimization problem,

$$\begin{aligned} \max / \min \ \alpha \overline{E}_m(x) + (1-\alpha)\underline{E}_m(x) \\ x \in \mathcal{X}, \end{aligned} \tag{23}$$

for some specified value of $\alpha \in [0, 1]$.

In the case of general focal sets, solving Problem (23) is usually much more challenging than solving its deterministic counterpart Problem LOP. For instance, the deterministic Problem SPP can be solved efficiently in polynomial time, but if $\alpha = 1$ the Problem (23) is weakly NP-hard already in the case when mass function m has a single focal set containing two elements [30]. The situation worsens if $\alpha = 0$, as the problem becomes strongly NP-hard and not approximable [12, Theorem 1]. However, under assumption RC, the complexity of Problem (23) remains unchanged compared to Problem LOP, since it is a direct consequence of the following characterization.

Proposition 3. *A solution x is in Opt^{α}_{hu} iff x is an optimal solution of Problem LOP with coefficients $c_i = \alpha\bar{u}_i + (1-\alpha)\bar{l}_i$.*

Proof. Using Proposition 2, the Problem (23) becomes

$$\max / \min \sum_{i=1}^{n} (\alpha\bar{u}_i + (1-\alpha)\bar{l}_i)x_i \tag{24}$$

$$x \in \mathcal{X} \tag{25}$$

□

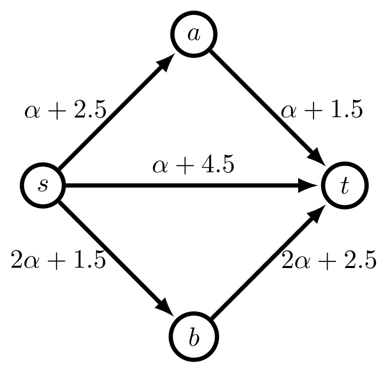


Figure 3: The parametric weighted graph associated with Opt^{α}_{hu}.

Example 7 (Example 5 continued)**.** *To find a best path in Opt^{α}_{hu} for the evidential weighted graph in Figure 2, we need to solve the deterministic SPP in the graph showed in Figure 3, for some specified value of α (we have for instance for arc s-a, using Example 6: $\alpha\bar{u}_{sa} + (1-\alpha)\bar{l}_{sa} = \alpha \cdot 3.5 + (1-\alpha) \cdot 2.5 = \alpha + 2.5$). For example, if $\alpha = 0$ then the corresponding shortest paths are s-a-t and s-b-t, while the shortest one is s-t, if $\alpha = 1$.*

Remark 2. *Thanks to Proposition 3, we can establish that best acts with respect to the generalized Hurwicz criterion for various α are solutions of a parametric LOP. Hence, methods from parametric optimization can help to solve a whole family of problems parameterized by α. For instance, the standard approach for solving parametric linear programming is the parametric simplex method [24, Chapter 7]. In the parametric SPP from Figure 3, as the DM varies his optimism/pessimism degree from 0 to 1, the break-point (point where a change in the parameter α causes a sudden change in the solutions) is 0.5. More precisely, for all $\alpha \in [0, 0.5]$ the best path is s-a-t, while for all $\alpha \in [0.5, 1]$ the optimal one is s-t. We refer to the work of Gusfield [13] for a comprehensive discussion of parametric combinatorial optimization problems.*

5.2. Strong dominance criterion

In the same spirit as Proposition 3, we give now a characterization for non-dominated solutions with respect to the strong dominance criterion when Problem LOP is a maximization problem.

Proposition 4. *A solution x is in Opt_{str} iff x is feasible with respect to the following constraints:*

$$x \in \mathcal{X} \tag{26}$$

$$\sum_{i=1}^{n} \bar{u}_i x_i \geq z \tag{27}$$

where z is the optimal value of Problem LOP in which $c_i = \bar{l}_i, i = 1, 2, \ldots, n$.

Proof. By definition,

$$x \in Opt_{str} \Leftrightarrow \nexists y \in \mathcal{X} \text{ such that } \underline{E}(y) > \overline{E}(x) \tag{28}$$

$$\Leftrightarrow \forall y \in \mathcal{X} \text{ then } \underline{E}(y) \leq \overline{E}(x) \tag{29}$$

$$\Leftrightarrow \max_{y \in \mathcal{X}} \underline{E}(y) \leq \overline{E}(x) \tag{30}$$

As a special case of Proposition 3, when $\alpha = 0$, $z = \max_{y \in \mathcal{X}} \underline{E}(y)$ is obtained by solving Problem LOP with $c_i = \bar{l}_i$. From Proposition 2, we have $\overline{E}(x) = \sum_{i=1}^{n} \bar{u}_i x_i$, and thus the result follows. □

We also have a similar result when Problem LOP is a minimization problem.

Proposition 5. *A solution x is in Opt_{str} iff x is a feasible with respect to the following constraints:*

$$x \in \mathcal{X} \tag{31}$$

$$\sum_{i=1}^{n} \bar{l}_i x_i \leq z \tag{32}$$

where z is the optimal value of Problem LOP in which $c_i = \bar{u}_i, (i = 1, 2, \ldots, n)$.

Problem (26 -27) is called a *lower bound feasibility problem* since it is the feasibility problem with the additional constraint $\sum_{i=1}^{n} \bar{u}_i x_i \geq z$ (see [28, Section I.5.5]).

Since the relation $\succeq_{str}$ is partial, it may be necessary to identify all solutions in the set Opt_{str}, meaning all feasible solutions of (26 -27). The complexity of this task depends on the structure of Problem LOP itself. In a specific case mentioned in our previous works [25], enumerating Opt_{str} for the SPP amounts to finding all paths in G with arc weights $c_{ij} = \bar{l}_{ij}$, whose weights are lower than or equal to the lowest weight of a s-t path in G with arc weights $c_{ij} = \bar{u}_{ij}$. Hence, we can use efficient algorithms such as the ones in [3, 4], where the authors studied a problem of determining near optimal paths; for example, they wished to find all *s*-*t* paths in a directed graph whose weights do not exceed more than 5% the lowest weight, which is equivalent to finding all paths whose weights are less than or equal to a given threshold.

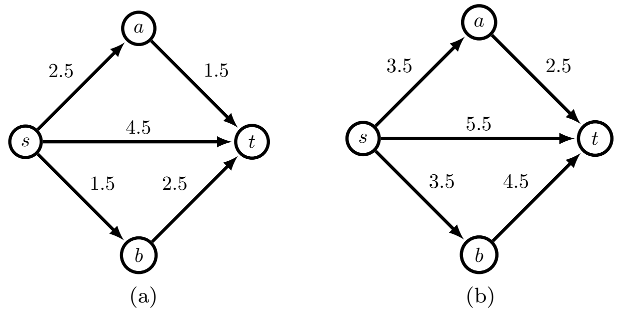


Figure 4: Two graphs associated with Opt_{str}.

Example 8 (Example 5 continued)**.** *To find the paths in Opt_{str} for the evidential weighted graph in Figure 2, according to Proposition 5 we first compute the lowest weight of a s-t path in the graph in Figure 4b, which is 5.5. The set Opt_{str} comprises then the s-t paths in the graph in Figure 4a that have weights no more than 5.5, which are the paths s-t, s-a-t, and s-b-t.*

5.3. Weak dominance criterion

There is a strong connection between the weak dominance criterion and bi-objective optimization. A bi-objective optimization problem can be expressed as

$$\max / \min \;\; f_1(x) \tag{33}$$
$$\max / \min \;\; f_2(x) \tag{34}$$
$$x \in \mathcal{X} \tag{35}$$

As the objectives (33-34) are typically conflicting, there is usually no solution x that maximizes (resp. minimizes) simultaneously $f_1(x)$ and $f_2(x)$. Instead, we seek to find all so-called efficient solutions of (33-35): a solution x is efficient if there is no feasible solution $y \in \mathcal{X}$ such that $f_1(y) \geq f_1(x)$ and $f_2(y) \geq f_2(x)$ (resp. $f_1(y) \leq f_1(x)$ and $f_2(y) \leq f_2(x)$) where at least one of the inequalities is strict.

Example 9. *The bi-objective SPP is a particular bi-objective optimization problem. Assume that each arc (i, j) in G has two deterministic attributes c_{ij} and t_{ij} that describes,* e.g.,, *the cost and the travel time from i to j, respectively. The* *goal is to find all efficient solutions,* i.e.*, s-t paths, of the following problem:*

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{36}$$

$$\min \sum_{(i,j)\in A} t_{ij}x_{ij} \tag{37}$$

$$x \textit{ is a s-t path} \tag{38}$$

We now give a characterization for solutions in Opt_{weak} in terms of efficient solutions of a bi-objective optimization problem.

Proposition 6. *A solution x is in Opt_{weak} iff x is a efficient solution of the problem:*

$$\begin{aligned} \max/\min \ & \sum_{i=1}^{n} \bar{l}_i x_i \\ \max/\min \ & \sum_{i=1}^{n} \bar{u}_i x_i \\ & x \in \mathcal{X} \end{aligned} \tag{39}$$

Proof. It is easy to see that $x \in Opt_{weak}$ iff x is an efficient solutions with objectives $f_1(x) := \overline{E}(x)$ and $f_2(x) := \underline{E}(x)$, which, using Proposition 2, leads to Problem (39). □

From Proposition 6, identifying solutions in Opt_{weak} is equivalent to finding solutions for Problem (39). Considering again Problem SPP as an example, we can remark that the bi-objective SPP has been extensively studied in the literature. Hence, we can apply off-the-shelf fast methods developed specifically for the bi-objective SPP, such as [11], to find solutions in Opt_{weak} for Problem SPP.

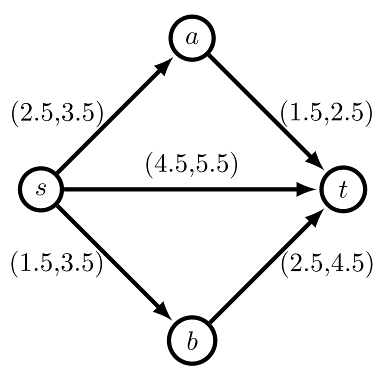


Figure 5: The graph associated with Opt_{weak} of which each arc (i, j) has two attributes $(\bar{l}_{ij}, \bar{u}_{ij})$.

Example 10 (Example 5 continued)**.** *Each path in Opt_{weak} is an efficient s-t path in the graph in Figure 5. Opt_{weak} consists of paths s-t and s-a-t (s-b-t is dominated by s-a-t).*

Remark 3. *It should be noted that any generalized Hurwicz optimal solution with $0 < \alpha < 1$ is also a solution of Opt_{weak}. As a result, determining such solutions for various α values can provide an inner approximation of Opt_{weak}. This stems from bi-objective optimization theory, where these solutions are known as* supported *efficient solutions: they are the solutions of $\min_{x\in\mathcal{X}} \lambda_1 f_1(x)+\lambda_2 f_2(x)$ for some $\lambda_1, \lambda_2 > 0$.*

5.4. Maximality and E-admissibility criteria

Contrary to the other criteria, identifying characterizations for maximality and E-admissibility relies on the nature of Problem LOP. As will be seen, solutions in Opt_{max} and Opt_{adm} are closely related to the notion of *possibly optimal solution* in robust optimization, where a solution x is referred to as possibly optimal if it is an optimal solution to a problem $\mathcal{P}$ for at least one scenario in the set of all possible scenarios Γ. This notion appears in various works in the realm of minimax regret optimization with interval data, such as in [16] for linear programming problems, in [29] for the minimum spanning tree problem (where the authors called a possibly optimal spanning tree a weak tree), and in [17] for other combinatorial optimization problems. To emphasize the importance of the notion, we frame it in the following definition.

Definition 2. *A solution x is a possibly optimal solution of Problem LOP with respect to the set $\mathcal{C} := \times_{i=1}^{n}[\bar{l}_i, \bar{u}_i]$ if x is an optimal solution for at least one vector c in $\mathcal{C}$. The set of these possibly optimal solutions is denoted by $Opt^{\mathcal{C}}_{pos}$.*

5.4.1. The general case

In the general case, *i.e.*, the Problem LOP with evidential coefficients, we are not able to provide similar characterizations for solutions in Opt_{max} and Opt_{adm} as for previous criteria. Instead, we offer partial answers by providing a sufficient condition for solutions of Opt_{max} (Proposition 7) and a necessary condition for solutions of Opt_{adm}(Proposition 8).

Proposition 7. *If $x \in Opt^{\mathcal{C}}_{pos}$ then $x \in Opt_{max}$.*

Proof. If x is optimal under c^o where $c^o_i \in [\bar{l}_i, \bar{u}_i]$, for all $i \in \{1, \dots, n\}$ then,

$$\forall y \in \mathcal{X},\ 0 \geq \sum_{i=1}^{n} c^o_i(y_i - x_i) = \sum_{i:y_i\geq x_i}^{n} c^o_i(y_i - x_i) + \sum_{i:y_i<x_i}^{n} c^o_i(y_i - x_i) \tag{40}$$

$$\Rightarrow 0 \geq \sum_{i:y_i\geq x_i} \bar{l}_i(y_i - x_i) + \sum_{i:y_i<x_i} \bar{u}_i(y_i - x_i) \tag{41}$$

On the other hand,

$$\underline{E}(y-x) = \sum_{r=1}^{K} m(F_r) \min_{c\in F_r} \sum_{i=1}^{n} c_i(y_i - x_i) \tag{42}$$

$$= \sum_{r=1}^{K} m(F_r)\Big(\sum_{i:y_i \geq x_i} l_i^r(y_i - x_i) + \sum_{i:y_i<x_i} u_i^r(y_i - x_i)\Big) \tag{43}$$

$$= \sum_{i:y_i\geq x_i} \bar{l}_i(y_i - x_i) + \sum_{i:y_i<x_i} \bar{u}_i(y_i - x_i) \tag{44}$$

From (41) and (44), we have that $\forall y \in \mathcal{X}$, $\underline{E}(y-x) \leq 0$, and thus $x \in Opt_{max}$. □

Proposition 8. *If $x \in Opt_{adm}$ then $x \in Opt_{pos}^{\mathcal{C}}$.*

Proof. Recall that an act x is a map from Ω to $\mathbb{R}$ such that $x(c) = \sum_{i=1}^n x_i c_i$. Note that $x(c) = \sum_{i=1}^n x_i \pi_i(c)$. Let $P \in \mathcal{P}(m)$. By linearity of integration, we have

$$E_P(x) = \int_\Omega x(c)\mathrm{d}P(c) = \sum_{i=1}^n x_i \int_\Omega \pi_i(c)\mathrm{d}P(c) = \sum_{i=1}^n x_i E_P(\pi_i). \tag{45}$$

Since $P \in \mathcal{P}(m)$, we have $\underline{E}(\pi_i) \leq E_P(\pi_i) \leq \overline{E}(\pi_i)$, *i.e.*,

$$\bar{l}_i \leq E_P(\pi_i) \leq \bar{u}_i. \tag{46}$$

If $x \in Opt_{adm}$ then $\exists P \in \mathcal{P}(m)$ such that $E_P(x) \geq E_P(y)\ \forall y$. From Equation (45), $\sum_{i=1}^n E_P(\pi_i)x_i \geq \sum_{i=1}^n E_P(\pi_i)y_i$, and thus x is optimal under c^o where $c_i^o := E_P(\pi_i)$. By Equation (46), we have $c_i^o \in [\bar{l}_i, \bar{u}_i]$. □

A direct consequence of Propositions 7 and 8 is the following result.

Corollary 1. *$Opt_{adm} \subseteq Opt_{pos}^{\mathcal{C}} \subseteq Opt_{max}$.*

In the important case of Problem CV the sets $Opt_{pos}^{\mathcal{C}}$, Opt_{adm}, and Opt_{max} coincide:

Proposition 9. *For Problem CV, $Opt_{adm} = Opt_{pos}^{\mathcal{C}} = Opt_{max}$.*

Proof. As the set of acts $\mathcal{X}$ is convex, by the result in [26, Section 3.9.5], $Opt_{adm} = Opt_{max}$. The result follows from Corollary 1. □

In the following two sections, we study these inclusions in Corollary 1 with respect to two other wide class of optimization problems besides Problem CV, namely Problems MIP and 01COP. As will be shown, the three sets also coincide for 01COP, whereas only the sets $Opt_{pos}^{\mathcal{C}}$ and Opt_{adm} coincide for MIP. Therefore, overall, our findings are that the inclusion between $Opt_{pos}^{\mathcal{C}}$ and Opt_{max} in Corollary 1 can be strict, whereas the inclusion between Opt_{adm} and $Opt_{pos}^{\mathcal{C}}$ is actually an equality for three important particular LOPs, *i.e.*, Problems CV, MIP and 01COP; it remains an open, non-trivial, question whether there exists an instance of Problem LOP for which the inclusion between Opt_{adm} and $Opt_{pos}^{\mathcal{C}}$ in Corollary 1 is strict.

5.4.2. Problem MIP

Let S be the feasible set of Problem MIP, consider the following optimization problem:

$$\begin{aligned} \max / \min \;\; & c^T x \\ \text{s.t.} \;\; & x \in \text{conv}(S) \end{aligned} \tag{CMIP}$$

where conv(S) is the convex hull of S.

A fundamental result in integer programming states that Problem CMIP is a linear programming problem and we can solve Problem MIP by solving Problem CMIP. To make the paper self-contained, we will state the result here without providing a proof. Further information and a detailed proof can be found in standard textbooks such as [28, Theorems 6.2 and 6.3].

Proposition 10. *Assume that Problem MIP is a maximization problem. For any $c \in \mathbb{R}^n$, if x^* is an optimal solution of Problem MIP, then x^* is an optimal solution of Problem CMIP.*

We can now provide a characterization of E-admissibility for Problem MIP by proving that the converse of Proposition 8 also holds.

Proposition 11. *For Problem MIP, $x \in Opt_{adm}$ iff $x \in Opt^{\mathcal{C}}_{pos}$.*

Proof. If x is an optimal solution of Problem MIP under some $c^o \in \mathcal{C}$ then by Proposition 10, x is also an optimal solution of Problem CMIP under c^o. As Problem CMIP is convex, by Proposition 9, x is an E-admissible act of Problem CMIP. Moreover, since $S \subseteq \text{conv}(S)$, then x is also an E-admissible act of Problem MIP. □

Corollary 1 states that if $x \in Opt^{\mathcal{C}}_{pos}$ then $x \in Opt_{max}$ for Problem LOP and thus also for Problem MIP. The next example shows that for Problem MIP, we can have $x \in Opt_{max}$ but $x \notin Opt^{\mathcal{C}}_{pos}$ (even when the mass function has a single focal set), *i.e.*, the inclusion between $Opt^{\mathcal{C}}_{pos}$ and Opt_{max} in Corollary 1 can be strict.

Example 11. *Consider the following optimization problem where each coefficient c_1, c_2, c_3 and c_4 in the objective is known to lie in an interval: $c_1 \in [1, 3]$,*

$c_2 \in [1,3]$, $c_3 = 0$ and $c_4 = 0$.

$$\begin{aligned}
\max\ & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
-2x_1 - x_2 &\leq -6 \\
x_1 + x_2 &\leq 5 \\
-x_1 - 2x_2 &\leq -6 \\
x_1 - 10x_3 &\leq 2 \\
-x_1 + 10x_3 &\leq 6 \\
x_2 - 10x_4 &\leq 2 \\
-x_2 + 10x_4 &\leq 6 \\
x_1, x_2 &\in \{1,2,3,4\} \\
x_3, x_4 &\in \{0,1\}
\end{aligned}$$

It can easily be checked that the set of feasible solutions $\mathcal{X}$ is $\mathcal{X} = \{x := (2,2,0,0), y := (1,4,0,1), z := (4,1,1,0)\}$. An easy computation gives $\underline{E}(y - x) = -1$ and $\underline{E}(z - x) = -1$, thus $x \in Opt_{max}$. Assume $x \in Opt_{pos}$, which means that there exists $c \in [1,3] \times [1,3] \times \{0\} \times \{0\}$ such that $c^T x \geq c^T y$ and $c^T x \geq c^T z$. It implies that

$$2c_1 + 2c_2 \geq c_1 + 4c_2 \text{ and } 2c_1 + 2c_2 \geq 4c_1 + c_2 \tag{47}$$

$$\Leftrightarrow c_1 \geq 2c_2 \text{ and } c_2 \geq 2c_1. \tag{48}$$

Since (48) cannot be true, we get a contradiction and thus $x \notin Opt_{pos}$.

5.4.3. Problem 01COP

We give the characterizations for non-dominated solutions with respect to the maximality and E-admissibility criteria for Problem 01COP. In this case the set of feasible acts $\mathcal{X}$ is not convex. Somewhat surprisingly, as we are going to show, the two sets of non-dominated solutions still coincide.

For any $x \in \mathcal{X}$, let $\bar{c}^{xr}$ be the scenario associated to x in focal set F_r, such that

$$\bar{c}_i^{xr} = u_i^r \text{ if } x_i = 1,\ \bar{c}_i^{xr} = l_i^r \text{ if } x_i = 0. \tag{49}$$

Lemma 1 is simple but it is the key element to uncover the characterization of the maximality criterion.

Lemma 1. *For any $x, y \in \mathcal{X}$,*

$$\min_{c \in F_r}\ c^T y - c^T x = (\bar{c}^{xr})^T y - (\bar{c}^{xr})^T x.$$

Proof. For any $c \in F_r$,

$$c^T y - c^T x = \sum_{i=1}^{n} c_i(y_i - x_i) = \sum_{i:x_i=0}^{n} c_i(y_i - x_i) + \sum_{i:x_i=1}^{n} c_i(y_i - x_i) \tag{50}$$

$$\geq \sum_{i:x_i=0}^{n} l_i^r(y_i - x_i) + \sum_{i:x_i=1}^{n} u_i^r(y_i - x_i) \tag{51}$$

$$= (\bar{c}^{xr})^T y - (\bar{c}^{xr})^T x \tag{52}$$

where the inequality (51) holds because if $x_i = 0$ then $y_i - x_i \geq 0$ and if $x_i = 1$ then $y_i - x_i \leq 0$. □

Denote by $\bar{c}^x$ the set of coefficients in which $\bar{c}_i^x = \sum_{r=1}^{K} m(F_r)\bar{c}_i^{xr}$. Hence, we have:

$$\bar{c}_i^x = \bar{u}_i \text{ if } x_i = 1,\ \bar{c}_i^x = \bar{l}_i \text{ if } x_i = 0. \tag{53}$$

A characterization of solutions in Opt_{max} is given as follows.

Proposition 12. *For Problem 01COP, a solution $x \in Opt_{max}$ iff x is an optimal solution under $\bar{c}^x$.*

Proof. By definition,

$$x \in Opt_{max} \Leftrightarrow \nexists y \text{ such that } y \succ_{max} x \Leftrightarrow \nexists y \text{ such that } \underline{E}(y - x) > 0 \tag{54}$$

$$\Leftrightarrow\ \forall y \in \mathcal{X},\ \sum_{r=1}^{K} m(F_r) \min_{c \in F_r}(c^T y - c^T x) \leq 0 \tag{55}$$

$$\Leftrightarrow\ \forall y \in \mathcal{X},\ \sum_{r=1}^{K} m(F_r)((\bar{c}^{xr})^T y - (\bar{c}^{xr})^T x) \leq 0 \ \ \text{(Lemma 1)} \tag{56}$$

$$\Leftrightarrow\ \forall y \in \mathcal{X},\ \sum_{r=1}^{K} m(F_r) \sum_{i=1}^{n} \bar{c}_i^{xr} y_i \leq \sum_{r=1}^{K} m(F_r) \sum_{i=1}^{n} \bar{c}_i^{xr} x_i \tag{57}$$

$$\Leftrightarrow\ \forall y \in \mathcal{X}, \sum_{i=1}^{n} \bar{c}_i^x y_i \leq \sum_{i=1}^{n} \bar{c}_i^x x_i \tag{58}$$

Hence, $x \in Opt_{max}$ iff x is an optimal solution under $\bar{c}^x$. □

Proposition 12 offers a method to check if a given feasible solution x belongs to Opt_{max}. To do so, one first calculates the optimal value, z_x, of Problem 01COP with $c_i = \bar{c}_i^x$ and then compares $\sum_{i=1}^{n} \bar{c}_i^x x_i$ with z_x. Moreover, the following characterization provides a way to identify a solution in Opt_{max} by solving Problem 01COP under some $c^o \in \mathcal{C}$.

Proposition 13. *For Problem 01COP, a solution $x \in Opt_{pos}^{\mathcal{C}}$ iff x is optimal under $\bar{c}^x$.*

Proof. One direction is obvious. We only need to show the other direction. Assume that x is optimal under $c^o \in \mathcal{C}$. Then for any y, we have

$$\sum_{i=1}^{n} c_i^o x_i \geq \sum_{i=1}^{n} c_i^o y_i \tag{59}$$

$$\Leftrightarrow \sum_{i:x_i=1,y_i=0} c_i^o \geq \sum_{i:y_i=1,x_i=0} c_i^o \tag{60}$$

$$\Rightarrow \sum_{i:x_i=1,y_i=0} \bar{u}_i \geq \sum_{i:y_i=1,x_i=0} \bar{l}_i \tag{61}$$

$$\Leftrightarrow \sum_{i:x_i=1,y_i=0} \bar{u}_i + \sum_{i:x_i=y_i=1} \bar{u}_i \geq \sum_{i:y_i=1,x_i=0} \bar{l}_i + \sum_{i:x_i=y_i=1} \bar{u}_i \tag{62}$$

$$\Leftrightarrow \sum_{i=1}^{n} \bar{c}_i^x x_i \geq \sum_{i=1}^{n} \bar{c}_i^x y_i. \tag{63}$$

Hence, x is optimal under $\bar{c}^x$. □

Remark 4. *The proof of Proposition 13 is essentially the same as the proof of [29, Theorem 2.1] where the authors characterize weak trees.*

We are now in the position to provide a characterization for E-admissibility. We remark here that although the feasible acts $\mathcal{X}$ of Problem 01COP may not be in the form $Mx \leq b$, the convex hull $\text{conv}(\mathcal{X})$ is still a bounded polyhedron as $\mathcal{X}$ is a finite set. Hence, it still follows from Propositions 10 and 11 that x is E-admissible iff $x \in Opt_{pos}^{\mathcal{C}}$. However, the nature of Problem 01COP makes it possible to derive a proof for this fact, without relying on the powerful Proposition 10. We feel that it is useful to present a simpler proof here.

Proposition 14. *For Problem 01COP, a solution x is in Opt_{adm} iff x is an optimal solution under $\bar{c}^x$.*

Proof. If $x \in Opt_{adm}$ then $x \in Opt_{max}$, by Proposition 12 x is a optimal solution under $\bar{c}^x$. Assume that x is an optimal solution with $c_i = \bar{c}_i^x$. We construct an allocation map a of m as:

$$a(\bar{c}^{xr}, F_r) = m(F_r),\ \forall r \in \{1, \dots, K\}. \tag{64}$$

We define a discrete probability measure P such that

$$P(\{c\}) = \sum_{\bar{c}^{xr}=c} a(\bar{c}^{xr}, F_r). \tag{65}$$

Thanks to [27, Theorem 1], we have $P \in \mathcal{P}(m)$. It is easy to see that $E_P(\pi_i) = \bar{u}_i$ if $x_i = 1$ and $E_P(\pi_i) = \bar{l}_i$ if $x_i = 0$. Since x is optimal and by Equation (45), $E_P(x) \geq E_P(y)$ for any y. Therefore, x is E-admissible.

□

Consequently, we arrive to the main result.

Proposition 15. *If Problem 01COP is a maximization problem then the following are equivalent:*
(i) $x \in Opt_{max}$.
(ii) $x \in Opt_{adm}$.
(iii) x is an optimal solution under $\bar{c}^x$.
(iv) $x \in Opt^{\mathcal{C}}_{pos}$.

Let $\underline{c}^x$ be the set of coefficients, defined as follows:

$$\underline{c}^x_i = \bar{l}_i \text{ if } x_i = 1,\ \underline{c}^x_i = \bar{u}_i \text{ if } x_i = 0. \tag{66}$$

Likewise, we have the next result.

Proposition 16. *If Problem 01COP is a minimization problem then the following are equivalent:*
(i) $x \in Opt_{max}$.
(ii) $x \in Opt_{adm}$.
(iii) x is an optimal solution under $\underline{c}^x$.
(iv) $x \in Opt^{\mathcal{C}}_{pos}$.

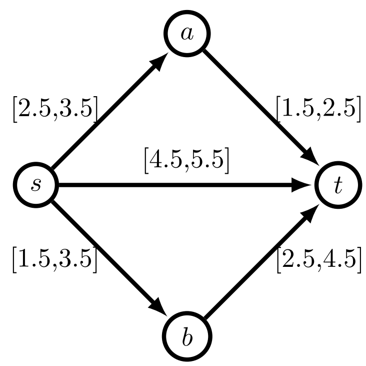


Figure 6: The graph associated with Opt_{max} and Opt_{adm} in which weights of arc (i, j) are in the interval $[\bar{l}_{ij}, \bar{u}_{ij}]$.

Example 12 (Example 5 continued)**.** *The graph in Figure 6 contains information about* Opt_{max} *(or, equivalently,* Opt_{adm}*). For instance, s-a-t* $\in Opt_{max}$ *since it is optimal under the set of arc weights* c_{sa}=2.5, c_{at}=1.5, c_{st}=5.5, c_{sb}=3.5, *and* c_{bt}=4.5. *By setting the arc weights to* c_{sa}=3, c_{at}=2.5, c_{st}=5, c_{sb}=3, *and* c_{bt}=4, *the optimal path is s-t, which also belongs to* Opt_{max}*. The set* Opt_{max} *consists of s-a-t, s-b-t, and s-t.*

The characterization we provided is particularly valuable for E-admissibility. As noted in [2], verifying whether an act is E-admissible typically involves solving a large linear programming problem. However, Propositions 15 and 16 imply that if Problem 01COP can be solved efficiently (e.g., Problem SPP), checking E-admissibility is also efficient.

Remark 5. *Since $\succeq_{max}$ is a partial relation, Opt_{max} may need to be enumerated. For some problems, such as the SPP, the size of Opt_{weak} (and therefore, the size of Opt_{max}) grows exponentially with $|V|$ [14], making the enumeration a very time-consuming process. Preprocessing can be applied to speed up the process by eliminating the elements x_i which are never in any solution of Opt_{max}. We note that determining whether $x_i = 1$ is part of a possibly optimal solution (i.e., solution in Opt_{max}) is NP-hard for many polynomially solvable problems such as the SPP or the assignment problem [17]. Nonetheless, for an important class of combinatorial optimization problems, i.e., the matroidal problem (which includes the minimum spanning tree problem), Kasperski et al. [18] showed that this determination can be done efficiently.*

6. Conclusion

In this paper, we have considered a very general optimization problem with a linear objective function (LOP). When coefficients of the objective are evidential, the notion of optimal solution is ill-defined. Therefore, we propose extensions of the notion of optimal solutions to this context, as the sets of non-dominated solutions according to the generalized Hurwicz, strong dominance, weak dominance, maximality and E-admissibility criteria. By considering the particular case where focal sets are Cartesian products of compact sets, we are able to characterize the non-dominated solutions in terms of various concepts in optimization. This makes it possible to find non-dominated solutions by solving known variants of the deterministic version of the LOP or even, in some cases, simply by solving the deterministic version. Specifically, non-dominated acts with respect to generalized Hurwicz are solutions of the *deterministic* LOP. Non-dominated acts with respect to generalized Hurwicz under unknown optimism/pessimism degree are solutions of the *parametric* LOP. Non-dominated acts with respect to strong dominance are solutions of a *lower-bound feasibility problem*. Non-dominated acts with respect to weak dominance correspond exactly to the *efficient* solutions of the bi-objective LOP problem. Lastly, non-dominated acts with respect to maximality and E-admissibility are linked to the robust optimization framework via the concept of *possibly optimal solutions* of the LOP.

Topics of future research include i) finding a characterization of the maximality criterion for linear mixed integer programming problems; ii) providing a polynomial representation of all non-dominated solutions with respect to maximality and E-admissibility for combinatorial optimization problems or at least for matroidal problems. Since these latter solutions are also possibly optimal, one possible direction is to expand the works of [9], in which a compact representation of possibly optimal solutions is given for the item selection problem (a special case of matroidal problems).

References

[1] Aguirre, F., Destercke, S., Dubois, D., Sallak, M., Jacob, C.: Inclusion–exclusion principle for belief functions. Int. J. Approx. Reason. **55**(8), 1708–1727 (2014)

[2] Augustin, T., Coolen, F.P., De Cooman, G., Troffaes, M.C.: Introduction to imprecise probabilities. John Wiley & Sons (2014)

[3] Byers, T.H., Waterman, M.S.: Determining all optimal and near-optimal solutions when solving shortest path problems by dynamic programming. Oper. Res. **32**(6), 1381–1384 (1984)

[4] Carlyle, W.M., Wood, R.K.: Near-shortest and k-shortest simple paths. Networks **46**(2), 98–109 (2005)

[5] Chanas, S., Zieliński, P.: The computational complexity of the criticality problems in a network with interval activity times. Eur. J. Oper. Res. **136**(3), 541–550 (2002)

[6] Couso, I., Moral, S.: Independence concepts in evidence theory. Int. J. Approx. Reason. **51**(7), 748–758 (2010)

[7] Denoeux, T.: Decision-making with belief functions: a review. Int. J. Approx. Reason. **109**, 87–110 (2019)

[8] Destercke, S.: A decision rule for imprecise probabilities based on pairwise comparison of expectation bounds. In: Borgelt, C., González-Rodríguez, G., Trutschnig, W., Lubiano, M.A., Gil, M.Á., Grzegorzewski, P., Hryniewicz, O. (eds.) Combining Soft Computing and Statistical Methods in Data Analysis. Advances in Intelligent and Soft Computing, vol. 77, pp. 189–197. Springer (2010)

[9] Destercke, S., Guillaume, R.: Necessary and possibly optimal items in selecting problems. In: Ciucci, D., Couso, I., Medina, J., Slezak, D., Petturiti, D., Bouchon-Meunier, B., Yager, R.R. (eds.) Information Processing and Management of Uncertainty in Knowledge-Based Systems - 19th International Conference, IPMU 2022, Part I. Communications in Computer and Information Science, vol. 1601, pp. 494–503. Springer (2022)

[10] Dias, L.C., Clímaco, J.N.: Shortest path problems with partial information: models and algorithms for detecting dominance. Eur. J. Oper. Res. **121**(1), 16–31 (2000)

[11] Duque, D., Lozano, L., Medaglia, A.L.: An exact method for the biobjective shortest path problem for large-scale road networks. Eur. J. Oper. Res. **242**(3), 788–797 (2015)

[12] Guillaume, R., Kasperski, A., Zieliński, P.: Robust optimization with scenarios using random fuzzy sets. In: Proc. of FUZZ-IEEE 2021. pp. 1–6. IEEE (2021)

[13] Gusfield, D.M.: Sensitivity analysis for combinatorial optimization. Tech. rep., EECS Department, University of California, Berkeley (May 1980)

[14] Hansen, P.: Bicriterion path problems. In: Fandel, G., Gal, T. (eds.) Multiple criteria decision making theory and application. pp. 109–127. Springer (1980)

[15] Helal, N., Pichon, F., Porumbel, D., Mercier, D., Lefèvre, E.: The capacitated vehicle routing problem with evidential demands. Int. J. Approx. Reason. **95**, 124–151 (2018)

[16] Inuiguchi, M., Sakawa, M.: Minimax regret solution to linear programming problems with an interval objective function. Eur. J. Oper. Res. **86**(3), 526–536 (1995)

[17] Kasperski, A.: Discrete optimization with interval data. Springer (2008)

[18] Kasperski, A., Zieliński, P.: On combinatorial optimization problems on matroids with uncertain weights. Eur. J. Oper. Res. **177**(2), 851–864 (2007)

[19] Kouvelis, P., Yu, G.: Robust discrete optimization and its applications, Nonconvex Optimization and Its Applications, vol. 14. Springer US (1996)

[20] Quaeghebeur, E., Shariatmadar, K., De Cooman, G.: Constrained optimization problems under uncertainty with coherent lower previsions. Fuzzy Sets Syst. **206**, 74–88 (2012)

[21] Shafer, G.: A mathematical theory of evidence. Princeton university press (1976)

[22] Tedjini, T., Afifi, S., Pichon, F., Lefèvre, E.: The vehicle routing problem with time windows and evidential service and travel times: A recourse model. In: Vejnarová, J., Wilson, N. (eds.) Proc. of ECSQARU 2021. pp. 381–395. Springer (2021)

[23] Troffaes, M.C.: Decision making under uncertainty using imprecise probabilities. Int. J. Approx. Reason. **45**(1), 17–29 (2007)

[24] Vanderbei, R.J.: Linear programming. Springer (2020)

[25] Vu, T.A., Afifi, S., Lefèvre, É., Pichon, F.: On modelling and solving the shortest path problem with evidential weights. In: Le Hégarat-Mascle, S., Bloch, I., Aldea, E. (eds.) Belief Functions: Theory and Applications. pp. 139–149. Springer International Publishing, Cham (2022)

[26] Walley, P.: Statistical reasoning with imprecise probabilities, vol. 42. Springer (1991)

[27] Wasserman, L.A.: Belief functions and statistical inference. Can. J. Stat. **18**(3), 183–196 (1990)

[28] Wolsey, L.A., Nemhauser, G.L.: Integer and combinatorial optimization, vol. 55. John Wiley & Sons (1999)

[29] Yaman, H., Karaşan, O.E., Pınar, M.Ç.: The robust spanning tree problem with interval data. Oper. Res. Lett. **29**(1), 31–40 (2001)

[30] Yu, G., Yang, J.: On the robust shortest path problem. Comput. Oper. Res. **25**(6), 457–468 (1998)