Relevance and Truthfulness in Information Correction and Fusion*

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February 24, 2011

*This paper is an extended and revised version of [6] and [20].

Abstract

A general approach to information correction and fusion for belief functions is proposed, where not only may the information items be irrelevant, but sources may lie as well. We introduce a new correction scheme, which takes into account uncertain metaknowledge on the source's relevance and truthfulness and that generalizes Shafer's discounting operation. We then show how to reinterpret all connectives of Boolean logic in terms of source behavior assumptions with respect to relevance and truthfulness. We are led to generalize the unnormalized Dempster's rule to all Boolean connectives, while taking into account the uncertainties pertaining to assumptions concerning the behavior of sources. Eventually, we further extend this approach to an even more general setting, where source behavior assumptions do not have to be restricted to relevance and truthfulness. We also establish the commutativity property between correction and fusion processes, when the behaviors of the sources are independent.

Keywords: Dempster-Shafer theory, Belief functions, Evidence theory, Boolean logic, Information fusion, Discounting.

Quand deux témoins me disent une chose, il faut, pour que je me trompe en ajoutant foi à leur témoignage, que l'un & l'autre m'induisent en erreur; si je suis sûr de l'un des deux, peu m'importe que l'autre soit croyable. Or la probabilité que l'un & l'autre me trompent, est une probabilité composée de deux probabilités, que le premier trompe, & que le second trompe. Celle du premier est 1/10 (puisque la probabilité que la chose est conforme à son rapport est 9/10); la probabilité que le second me trompe aussi, est encore 1/10: donc la probabilité composée est la dixième d'une dixième ou 1/100; donc la probabilité du contraire, c'est-à-dire celle que l'un ou l'autre dit vrai, est 99/100. Entry "Probabilité", Encyclopedia of D'Alembert and Diderot, XVIIIth century.

1 Introduction

The problem of constructing an agent's knowledge on the value taken by a parameter x defined on a domain X, where the agent's sole information on the parameter comes from one or many sources, has gained increased interest in the last twenty years with the development of various kinds of information systems. This problem is actually as old as probability theory: its roots can be traced at least back to the formalization of the reliability of testimonies (see, for instance, the entry "Probabilité" in D'Alembert and Diderot's famous XVIIIth century Encyclopedia[1]).

It is not possible for an agent to evaluate the pieces of information provided by several sources, unless some meta-knowledge on the sources is available to this agent. Typically, meta-knowledge on the sources amounts to assumptions about their relevance. If a source providing a testimony of the form $x \in A$ is relevant with probability p, then one assumes that the corresponding information is not useful with probability $1 - p$. In the context of the theory of belief functions [4, 22, 31], this is known as the *discounting* of a piece of information [22, 27] and the resulting state of knowledge is represented by a simple support function [22]: the weight p is allocated to the fact of being able to state $x \in A$ with certainty, and the weight $1 - p$ is allocated to the tautology (it becomes the probability of knowing nothing from the source). If the agent receives the piece of information $x \in A$ from two independent sources, with respective reliabilities p_1 and p_2, then Dempster's rule of combination [4, 22], justifies attaching reliability $p_1 + p_2$ - $p_1 p_2$ to the statement $x \in A$ (this was already explained in full details in the D'Alembert and Diderot Encyclopedia, see the above-mentioned entry).

In this paper, it is proposed to also take into account some meta-knowledge on the truthfulness of the sources. We study how the information provided by a single source is modified, or *corrected* [16], when the agent has some uncertain meta-knowledge on relevance and truthfulness of the source. The case where multiple sources provide information is also thoroughly investigated. This study is performed in the framework of the theory of belief functions. It leads to a general approach to the correction (single source case) and fusion (multiple sources case) of belief functions. This exploration is then pushed forward and further extended to an even more general setting, where

[1] Encyclopédie, ou dictionnaire raisonné des sciences, des arts et des métiers. D. Diderot and J. le Rond D'Alembert, editors. University of Chicago: ARTFL Encyclopédie Projet (Winter 2008 Edition), Robert Morrissey (ed).
http://artflx.uchicago.edu/cgi-bin/philologic/getobject.pl?c.99:87.encyclopedie0110.362669

assumptions about information sources do not have to be restricted to relevance and truthfulness.

The rest of this paper is organized as follows. The notion of truthfulness is added to the notion of relevance in Section 2, where a thorough study of what this addition brings to the problems of information correction and fusion is conducted. In Section 3, this investigation is pursued by allowing for general source behavior assumptions that go beyond the notions of relevance and truthfulness. A link between information correction and fusion processes, when the behaviors of the sources are independent, is exhibited in Section 4. Some relationships with previous works are outlined in Section 5. Section 6 concludes the paper.

2 Relevance and truthfulness

It is assumed here that the reliability of a source of information involves two dimensions: its *relevance* and its *truthfulness*. A source is said to be relevant if it provides useful information regarding a given question of interest. If the source is a human agent, irrelevance means that the provided information does not pertain to the question it answers, for instance because the agent is actually ignorant. If the source is a sensor, the sensor response is typically irrelevant when it is out of order. For instance, it is useless to try and find the time it is from a clock that is not working since there is no way to know whether the supplied information is correct or not (the hour read on a broken watch can even be correct). In contrast, a source is said to be truthful if it actually supplies the information it possesses. There are various forms of lack of truthfulness. A source may declare the contrary of what it knows, or just say less, or something different, even if consistent with its knowledge. Lack of truthfulness for a sensor may take the form of a systematic bias. Note that if the the agent receiving information does not know in which way the source lies, the difference between irrelevance and lack of truthfulness of a source becomes itself less significant from the standpoint of this agent.

2.1 The case of a single source

Suppose a single source provides information on the value of some deterministic parameter x ranging on a set X of possible values (for instance, somebody's birth-date). Such a piece of information may be of the form "All the source knows is that $x \in A$" where A is a proper non-empty subset of X, supposedly containing the actual value of x. We assume that $\emptyset \subset A \subset X$ because we consider as a source any entity that supplies a non-trivial and non-self-contradictory input. If the source declares not to know the value of x, this would be modeled by $A = X$. However, such information is immaterial for the purpose of information fusion. For simplicity, in the following, we shall assume a crude description of the lack of truthfulness, namely that the source declares the opposite of what it knows to be true. The difference between a source known to lie in this way, and a source known to be irrelevant is that it is possible to retrieve the actual information from the former, while the latter is totally useless.

Knowledge about whether a source is reliable or not, truthful or not differs from the knowledge supplied by the source. It is higher order knowledge and is called *meta-knowledge*. If the source that declares $x \in A$ is known to be irrelevant, the agent receiving this information can always replace it by the trivial information $x \in X$, whether the source is truthful or not. If the source is relevant and is known to lie in the way assumed above, the agent should replace it by $x \in \overline{A}$, where $\overline{A}$ is the complement of A.

2.1.1 Crisp testimony and uncertain meta-knowledge

However, the difficulty is that, in general, the meta-information is uncertain. Consider the frame of discernment $\mathcal{H}$ describing the possible states of the source. Define $\mathcal{H} = \mathcal{R} \times \mathcal{T}$, with $\mathcal{R} = \{R, \neg R\}$ and $\mathcal{T} = \{T, \neg T\}$, as the domain of the pair of Boolean variables (h_R, h_T), where R means relevant and T means truthful. Meta-knowledge about a source may take the form of subjective probabilities $prob(h_R, h_T)$ about the state of the source. Following Dempster's approach [4], a multiple-valued function Γ_A from $\mathcal{H}$ to X can be defined such that:

$$\begin{aligned} \Gamma_A(R, T) &= A; \\ \Gamma_A(R, \neg T) &= \overline{A}; \\ \Gamma_A(\neg R, T) &= \Gamma(\neg R, \neg T) = X. \end{aligned}$$

$\Gamma_A(h)$ interprets the testimony $x \in A$ in each configuration h of the source. Hence, this piece of information will be systematically interpreted by a belief function in the sense of Shafer [22], with mass function m^X on X defined by

$$\begin{aligned} m^X(A) &= prob(R, T) \\ m^X(\overline{A}) &= prob(R, \neg T) \\ m^X(X) &= prob(\neg R) = prob(\neg R, T) + prob(\neg R, \neg T). \end{aligned}$$

A mass function m^X on X is formally a probability distribution on the power set of X (hence $\sum_{A \subseteq X} m^X(A) = 1$). In this uncertainty theory, the mass $m^X(A)$ is assigned to the possibility of stating $x \in A$ as a faithful representation of the available knowledge; it does not evaluate the likelihood of event A like does a subjective probability $prob(A)$. Philosophically, and in analogy to modal logic, the probability $prob(A)$ could be called a *de re* probability, while $m^X(A)$ can be understood as a *de dicto* probability (in opposition to the usual probabilistic tradition).

Let $q = prob(T|R)$ and $p = prob(R)$. Assuming $\emptyset \subset A \subset X$, it is easily found that

$$m^X(A) = p \cdot q; \tag{1}$$
$$m^X(\overline{A}) = p \cdot (1 - q); \tag{2}$$
$$m^X(X) = 1 - p, \tag{3}$$

corresponding, respectively, to the cases where the source is relevant and truthful, relevant and untruthful, and irrelevant. In practice, it can be assumed that the relevance of a source is independent of its truthfulness, although equations (1)-(3) show that this is not necessary in our approach. In this case, the probability distribution on $\mathcal{H}$ is defined from the probability $p = prob(R)$ that it is relevant and $q = prob(T)$ the probability of its being truthful.

2.1.2 Uncertain testimony and meta-knowledge

More generally, one may assume that the information supplied by a source already takes the form of any kind of mass function m_S^X on X (especially, $m_S^X(X) > 0$ and/or

$m_S^X(\emptyset) > 0$ could be allowed). Assuming that the source is in a given state h, then each mass $m_S^X(A)$ should be transferred to $\Gamma_A(h)$, yielding the following mass function:

$$m^X(B|h) = \sum_{A:\Gamma_A(h)=B} m_S^X(A), \quad \forall B \subseteq X. \tag{4}$$

When meta-knowledge on the source is uncertain and each state h has a probability $prob(h)$, then (4) implies that:

$$m^X(B) = \sum_h m^X(B|h)prob(h) = \sum_h prob(h) \sum_{A:\Gamma_A(h)=B} m_S^X(A). \tag{5}$$

Let us already remark that (5) may also be recovered using standard operations of belief function theory (i.e., vacuous extension, Dempter's rule of combination and marginalization) on the considered pieces of evidence (namely the uncertain testimony and meta-knowledge), as will be shown in Section 4.2 (Lemma 1).

Assuming the uncertain meta-knowledge of the preceding section, i.e., $prob(R, T) = pq$, $prob(R, \neg T) = p(1 - q)$, $prob(\neg R, T) = (1 - p)q$ and $prob(\neg R, \neg T) = (1 - p)(1 - q)$, leads then to transforming the mass function m_S^X into a new mass function denoted by m^X and defined by:

$$m^X = pq\, m_S^X + p(1 - q)\, \overline{m}_S^X + (1 - p)\, m_X^X, \tag{6}$$

where $\overline{m}_S^X$ is the (random) set complement of m [7], defined by $\overline{m}_S^X(A) = m_S^X(\overline{A}), \forall A \subseteq X$, and m_X^X the vacuous mass function defined by $m_X^X(X) = 1$. We thus get

$$m^X(A) = pq\, m_S^X(A) + p(1 - q)\, m_S^X(\overline{A})$$

for all $A \neq X$ and

$$m^X(X) = pq\, m_S^X(X) + p(1 - q)m_S^X(\emptyset) + 1 - p.$$

This is clearly a generalization of the notion of discounting of a belief function proposed by Shafer [22] to integrate the reliability of information sources. In the model underlying the discounting operation, the lack of reliability of a source is assumed to originate in some flaw making it irrelevant. Our approach adds the possibility of the source lacking truthfulness, i.e., lying[2]. Let us also remark that the complement of a mass function is recovered as a special case of this approach: it corresponds to a relevant source that is lying.

One may as well consider more complex assumptions corresponding to subsets of $\mathcal{H}$, representing epistemic states of the receiving agent about the source state. For instance, the agent may know that:

- The source is either relevant or truthful but not both, that is $R \oplus T = (R \wedge \neg T) \vee (\neg R \wedge T)$, corresponding to the disjunction of two mutually exclusive states.
- The source is relevant or truthful, which is the disjunction $R \vee T$ of three possible mutually exclusive states $\neg R \wedge T$, $R \wedge \neg T$ and $R \wedge T$.

[2]We use the term "lying" here as a synonym of "not telling the truth", irrespective of the existence of any intention of an agent to deceive.

Let $H \subseteq \mathcal{H}$ be some assumption of this type about the source. Using a common abuse of notation, the image of H under Γ_A will be denoted as $\Gamma_A(H)$. It is defined as

$$\Gamma_A(H) = \bigcup_{h \in H} \{\Gamma_A(h)\}.$$

The above results can be applied to such non-elementary assumptions. However, this is not so useful in the case of a single source, since $\Gamma_A(H) = X$ as long as H is not elementary. The major appeal of non-elementary assumptions in the case of several sources will become patent in the sequel.

2.2 The case of multiple sources

If there are two sources of information, two approaches can be envisaged:

1. Modifying information items supplied by each source, then merging the resulting belief functions (using Dempster's rule [22] or its unnormalized version [29]).
2. Embedding meta-knowledge about the source state inside the merging process.

It is clear that the latter option looks more general and more convincing. In this case, a joint assumption on the relevance and truthfulness of sources is in order. Denoting by $\mathcal{H}_1$ and $\mathcal{H}_2$, the set of possible state configurations of each source, the set of elementary joint state assumptions on sources will be $\mathcal{H}_{12} = \mathcal{H}_1 \times \mathcal{H}_2$. Hence, there are 16 possible states of the pair of sources $(h^1_R, h^1_T, h^2_R, h^2_T)$. Uncertain meta-knowledge about the state of sources must be expressed on $\mathcal{H}_{12}$. In the following we describe the result of making elementary assumptions on sources; then we consider the case of uncertain meta-knowledge and uncertain sources, and we are led to equip the assumption space itself with a belief structure.

2.2.1 Crisp testimonies and precise meta-knowledge

Suppose that source 1 asserts $x \in A$ and source 2 asserts $x \in B$ where $A, B \neq X$. How to combine these pieces of information depends on the chosen assumption on the state of sources. Namely, there is a multiple-valued mapping $\Gamma_{A,B} : \mathcal{H}_{12} \to 2^X$ prescribing, for each elementary assumption, the result of the process of merging the two information items.

1. Suppose both sources are truthful.
   (a) If they are both relevant, then one must conclude that $x \in A \cap B$;
   (b) If source 2 (resp. 1) is irrelevant, then one must conclude that $x \in A$ (resp. B).
   (c) Else $x \in X$.
2. Suppose source 1 truthful and source 2 lies.
   (a) If they are both relevant, then one must conclude that $x \in A \cap \overline{B}$;
   (b) If source 2 (resp. 1) is irrelevant, then one must conclude that $x \in A$ (resp. $\overline{B}$).

(c) Else $x \in X$.

3. Suppose source 2 is truthful and source 1 lies.

   (a) If they are both relevant, then one must conclude that $x \in \overline{A} \cap B$;

   (b) If source 2 (resp. 1) is irrelevant, then one must conclude that $x \in \overline{A}$ (resp. B).

   (c) Else $x \in X$.

4. Suppose both sources lie.

   (a) If they are both relevant, then one must conclude that $x \in \overline{A} \cap \overline{B}$;

   (b) If source 2 (resp. 1) is irrelevant, then one must conclude that $x \in \overline{A}$ (resp. $\overline{B}$).

   (c) Else $x \in X$.

Obviously, the four binary connectives $A \cap B$, $\overline{A} \cap B$, $A \cap \overline{B}$, and $\overline{A} \cap \overline{B}$ are obtained, depending on the truthfulness of supposedly relevant sources. Note that elementary assumptions may be incompatible with some available pieces of information. For instance, in case of conflicting information ($A \cap B = \emptyset$), case 1a is obviously impossible: either one of the sources is irrelevant, or one of them lies. Likewise, case $A \cap \overline{B} = \emptyset$ excludes assumption 2a, and case $A \cup B = X$ excludes the assumption that both sources lie.

2.2.2 Crisp testimonies and incomplete meta-knowledge

Other Boolean binary connectives can be retrieved by considering non-trivial non-elementary assumptions $H \subset \mathcal{H}_{12}$ on the state of sources, namely disjunctions of elementary assumptions. In theory, the number of such composite assumptions is huge (2^{16}). In practice, only a few assumptions are interesting to study. Indeed, the resulting information is trivial ($x \in X$ because $\Gamma_{A,B}(H) = X$) as soon as H contains an elementary assumption of the form $(\neg R, h_T^1, \neg R, h_T^2)$, for instance. Some forms of non trivial incomplete meta-knowledge are worth considering.

A first kind of non-trivial meta-knowledge consists in guessing the number of truthful and/or relevant sources, by lack of knowledge on the reliability of individual sources. The interesting cases are as follows (alternative weaker assumptions of this form generate no information):

- Both sources are relevant, and at least one of them is truthful. This is the disjunction of assumptions 1a, 2a, and 3a. Then, $x \in A \cup B$ follows.

- Both sources are relevant, exactly one of which is truthful. This is the disjunction of assumptions 2a and 3a. Then, $x \in A \triangle B$ (*exclusive or*).

- Both sources are relevant, at most one of which truthful. This is the disjunction of assumptions 2a, 3a, and 4a. Then, $x \in \overline{A} \cup \overline{B}$.

- Both sources are truthful, and at least one of them is relevant. This is the disjunction of assumptions 1b and 1a. Then, again $x \in A \cup B$. The same results would be obtained by assuming truthful sources truthful, *exactly* one of them being relevant.

Another kind of meta-knowledge pertains to logical dependence between source states. For instance, one may know that both sources are relevant, but source 1 is truthful if and only if source 2 is so too. This is the disjunction of assumptions 1a and 4a, yielding $x \in (A \cap B) \cup (\overline{A} \cap \overline{B})$, which corresponds to the Boolean equivalence connective. One could likewise retrieve the connective $A \cup \overline{B}$ postulating that:

- Both sources are relevant, but it is impossible that at the same time source 1 lies and source 2 is truthful;
- Or yet that either source 1 is truthful while the other is irrelevant, or source 2 lies and the first one is irrelevant.

This assumption is captured by the implication *if B then A*, which boils down to the following piece of meta-knowledge: "If source 2 is truthful, then source 1 is truthful too" (in the case of relevant sources).

It is possible to retrieve *almost all* binary Boolean connectives of propositional logic (except $A \perp B = \emptyset$, already ruled out in the case of a single source, if one requires that the result of the merging process should be logically consistent). This is not surprising at all, in some sense. However the point here is that *each* logical connective can be derived from an assumption about the global quality of information sources, in terms of truthfulness and relevance. This kind of interpretation has been known for a long time for union and intersection only [8].

Actually, when modeling a complex assumption on quality of sources by means of the appropriate connective yielding the correct ensuing information drawn from these sources, part of the actual meta-information is lost. For instance, $x \in A \cup B$ is obtained in several distinct situations. However, the information that would result from a finer representation of the complex assumptions would be different in each case. Namely:

- If sources are both truthful, and *exactly* one is relevant, then either one should know that $x \in A$ or one should know that $x \in B$ (had we known which source is relevant);
- If both sources are relevant, and at least one is truthful, then either one should know that $x \in A \cap \overline{B}$, or one should know that $x \in \overline{A} \cap B$ or yet that $x \in A \cap B$ (had we known which source is truthful).

In both cases, *one can derive* that we know $x \in A \cup B$, which is weaker that the most precise pieces of information one could derive in each case. In order to express these subtle distinctions, modal logic could be instrumental since it is more expressive than propositional logic. Denoting $\Box$ the modality "to know", it is widely known that $\Box A \vee \Box B$ is not equivalent to (and weaker than) the formula $\Box(A \cap \overline{B}) \vee \Box(\overline{A} \cap B) \vee \Box(A \cap B)$ in a standard modal logic. This line of study would require an investigation in epistemic logic [13], and is left for further research. Nevertheless, the reader is referred to Banerjee and Dubois [1] for a more refined representation in a modal logic framework (and in terms of subsets of the power set of X) of what an agent knows about the epistemic state of another agent acting as a source of information.

2.2.3 Uncertain testimonies and sure meta-knowledge

Suppose now that one incomplete assumption $H \subset \mathcal{H}_{12}$ on the quality of the sources is known to be true. Let $\otimes_H$ be the set-theoretic connective associated to the assumption H in agreement with the above described assignment. Suppose that source 1 (resp. 2) supplies a mass function m_1^X (resp. m_2^X). Moreover we postulate that sources are *independent* in the following sense: interpreting $m_i^X(A)$ as the probability that source i supplies information item $x \in A$, then the probability that source 1 supplies information item $x \in A$ and source 2 supplies at the same time information item $x \in B$ is the product $m_1^X(A) \cdot m_2^X(B)$.

In this framework, the probability that should be assigned to the possibility of interpreting the joint information supplied by the sources by the statement $x \in C \subseteq X$ is equal to

$$m^X(C) = \sum_{A,B:C=A\otimes_H B} m_1^X(A) \cdot m_2^X(B). \tag{7}$$

This result is a straightforward consequence of the claim that if source 1 asserts $x \in A$ and source 2 asserts $x \in B$, then under assumption H, the conclusion should be that $x \in A \otimes_H B$ is what we actually know. There are 15 variants of this combination rule including the unnormalized version of Dempster's rule (also called conjunctive rule) (Smets [29]) and the disjunctive rule (Dubois et Prade [7]). Observe that when $A \otimes_H B = \emptyset$ for two focal sets A and B, each coming from a distinct source, this conflict no longer pertains to a disagreement inside X between the two sources, but to a conflict between the information items supplied by the two sources and the meta-assumption H, in space $\mathcal{H}_{12} \times X$. Several approaches make sense to cope with this conflict:

- Either renormalize the resulting belief function like with Dempster's rule, which amounts to assuming the correctness of assumption H, and conditioning on the assumption that sources should not contradict each other;
- Or reject assumption H and prefer one that is compatible with the information supplied by the sources.

Remark: When belief functions built from mass functions m_i^X are consonant, hence fully represented by their contour functions considered as possibility distributions $\pi_i^X : X \to [0, 1]$, one could choose to perform the fusion operation inside the possibilistic framework [9], replacing combination rule (7) by a fuzzy logic connective that extends $\otimes_H$ from the Boolean to the multiple-valued setting.

2.2.4 Crisp testimonies and uncertain meta-knowledge

Now we assume some uncertainty about the meta-knowledge regarding source quality. It is natural to try and represent this meta-uncertainty by means of a mass function $m^{\mathcal{H}}$ on the space $\mathcal{H}$ of incomplete assumptions, rather than a probability distribution on $\mathcal{H}_{12}$. At this point, we limit ourselves to the case where information supplied by sources are simple testimonies of the form $x \in A$ and $x \in B$ respectively. The result of the merging is a mass function m^X on X defined by:

$$m^X(C) = \sum_{H:A\otimes_H B=C} m^{\mathcal{H}_{12}}(H). \tag{8}$$

This mass function actually induces a probability distribution over the 15 Boolean binary connectives attached to assumptions H:

$$p^{\mathcal{H}_{12}}(\otimes) = \sum_{H:\otimes_H=\otimes} m^{\mathcal{H}_{12}}(H). \tag{9}$$

If one of the sources is non-informative ($B = X$), only three connectives remain possible (they reduce to $C = A, \overline{A}$ or X) and the interpretation of information supplied by a single source is recovered (Section 2.1.1).

This approach is general in the sense that, even if the information supplied by each source is independent from the information supplied by the other one, pieces of meta-knowledge regarding the states of each source may not be independent. Such a "meta-independence" between sources may be modeled by assuming that

$$m^{\mathcal{H}_{12}}(H) = \begin{cases} m^{\mathcal{H}_1}(H_1)m^{\mathcal{H}_2}(H_2) & \text{if } H = H_1 \times H_2 \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

which corresponds to evidential independence [22] between frames $\mathcal{H}_1$ and $\mathcal{H}_2$ with respect to $m^{\mathcal{H}_{12}}$. We note that this notion should not be confused with other notions of independence in evidence theory, as outlined, e.g., in [3] and [5].

For instance, assume truthful sources with independent probabilities of relevance p_1 and p_2: for $i = 1, 2$,

$$m^{\mathcal{H}_i}(\{(R_i, T_i)\}) = p_i, \tag{11}$$

$$m^{\mathcal{H}_i}(\{(\neg R_i, T_i)\}) = 1 - p_1. \tag{12}$$

We then have

$$m^{\mathcal{H}_{12}}(\{(R_1, T_1, R_2, T_2)\}) = m^{\mathcal{H}_1}(\{(R_1, T_1)\})m^{\mathcal{H}_2}(\{(R_2, T_2)\}) = p_1p_2 \tag{13}$$

and, similarly,

$$m^{\mathcal{H}_{12}}(\{(R_1, T_1, \neg R_2, T_2)\}) = p_1(1 - p_2), \tag{14}$$

$$m^{\mathcal{H}_{12}}(\{(\neg R_1, T_1, R_2, T_2)\}) = (1 - p_1)p_2, \tag{15}$$

$$m^{\mathcal{H}_{12}}(\{(\neg R_1, T_1, \neg R_2, T_2)\}) = (1 - p_1)(1 - p_2), \tag{16}$$

and $m^{\mathcal{H}_{12}}(H) = 0$ for all other $H \subseteq \mathcal{H}_{12}$.

Furthermore, it is easy to verify, under these specific hypotheses, that it is equivalent to combine discounted testimonies from each source (with discounting factors p_1 and p_2) by means of the unnormalized Dempster's rule, or to use the combination rule (8) proposed above using the mass function $m^{\mathcal{H}_{12}}$ defined by (13)-(16). Indeed, both methods yield the same mass function m^X:

$$\begin{array}{ll} m^X(A \cap B) = p_1p_2 & (H = (R_1, T_1, R_2, T_2)); \\ m^X(A) = p_1(1 - p_2) & (H = (R_1, T_1, \neg R_2, T_2)); \\ m^X(B) = (1 - p_1)p_2 & (H = (\neg R_1, T_1, R_2, T_2)); \\ m^X(X) = (1 - p_1)(1 - p_2) & (H = (\neg R_1, T_1, \neg R_2, T_2)). \end{array}$$

A more general form of this property will be studied in Section 4.

2.2.5 General case

Consider now the general case where information forwarded by independent sources are belief functions defined by independent mass functions m_1^X and m_2^X. The two merging operations (7) and (8) can be extended jointly by first selecting a merging operation $\otimes$ with probability $p^{\mathcal{H}_{12}}(\otimes)$, and then applying combination $\otimes$ between focal sets of m_1^X and m_2^X:

$$m^X(C) = \sum_H m^{\mathcal{H}_{12}}(H) \sum_{A,B:C=A\otimes_H B} m_1^X(A)m_2^X(B) \quad (17)$$

$$= \sum_{\otimes} p^{\mathcal{H}_{12}}(\otimes) \sum_{A,B:C=A\otimes B} m_1^X(A)m_2^X(B). \quad (18)$$

Here again, we may remark that a more formal derivation of the above result in a more general setting will be presented in Section 4.2 (Lemma 2).

The extension of this approach to the case of $n > 2$ sources that are more or less certainly truthful and/or relevant does not raise any theoretical issue. However the computational complexity will increase exponentially (since there will be 4^n elementary assumptions on the global state of sources, hence a 2^{4^n} complexity for the belief function expressing meta-knowledge on the sources, in the general case).

3 Beyond relevance and truthfulness: a general model of meta-knowledge

In the preceding section, we have seen that considering meta-knowledge on the relevance and truthfulness of information sources leads to some interesting results. In particular, a new correction scheme has been introduced, which generalizes the notions of discounting and complement of a belief function. It also becomes possible to reinterpret all connectives of Boolean logic in terms of assumptions with respect to the relevance and truthfulness of information sources. Furthermore, a general combination rule has been derived, which generalizes the unnormalized version of Dempster's rule to all Boolean connectives and that integrates the uncertainties pertaining to assumptions concerning the possible behavior or state of the sources in the fusion process itself.

In some applications, it may happen that one has finer or even different meta-knowledge on the sources than knowing their relevance and truthfulness. It seems interesting to be able to use such meta-knowledge. In this section, an approach to account for general source behavior assumptions is proposed, through a generalization of the preceding section. We study first the single source case before continuing with the multiple sources case.

3.1 The case of a single source

The notions of relevance and truthfulness were formalized in Section 2.1 using multivalued mappings Γ_A from $\mathcal{H} = \mathcal{R} \times \mathcal{T}$ to X, for each $A \subseteq X$. In this section, we propose a generalization of this setting to account for general source behavior assumptions.

3.1.1 Crisp testimony and certain meta-knowledge

Let us suppose that a source S provides a piece of information on the value taken by a variable y defined on a domain Y. We suppose that this piece of information takes the form $y \in A$, for some $A \subseteq Y$. Let us further assume that the source may be in N elementary states instead of four (as is the case in Section 2.1.1), i.e., we generalize the frame from $\mathcal{H} = \{(R,T),(R,\neg T),(\neg R,T),(\neg R,\neg T)\}$ to $\mathcal{H} = \{h_1,...,h_N\}$ (N does not need to be greater than or equal to four, as illustrated in Example 1 below). In addition, we consider that we are not so much interested in the value taken by y, as by the related value taken by a variable x defined on a domain X (x and y may or may not be the same parameter). Let us also assume that we have at our disposal some meta-knowledge that relate the piece of information $y \in A$ provided by the source on Y to an information of the form $x \in B$, for some $B \subseteq X$, for each possible state $h \in \mathcal{H}$ of the source.

The reasoning described in the previous paragraph can be formalized as follows. For each $A \subseteq Y$, we define a multivalued mapping Γ_A from $\mathcal{H}$ to X. $\Gamma_A(h)$ indicates how to interpret on X the piece of information $y \in A$ provided by the source in each configuration h of the source. As done in Section 2.1.2, we may also consider non elementary hypotheses $H \subseteq \mathcal{H}$, whose image by Γ_A is $\Gamma_A(H) = \cup_{h \in H}\Gamma_A(h)$.

It is easy to see that the setting introduced in Section 2.1.1 is a particular case of this general scheme, with $N = 4$ and $y = x$ and where the multivalued mappings Γ_A are defined by, for all $A \subseteq X$:

$$\begin{aligned} \Gamma_A(h_1) &= A, \\ \Gamma_A(h_2) &= \overline{A}, \\ \Gamma_A(h_3) &= \Gamma_A(h_4) = X. \end{aligned} \tag{19}$$

The states h_1, h_2, h_3 and h_4 then respectively correspond to the hypotheses (R,T), $(R,\neg T)$, $(\neg R,T)$ and $(\neg R,\neg T)$. More generally this framework also covers known canonical examples for belief function design such as the randomly coded message example, provided by Shafer and Tversky[24].

Furthermore, let us illustrate this general setting using two examples, where meta-knowledge on sources is not limited to notions of relevance and truthfulness.

Example 1 (Case $y = x$, inspired from Shafer [23])**.** *Let us assume that we are interested by the amount of money Glenn paid for his coffee dues. Besides, we consider that there are only four possible amounts:* 0*,* \$1*,* \$5 *or* \$10*. The only information we have on this amount comes from a person, named Bill, that we do not know very well and that may be informed, approximately informed or unreliable. If Bill is informed, whatever amount he provides should be accepted. If Bill is approximately informed, the amount he provides should be expanded using the lowest and highest closest amounts (e.g.,* \$1 *is expanded to* $\{0, \$1, \$5\}$*). If Bill is unreliable, the amount he provides cannot be used and we are left in our state of ignorance.*

Using the general scheme proposed above, we may formalize this problem as follows. We have $\mathcal{H} = \{\textit{informed, approximately informed, unreliable}\} = \{h_1, h_2, h_3\}$ *and* $X = \{0, \$1, \$5, \$10\} = \{x_1, x_2, x_3, x_4\}$*. Let* $A_{k,r}$ *denote the subset* $\{x_k, ..., x_r\}$*, for* $1 \le k \le r \le 4$ *and let* I *denote the set of intervals of* X*:* $I = \{A_{k,r}, 1 \le k \le r \le 4\}$*. By convention, we consider that the piece of information provided by Bill is one of the intervals in* I*.*

We may then define the various states of the source as follows:

$$\begin{aligned}
\Gamma_{A_{k,r}}(\textit{informed}) &= A_{k,r}, \\
\Gamma_{A_{k,r}}(\textit{ap-informed}) &= \begin{cases} \{x_{k-1}\} \cup A_{k,r} \cup \{x_{r+1}\} & \text{if } k > 1 \text{ and } r < 4, \\ A_{k,r} \cup \{x_{r+1}\} & \text{if } k = 1 \text{ and } r < 4, \\ \{x_{k-1}\} \cup A_{k,r} & \text{if } k > 1 \text{ and } r = 4, \\ A_{k,r} & \text{if } k = 1 \text{ and } r = 4, \end{cases} \\
\Gamma_{A_{k,r}}(\textit{unreliable}) &= X.
\end{aligned}$$

Example 2 (Case $Y \neq X$, inspired from Janez and Appriou [14])**.** *Let us assume that we are interested in finding the type of a given road, which can only be a track, a lane or a highway. We have a source at our disposal that provides information on this type. However, the source has but a limited perception of the possible types of road and in particular is not aware of the existence of the type "lane". In addition, we know that this source discriminate between roads either using their width or their texture (width and texture are called attributes in [14]). If the source uses the road width, then when it says "track", it is clear that we may only safely infer that the type is "track or lane" since tracks and lanes have similar width, and when it says "highway", we may infer "highway". On the other hand, if the source uses the road texture, then when it says "track", we may infer "track", and when it says "highway", we may only infer "highway or lane" since highways and lanes have similar textures.*

Using the approach proposed above, we may formalize this problem as follows. We have $Y = \{\textit{track}, \textit{highway}\}$, $X = \{\textit{track}, \textit{lane}, \textit{highway}\}$, $\mathcal{H} = \{\textit{width}, \textit{texture}\}$ *and*

$$\begin{aligned}
\Gamma_{\textit{track}}(\textit{width}) &= \{\textit{track}, \textit{lane}\}, \\
\Gamma_{\textit{highway}}(\textit{width}) &= \{\textit{highway}\}, \\
\Gamma_{Y}(\textit{width}) &= X, \\
\Gamma_{\textit{track}}(\textit{texture}) &= \{\textit{track}\}, \\
\Gamma_{\textit{highway}}(\textit{texture}) &= \{\textit{lane}, \textit{highway}\}, \\
\Gamma_{Y}(\textit{texture}) &= X.
\end{aligned}$$

3.1.2 Behavior-based correction scheme

The approach described in the previous section may be generalized to the case where the source provides uncertain information in the form of a mass function m_S^Y and meta-knowledge on the source are uncertain. Assuming some hypothesis $H \subseteq \mathcal{H}$ on the behavior of the source, then each mass $m_S^Y(A)$ should be transferred to $\Gamma_A(H)$, yielding the following mass function:

$$m^X(B|H) = \sum_{A:\Gamma_A(H)=B} m_S^Y(A), \tag{20}$$

for all $B \subseteq X$.

In the more general situation where we have uncertain meta-knowledge described by a mass function $m^{\mathcal{H}}$ on $\mathcal{H}$, then we get

$$m^X(B) = \sum_H m^X(B|H) m^{\mathcal{H}}(H) = \sum_H m^{\mathcal{H}}(H) \sum_{A:\Gamma_A(H)=B} m_S^Y(A), \tag{21}$$

for all $B \subseteq X$, which clearly generalizes (5). The correction mechanism defined by (21) will be hereafter referred to as *Behavior-Based Correction* (BBC). A more formal derivation of (21) will be provided in Section 4.2 (Lemma 1).

In addition, let us remark that the BBC procedure generalizes a familiar operation of Dempster-Shafer theory, called *conditional embedding* or *ballooning extension* [26, 27]. Let us explicitly reinterpret this operation in terms of source behavior assumptions. The ballooning extension is the process that transforms a mass function m^Y defined on a domain Y into a mass function on an extended space X, where $X \supseteq Y$. Let $m^{Y\Uparrow X}$ denote the ballooning extension of m^Y to X. It is defined as $m^{Y\Uparrow X}(B) = m^Y(A)$ if $B = A \cup (X\backslash Y)$ and $m^{Y\Uparrow X}(B) = 0$ otherwise. Suppose that a source S provides a piece of information on the value taken by a parameter x defined on a domain X. We assume further that the information provided by S takes the form of a mass function m_S^Y on the domain $Y \subseteq X$. We consider that there may be two reasons why the source provides a piece of information on the value taken by x on the domain Y instead of X: either the source has a limited perception of the actual domain of x or it knows that the values in $X\backslash Y$ are impossible. Let h_1 denote the state where the source has a limited perception of the actual domain of x and let h_2 denote the state where the source knows the values in $X\backslash Y$ to be impossible. We associate to these two states the multivalued mappings Γ_A, $A \subseteq Y$, from $\mathcal{H} = \{h_1, h_2\}$ to X defined by, for all $A \subseteq Y$:

$$\Gamma_A(h_1) = A \cup (Y\backslash X), \tag{22}$$

$$\Gamma_A(h_2) = A. \tag{23}$$

$\Gamma_A(h_1)$ translates the idea that when the source states $x \in A$, $A \subseteq Y$, we may only safely conclude that $x \in A \cup (X\backslash Y)$, due to the limited perception of the source. Let $m^{\mathcal{H}}$ represent our meta-knowledge on the behavior of the source. If $m^{\mathcal{H}}$ is such that $m^{\mathcal{H}}(\{h_1\}) = 1$ and if we use the BBC procedure to transform m_S^Y into a mass function on X, then the ballooning extension is recovered. The ballooning extension can thus be seen as a correction scheme corresponding to a particular assumption on the behavior of the source with respect its limited perception of the actual domain of a variable.

The ballooning extension is the most well-known representative of so-called deconditioning methods [15]. To complete the picture on the relationship between the BBC and these methods, we may remark that another deconditioning method, known as the method by association of highest compatible hypotheses [15] and that generalizes the ballooning extension, can also be seen as a particular case of the BBC scheme. Similarly to the ballooning extension, this method transforms a mass function m^Y defined on a domain Y into a mass function on an extended space X, where X contains all the elements of Y and some new elements. However, this transformation is guided by a compatibility relation $\omega : 2^Y \to 2^{X\backslash Y}$, where $\omega(A)$, $A \subseteq Y$, represents the set of hypotheses in $X\backslash Y$ with which the hypotheses of Y contained in A are strongly compatible. For instance, in Example 2 above, the transformation based on road width of the type "track" to "track or lane" and of the type "highway" to "highway" relies on such compatibility relation

ω, where $Y = \{\text{track}, \text{highway}\}$, $X = \{\text{track}, \text{lane}, \text{highway}\}$, and $\omega(\text{track}) = \text{lane}$ and $\omega(\text{highway}) = \emptyset$.

Let $m^{Y \Uparrow_\omega X}$ denote the extension of m^Y to X using the method by association of highest compatible hypotheses. It is defined as $m^{Y \Uparrow_\omega X}(B) = m^Y(A)$ if $B = A \cup \omega(A)$ and $m^{Y \Uparrow_\omega X}(B) = 0$ otherwise. The ballooning extension is recovered when $\omega(A) = X \backslash Y$, for all $A \subseteq Y$. As the ballooning extension, it is easy to see that the method by association of highest compatible hypothesesis is a particular case of the BBC (simply replace $\Gamma_A(h_1) = A \cup (Y \backslash X)$ in (22) by $\Gamma_A(h_1) = A \cup \omega(A)$). The state h_1 then corresponds to a particular attribute, such as road width, used by the source to discriminate the hypotheses in Y, as an attribute defines a particular compatibility relation. This deconditioning method can thus also be seen as a correction scheme corresponding to a hypothesis on the behavior of the source.

3.2 The case of multiple sources

Let us now consider that we have two sources S_1 and S_2, each of which may be in one of N elementary states (those N states are the same for both sources). It is convenient to denote by h_j^i the state j of source S_i, for $i = 1, 2$ and $j = 1, ..., N$. Accordingly, let $\mathcal{H}_i = \{h_1^i, ..., h_N^i\}$ denote the possible states of source S_i, $i = 1, 2$. The set of elementary hypotheses on the source behaviors will be $\mathcal{H}_{12} = \mathcal{H}_1 \times \mathcal{H}_2$.

3.2.1 Crisp testimonies and certain meta-knowledge

Let us assume that source S_1 states $y \in A$ and S_2 states $y \in B$, $A, B \subseteq Y$. What can be concluded about X after merging these pieces of information will depend on the hypothesis made on the behavior of the sources. We can define a multivalued mapping $\Gamma_{A,B}$ from $\mathcal{H}_{12}$ to X, which assigns to each elementary hypothesis $h = (h^1, h^2), h \in \mathcal{H}_{12}$, the result of the fusion of the two pieces of information $y \in A$ and $y \in B$. As we must conclude $\Gamma_A(h^1)$ when S_1 is in state $h^1 \in \mathcal{H}_1$, and we must conclude $\Gamma_B(h^2)$ when S_2 is in state $h^2 \in \mathcal{H}_2$, where Γ_A and Γ_B are the mappings defined in Section 3.1.1, it is clear that we must conclude $\Gamma_A(h^1) \cap \Gamma_B(h^2)$ when the sources are in state $(h^1, h^2) \in \mathcal{H}_{12}$. Hence, the mapping $\Gamma_{A,B}$ is defined by

$$\Gamma_{A,B}(h) = \Gamma_A(h^1) \cap \Gamma_B(h^2),$$

for all $h \in \mathcal{H}_{12}$.

3.2.2 Behavior-based fusion scheme

Following the same path as that of Section 2.2, we can generalize the above approach by allowing both the information provided by the source and our meta-knowledge about the source to be uncertain.

Let us assume that S_1 and S_2 provide information on Y in the form of two mass functions m_1^Y and m_2^Y, respectively, and that they are independent. If we know that hypothesis $H \subseteq \mathcal{H}_{12}$ holds, then the mass $m_1^Y(A)m_2^Y(B)$ should be transferred to the set

$$C = \Gamma_{A,B}(H) = \bigcup_{(h^1,h^2)\in H} (\Gamma_A(h^1) \cap \Gamma_B(h^2)).$$

The result of the fusion of m_1^Y and m_2^Y given $H \subseteq \mathcal{H}_{12}$ is then the mass function $m^X(\cdot|H)$ defined by:

$$m^X(C|H) = \sum_{A,B:C=\Gamma_{A,B}(H)} m_1^Y(A)\, m_2^Y(B), \tag{24}$$

for all $C \subseteq X$. When meta-knowledge on $\mathcal{H}_{12}$ is represented by a mass function $m^{\mathcal{H}_{12}}$, we then get:

$$\begin{aligned} m^X(C) &= \sum_H m^X(C|H) m^{\mathcal{H}_{12}}(H) \\ &= \sum_H m^{\mathcal{H}_{12}}(H) \sum_{A,B:C=\Gamma_{A,B}(H)} m_1^Y(A)\, m_2^Y(B), \end{aligned} \tag{25}$$

for all $C \subseteq X$. Equation (25) will be referred to as *Behavior-Based Fusion* (BBF). It is clearly a generalization of the general combination rule proposed in Section 2.2.5. A more formal derivation of this rule will be presented in Section 4.2 (Lemma 2).

As a final remark in this section, we may note that the fusion process can be cast in more general settings than those considered here. In particular, one may face problems where sources S_i, $i = 1, ..., n$, provide information on different frames Y_i and admit different numbers N_i of elementary states. It is interesting to note that in such a setting, an equation similar to (25) can easily be obtained and a result similar to the one that will be shown in the next section still holds. Although this setting is more general, we have refrained from introducing it in this paper in order to improve readability.

4 Commutativity between correction and fusion schemes with meta-independent sources

As we did at the end of Section 2.2.4, let us now assume sources are meta-independent, i.e., that the mass function $m^{\mathcal{H}_{12}}$ expressing our uncertain meta-knowledge satisfies (10). Under this assumption, we may wonder whether it is equivalent to combine two mass functions m_1^Y and m_2^Y using the BBF rule, or to apply the BBC procedure to m_1^Y and m_2^Y, and combine the transformed mass functions by the unnormalized Dempster's rule (Figure 1). In order to answer this question, we need first to recall the definitions of some operations related to the use of belief functions defined on product spaces.

4.1 Operations on product spaces

Let $m^{X \times Y}$ denote a mass function defined on the Cartesian product $X \times Y$ of the domains of two parameters x and y. The marginal mass function $m^{X \times Y \downarrow X}$ is defined as

$$m^{X \times Y \downarrow X}(A) = \sum_{\{B \subseteq X \times Y, (B \downarrow X) = A\}} m^{X \times Y}(B), \quad \forall A \subseteq X,$$

where $(B \downarrow X)$ denotes the projection of B onto X.

Conversely, let m^X be a mass function defined on X. Its vacuous extension [22] on $X \times Y$ is defined as:

$$m^{X \uparrow X \times Y}(B) = \begin{cases} m^X(A) & \text{if } B = A \times Y \text{ for some } A \subseteq X, \\ 0 & \text{otherwise.} \end{cases}$$

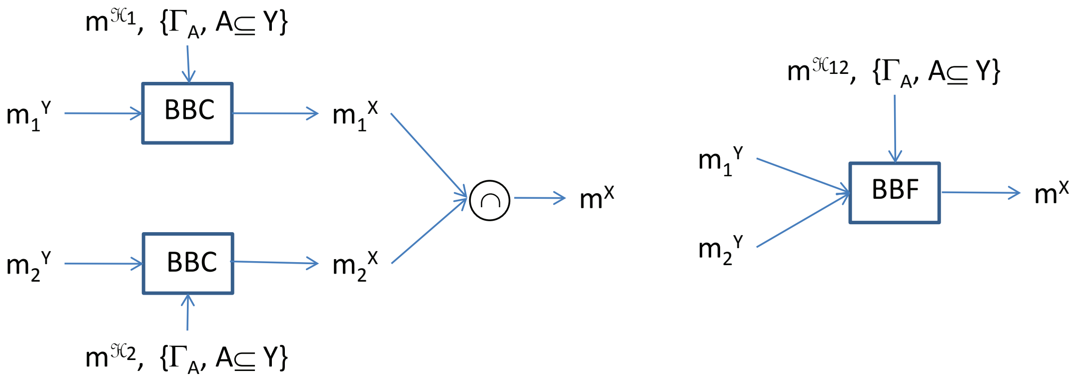

Figure 1: Two ways of combining two mass functions m_1^Y and m_2^Y using meta-knowledge about the sources: using the BBC procedure (left) and using the BBF rule (right). The equivalence between these two methods under the meta-independence assumption is proved in this section.

Given two mass functions m_1^X and m_2^Y, their combination by the unnormalized Dempster's rule on $X \times Y$ can be obtained by combining their vacuous extensions on $X \times Y$. Formally:

$$m_1^X \textcircled{\cap} m_2^Y = m_1^{X\uparrow X\times Y} \textcircled{\cap} m_2^{Y\uparrow X\times Y}.$$

Let m^U and m^V be two mass functions on product spaces U and V. The following property, referred to as "Distributivity of marginalization over combination" [25], holds:

$$(m^U \textcircled{\cap} m^V)^{\downarrow U} = m^U \textcircled{\cap} m^{V\downarrow U\cap V}, \tag{26}$$

where $U \cap V$ denotes, by convention, the Cartesian product of frames common to U and V.

4.2 Meta-independence result

Let us consider again the setting of Section 3.1, in which three distinct pieces of evidence are defined:

1. A mass function m_S^Y on Y provided by source S;
2. A mass function $m^{\mathcal{H}}$ on $\mathcal{H} = \{h_1, ..., h_N\}$ representing our uncertain meta-knowledge on the source;
3. For each $A \subseteq Y$, a multivalued mapping Γ_A from $\mathcal{H}$ to X indicating how to interpret on X the piece of information $y \in A \subseteq Y$ provided by the source in each configuration $h \in \mathcal{H}$.

The last piece of evidence defines a relation between spaces $\mathcal{H}$, Y and X, which may be represented by the following categorical mass function on $\mathcal{H} \times Y \times X$:

$$m_\Gamma^{\mathcal{H}\times Y\times X}\left[\bigcup_{h\in\mathcal{H}, A\subseteq Y} (\{h\} \times A \times \Gamma_A(h))\right] = 1. \tag{27}$$

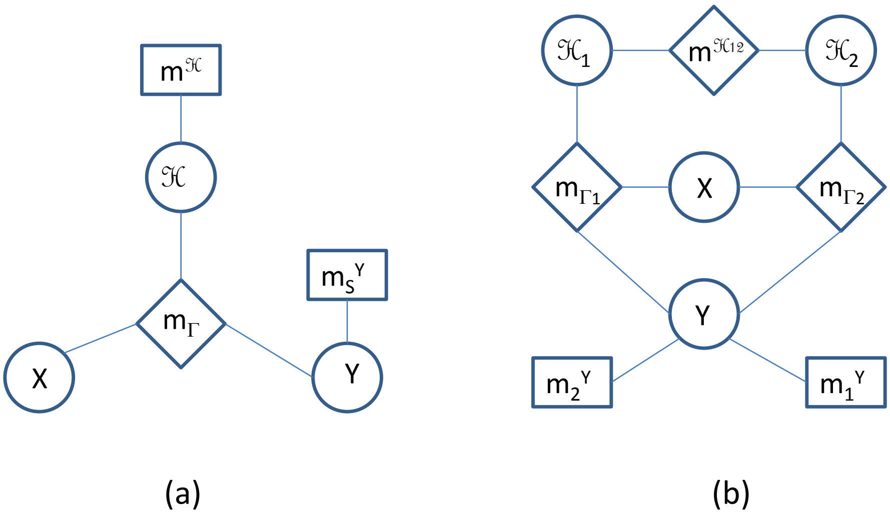

Figure 2: Evidential networks corresponding to the BBC procedure (a) and the BBF rule (b).

The three mass functions m_S^Y, $m^{\mathcal{H}}$ and $m_\Gamma^{\mathcal{H}\times Y\times X}$ can be seen as defining an evidential network, as shown in Figure 2a. As will be shown below, the BBC procedure is equivalent to combining these three mass functions using Dempter's rule, and marginalizing the result on X.

By combining m_S^Y with $m_\Gamma^{\mathcal{H}\times Y\times X}$ and marginalizing on $\mathcal{H}\times X$, we get a mass function $m_{S\Gamma}^{\mathcal{H}\times X}$ on $\mathcal{H}\times X$, defined by:

$$m_{S\Gamma}^{\mathcal{H}\times X}\left[\bigcup_{h\in\mathcal{H}}(\{h\}\times\Gamma_A(h))\right]=m_S^Y(A),\quad \forall A\subseteq Y. \tag{28}$$

For instance, let $\mathcal{H}$ be the space $\mathcal{H}=\mathcal{R}\times\mathcal{T}$, let $Y=X$ and let the multivalued mappings Γ_A be defined by (19) for all $A\subseteq Y$. The mass function $m_{S\Gamma}^{\mathcal{H}\times X}$ is then given by

$$m_{S\Gamma}^{\mathcal{H}\times X}((\{h_1\}\times A)\cup(\{h_2\}\times\overline{A})\cup(\{h_3\}\times X)\cup(\{h_4\}\times X))=m_S^Y(A),$$

for all $A\subseteq Y$.

The following lemma states that the mass function given by the BBC (21) can be obtained by combining $m_{S\Gamma}^{\mathcal{H}\times X}$ with $m^{\mathcal{H}}$, and marginalizing on X.

Lemma 1. *We have, for all $B\subseteq X$*

$$\left(m_{S\Gamma}^{\mathcal{H}\times X}\circledcap m^{\mathcal{H}}\right)^{\downarrow X}(B)=m^X(B),$$

where m^X is the mass function defined by (21).

Proof. Let $m^{\mathcal{H}\times X}=m_{S\Gamma}^{\mathcal{H}\times X}\circledcap m^{\mathcal{H}}$. It can be computed as follows:

$$m^{\mathcal{H}\times X}(C)=\begin{cases} m^{\mathcal{H}}(H)\cdot m_S^Y(A) & \text{if } C=\left(\bigcup_{h\in\mathcal{H}}\{h\}\times\Gamma_A(h)\right)\cap(H\times X),\\ 0 & \text{otherwise.}\end{cases}$$

Now, for all $H \subseteq \mathcal{H}$ and all $A \subseteq Y$,

$$\left[\left(\bigcup_{h \in \mathcal{H}} \{h\} \times \Gamma_A(h)\right) \cap (H \times X)\right] \downarrow X = \bigcup_{h \in H} \Gamma_A(h) = \Gamma_A(H).$$

Therefore, $m^{\mathcal{H} \times X \downarrow X}(B)$ for any $B \subseteq X$ can be obtained by summing over all $H \subseteq \mathcal{H}$ and all $A \subseteq Y$ such that $\Gamma_A(H) = B$:

$$m^{\mathcal{H} \times X \downarrow X}(B) = \sum_{H, A : \Gamma_A(H) = B} m^{\mathcal{H}}(H) \cdot m_S^Y(A),$$

which is equivalent to (21). □

This lemma shows that BBC, and thus deconditioning methods, the discounting operation, the complement of a belief function and the correction scheme introduced in Section 2, can be obtained by defining an evidential network on $\mathcal{H} \times Y \times X$ and by propagating uncertainty in this network using the unnormalized Dempster's rule.

Let us now consider the setting of Section 3.2. We consider two sources S_1 and S_2, which provide items of evidence m_1^Y and m_2^Y on Y, respectively. Let $m^{\mathcal{H}_{12}}$ be a mass function on $\mathcal{H}_{12} = \mathcal{H}_1 \times \mathcal{H}_2$ representing our uncertain meta-knowledge on the sources. As before, the mappings Γ_A for all $A \subseteq Y$ induce mass functions $m_{\Gamma i}^{\mathcal{H}_i \times Y \times X}$, $i = 1, 2$ of the form (27). These mass functions define the evidential network shown in Figure 2b. By combining $m_{\Gamma i}^{\mathcal{H}_i \times Y \times X}$ with m_i^Y and marginalizing on $\mathcal{H}_i \times X$, we get mass functions $m_{i\Gamma}^{\mathcal{H}_i \times X}$, $i = 1, 2$ with the following expressions:

$$m_{i\Gamma}^{\mathcal{H}_i \times X}\left[\bigcup_{h \in \mathcal{H}_i} (\{h\} \times \Gamma_A(h))\right] = m_i^Y(A), \quad \forall A \subseteq Y.$$

As expressed by the following lemma, the mass function m^X computed by the BBF rule (25) can be obtained by combining mass functions $m_{i\Gamma}^{\mathcal{H}_i \times X}$, $i = 1, 2$ with $m^{\mathcal{H}_{12}}$, and marginalizing the result on X:

Lemma 2. *We have, for all $B \subseteq X$*

$$\left(m_{1\Gamma}^{\mathcal{H}_1 \times X} \circledcirc m_{2\Gamma}^{\mathcal{H}_2 \times X} \circledcirc m^{\mathcal{H}_{12}}\right)^{\downarrow X}(B) = m^X(B), \tag{29}$$

where m^X is the mass function defined by (25).

Proof. Let $m_{12\Gamma}^{\mathcal{H}_{12} \times X}$ be the mass function obtained by combining the first two mass functions in (29). We have, for all $A, B \subseteq Y$:

$$m_{12\Gamma}^{\mathcal{H}_{12} \times X}(C) = \begin{cases} m_1^Y(A) \cdot m_2^Y(B) & \text{if } C = \bigcup_{(h^1, h^2) \in \mathcal{H}_{12}} \{(h^1, h^2)\} \times (\Gamma_A(h^1) \cap \Gamma_B(h^2)) \\ 0 & \text{otherwise.} \end{cases}$$

By combining the above mass function with $m^{\mathcal{H}_{12}}$, we get a new mass function $m^{\mathcal{H}_{12} \times X}$ defined by

$$m^{\mathcal{H}_{12} \times X}(C) = m^{\mathcal{H}_{12}}(H) \cdot m_1^Y(A) \cdot m_2^Y(B)$$

if

$$C = \left(\bigcup_{(h^1,h^2)\in\mathcal{H}_{12}} \{(h^1,h^2)\} \times (\Gamma_A(h^1) \cap \Gamma_B(h^2)) \right) \cap (H \times X)$$

and $m^{\mathcal{H}_{12}\times X}(C) = 0$ otherwise.

Now, for all $H \subseteq \mathcal{H}_{12}$ and for all $A, B \subseteq Y$,

$$\left[\left(\bigcup_{(h^1,h^2)\in\mathcal{H}_{12}} \{(h^1,h^2)\} \times (\Gamma_A(h^1) \cap \Gamma_B(h^2)) \right) \cap (H \times X) \right] \downarrow X$$
$$= \bigcup_{(h^1,h^2)\in H} (\Gamma_A(h^1) \cap \Gamma_B(h^2)) = \Gamma_{A,B}(H).$$

Therefore, $m^{\mathcal{H}_{12}\times X\downarrow X}(C)$ for $C \subseteq X$ can be obtained by summing over all $H \subseteq \mathcal{H}_1 \times \mathcal{H}_2$ and all $A, B \subseteq Y$ such that $\Gamma_{A,B}(H) = C$:

$$m^{\mathcal{H}_{12}\times X\downarrow X}(C) = \sum_{H,A,B:\Gamma_{A,B}(H)=C} m^{\mathcal{H}_{12}}(H) \cdot m_1^Y(A) \cdot m_2^Y(B)$$

which is equivalent to (25). □

This lemma shows that the BBF rule, and thus the generalization of the unnormalized version of Dempster's rule to all Boolean connectives, can be obtained by defining an evidential network on $\mathcal{H}_{12} \times Y \times X$ and by propagating uncertainty in this network using the unnormalized Dempster's rule. In addition, this implies that the fusion schemes studied in this paper can be recovered using the unnormalized Dempster's rule and marginalization.

Theorem 1. *With meta-independent sources, it is equivalent to combine the uncertain information m_1^Y and m_2^Y by the BBF rule or to combine by the unnormalized Dempster's rule each of these pieces of information corrected using the BBC procedure.*

Proof. Let $m^{\mathcal{H}_1}$ and $m^{\mathcal{H}_2}$ represent our uncertain meta-knowledge on the behaviors of two sources S_1 and S_2, respectively. Meta-independence of S_1 and S_2 is equivalent to $m^{\mathcal{H}_{12}} = m^{\mathcal{H}_1} \circledcap m^{\mathcal{H}_2}$, where $m^{\mathcal{H}_{12}}$ represent our uncertain meta-knowledge on the sources. Under this assumption, we thus have, with the same notations as above:

$$m_{1\Gamma}^{\mathcal{H}_1\times X} \circledcap m_{2\Gamma}^{\mathcal{H}_2\times X} \circledcap m^{\mathcal{H}_{12}} = m_{1\Gamma}^{\mathcal{H}_1\times X} \circledcap m_{2\Gamma}^{\mathcal{H}_2\times X} \circledcap m^{\mathcal{H}_1} \circledcap m^{\mathcal{H}_2}.$$

Marginalizing on $\mathcal{H}_1 \times X$, we get, using (26):

$$\left(m_{1\Gamma}^{\mathcal{H}_1\times X} \circledcap m_{2\Gamma}^{\mathcal{H}_2\times X} \circledcap m^{\mathcal{H}_1} \circledcap m^{\mathcal{H}_2} \right)^{\downarrow \mathcal{H}_1\times X} = m_{1\Gamma}^{\mathcal{H}_1\times X} \circledcap m^{\mathcal{H}_1} \circledcap \left(m_{2\Gamma}^{\mathcal{H}_2\times X} \circledcap m^{\mathcal{H}_2} \right)^{\downarrow X},$$

which, after further marginalization on X, becomes:

$$\left(m_{1\Gamma}^{\mathcal{H}_1\times X} \circledcap m_{2\Gamma}^{\mathcal{H}_2\times X} \circledcap m^{\mathcal{H}_1} \circledcap m^{\mathcal{H}_2} \right)^{\downarrow X} = \left(m_{1\Gamma}^{\mathcal{H}_1\times X} \circledcap m^{\mathcal{H}_1} \right)^{\downarrow X} \circledcap \left(m_{2\Gamma}^{\mathcal{H}_2\times X} \circledcap m^{\mathcal{H}_2} \right)^{\downarrow X}.$$

The theorem then follows from Lemmas 1 and 2. □

Let us note that a similar theorem holds for the case of N sources instead of 2. This is a direct consequence of Lemma 2 being straightforwardly generalizable to the case of N sources.

This theorem leads to an interesting remark: the method, often used in applications, that consists in discounting sources and then combining them by the unnormalized Dempster's rule, can be seen as a particular case of the BBF rule. Indeed, without lack of generality, consider the case of two sources S_1 and S_2. The discount and combine method corresponds to truthful sources with independent probabilities p_1 and p_2 of relevance, i.e., to a meta-knowledge $m^{\mathcal{H}_{12}}$ on the sources such that $m^{\mathcal{H}_{12}} = m^{\mathcal{H}_1} \textcircled{\cap} m^{\mathcal{H}_2}$, with $m^{\mathcal{H}_1}$, $m^{\mathcal{H}_2}$ and $m^{\mathcal{H}_{12}}$ defined by (11)-(16).

Another popular method for taking into account meta-knowledge on the reliability of the sources is to compute a weighted average of the mass functions to be combined. Indeed, as remarked by Shafer [22, p. 253], this methods yield results similar to those obtained by Dempster's rule applied to equally discounted mass functions, when the discount rate tends to 1. Interestingly, the weighted average rule is also a special case of the BBF rule. The mass function m resulting from the weighted average of two mass functions m_1 and m_2 provided by two sources S_1 and S_2 is defined by $m = w \cdot m_1 + (1 - w) \cdot m_2$, $w \in [0, 1]$, where w is the relative reliability of S_1. It is clear that the weighted average results from meta-knowledge on the sources described by the following mass function

$$\begin{aligned} m^{\mathcal{H}_{12}}(\{(R_1, T_1, \neg R_2, T_2)\}) &= w, \\ m^{\mathcal{H}_{12}}(\{(\neg R_1, T_1, R_2, T_2)\}) &= 1 - w, \end{aligned}$$

which is clearly different from that defined by (13)-(16) and associated to the discount and combine method.

5 Relation to previous work

The idea of exploiting meta-knowledge about the sources of information for correcting or combining belief functions has been explored by several researchers. This section discusses the relation between the notions introduced in Sections 2 and 3 and previous work on similar topics.

5.1 Related work on information correction

As already mentioned, the approach developed in Section 2.1 extends the discounting operation, introduced by Shafer [22] and formalized by Smets [27]. This basic model corresponds to the case where the source is known to be truthful, but has only a probability of being relevant. In [30], Smets proposed a counterpart to this model, in which the source is relevant but may not be truthful. Smets described a scenario in which a "deceiver agent" may replace a belief function by its complement, and he proposed solutions to detect and remedy such a situation. The model introduced in Section 2.1 clearly subsumes these two basic models.

An extension of the discounting operation, called *contextual discounting*, was introduced by Mercier et al. in [18]. In this approach, a binary frame $\mathcal{R}$ for the relevance of the source is introduced as in classical discounting. Additionally, a coarsening Θ of X is defined, and conditional mass functions $m^{\mathcal{R}}(\cdot|\theta)$ on $\mathcal{R}$ given θ, for each $\theta \in \Theta$, are postulated. A discounted mass function on X is obtained by combining the mass function m_S^X provided by the source with the conditional mass functions $m^{\mathcal{R}}(\cdot|\theta)$, $\theta \in \Theta$. In [17], Mercier et al. further generalize this model by allowing the user to specify conditional mass functions $m^{\mathcal{R}}(\cdot|A)$ for any $A \subseteq X$. A crucial assumption in the contextual discounting model and its variants is that of independence between the items of evidence introduced in the model. This correction scheme is, in a sense, simpler than the one introduced in Section 2.1, in that it has no "truthfulness" component. On the other hand, it is based on more complex meta-knowledge about the source, as beliefs on $\mathcal{R}$ are assessed conditionally on different contexts, corresponding to different hypotheses about the variable x of interest. A more complex model incorporating both $\mathcal{R}$ and $\mathcal{T}$ components, and conditional mass functions on $\mathcal{R} \times \mathcal{T}$ given hypotheses about X could obviously be defined, if required by applications.

In [16], Mercier et al. also proposed another extension of the discounting operation, in which uncertain meta-knowledge on the source S is quantified by a mass function $m^{\mathcal{H}}$ on the space $\mathcal{H} = \{h_1, \ldots, h_N\}$ of possible states of the source. The interpretation of those states $h \in \mathcal{H}$ is given by transformations m_h^X of m_S^X: if the source is in state h and if it provides the mass function m_S^X, then we must adopt m_h^X as the representation of our state of belief. This is formalized using conditional mass function by postulating that $m^X(\cdot|h, m_S^X) = m_h^X$, where $m^X(\cdot|h, m_S^X)$ represents our uncertainty on X in a context where h holds and the source provides information m_S^X. This correction scheme is comparable to BBC introduced in Section 3.1, in that it expresses meta-knowledge about the source in a frame of N arbitrary states. The two models coincide in the special case where the mass function $m^{\mathcal{H}}$ on $\mathcal{H}$ is Bayesian, and m_h^X is defined as from m_S^X using multivalued mappings Γ_A as:

$$m_h^X(B) = \sum_{A:\Gamma_A(h)=B} m_S^Y(A), \quad \forall B \subseteq X,$$

in which case both models yield

$$m^X = \sum_{h \in \mathcal{H}} m^{\mathcal{H}}(\{h\}) \cdot m_h^X.$$

However, the two models are distinct in the general case, and the choice of one model or another should be guided by the nature of available knowledge in each specific application.

Finally, we should also mention in this section the work of Haenni and Hartmann [12], who proposed a model of partially relevant information sources. In this model, each source S_i is assumed to provide information on a binary variable HYP in the form of a binary report REP_i. Each source generates its report according to s independent variables, possibly including the hypothesis HYP in question. Based on various hypotheses about the relation between REP_i and the underlying variables, a taxonomy of models is generated. Although, at first glance, this formalism seems to be different from ours, our approach happens upon closer examination to be more general. Consider, for instance, the PD model, which is one of the most complex models described in [12]. In this model, the report is generated by the source as follows: if the source is reliable ($REL = 1$), then $REP = HYP$. IF $REL = 0$, then REP is equal to random variable P if $HYP = 1$,

Table 1: Mappings Γ_A corresponding to the PD model of Haenni and Hartmann [12].

h	$\Gamma_{\{0\}}(h)$	$\Gamma_{\{1\}}(h)$
100	$\{0\}$	$\{1\}$
110	$\{0\}$	$\{1\}$
101	$\{0\}$	$\{1\}$
111	$\{0\}$	$\{1\}$
000	$\{0,1\}$	$\emptyset$
010	$\{0\}$	$\{1\}$
001	$\{1\}$	$\{0\}$
011	$\emptyset$	$\{0,1\}$

and it is equal to a random variable Q if $HYP = 0$. The three random variables REL, P and Q are assumed to be independent, and the model has three parameters $\rho = \Pr(REL = 1)$, $p = \Pr(P = 1)$ and $q = \Pr(Q = 1)$. With our notations, this model can be translated as follows. Let $Y = \{0,1\}$ the frame of REP, and $\mathcal{H} = \{0,1\}^3$ the frame of the triple (REL, P, Q). The joint probability distribution of this triple defines a Bayesian mass function $m^{\mathcal{H}}$ on $\mathcal{H}$; for instance, $m^{\mathcal{H}}(\{(1,0,1)\}) = \rho(1-p)q$. Finally, the mappings Γ_A for $A = \{0\}$ and $A = \{1\}$ are given in Table 1. All other models described by Haenni and Hartmann could be translated in a similar way.

5.2 Related work on information fusion

The idea of defining alternatives to Dempster's rule by replacing intersection with other set-theoretic operations can be traced to Smets' 1978 thesis [26], in which he introduced the disjunctive rule of combination together with the Generalised Bayes Theorem (see also [27] for a more accessible reference). This approach was generalized to arbitrary set operations by Dubois and Prade [7] and Yager [32]. In [11], Haenni noticed that the disjunctive rule could be deduced by defining an evidential network with two binary frames $\mathcal{R}_1$ and $\mathcal{R}_2$ for the reliability of the two sources, and combining mass functions m_1^X and m_2^X with a categorical mass function on $\mathcal{R}_1 \times \mathcal{R}_2$ expressing that at least one of the two sources is reliable. The model defined in Section 2.2 is clearly an extension of this simple framework.

In [28], Smets introduced two families of combination rules depending on a parameter α, which he called α-conjunctions and α-disjunctions. These two families are basically the only sets of linear operators with a commutative monoid structure. The α-conjunctions include the unnormalized Dempster's rule (for $\alpha = 1$) and admit the vacuous mass function as neutral element. The α-disjunctions range between the disjunctive rule and a rule corresponding to the exclusive OR, and admit the contradiction ($m(\emptyset) = 1$) as neutral element. In [28], Smets derived these rules from axiomatic requirements, but admitted that they lacked a clear interpretation for $\alpha \in (0,1)$. In [19, 21], Pichon provided such an interpretation in terms of truthfulness of the sources. For instance, he showed that the α-conjunction results from the following assumptions:

1. The two sources are relevant, and either both truthful, or both non truthful (i.e., the operator $\otimes$ is logical equivalence);

2. The degree of belief in the hypothesis that at least one of sources is truthful, conditionally to each value $x \in X$, is equal to α.

Under these assumptions, the α-conjunction can be obtained by defining a belief network in $\mathcal{T}_1 \times \mathcal{T}_2 \times X$, and combining all pieces of evidence, assumed to be independent. The α-disjunction can be obtained in a similar way, starting from different assumptions about the truthfulness of the sources.

As shown in Section 4, the approach developed in Section 2.2, and extended in Section 3.2 can also be derived from uncertainty propagation in an evidential network, in which some variables may be related to the truthfulness of the sources. Although each of the two families of α-junctions relies on a single parameter, the interpretation of this parameter is not easy to disclose, and the independence assumptions involved in the model do not seem very natural. In contrast, the model developed in this paper allows us to represent richer forms of meta-knowledge and it lends itself to easier interpretation.

6 Conclusion

We have proposed a general approach to the correction and fusion of belief functions, which integrates an agent's meta-knowledge on the truthfulness and relevance of the sources of information. This formalism considerably extends Shafer's discounting operation, which deals only with the relevance of sources, as well as the unnormalized Dempster's rule. The obtained results can be applied, in particular, to all domains where information sources are intelligent agents able to lie, independently of their competence to provide information.

We have further extended this approach by allowing for general source behavior assumptions that go beyond the notions of relevance and truthfulness. This extension is potentially useful for various applications and, in particular, those involving information sources defined on different frames.

We have then shown that the correction and fusion schemes introduced in this paper can be obtained by defining particular evidential networks and by propagating uncertainty in these networks using the unnormalized Dempster's rule. Using a well-known property of belief functions defined on product spaces, we have proved that commutativity between correction and fusion processes holds, when the behaviors of the sources are independent.

Finally, the proposed formal representation of meta-knowledge on the behavior of information sources turns out to be somewhat similar to, but arguably more general and flexible than other approaches introduced in the Dempster-Shafer framework.

One line for further research is to extend the framework to the case of sources reporting to the agent what other sources reported to them. In other words, instead of considering several parallel testimonies, one may consider a series of agents, each reporting to the next one what the previous agent reported. There are then several uncertain information distortion steps in a row by sources having uncertain behavior. Interestingly, this alternative line of research is already present in the entry "Probabilité" in D'Alembert and Diderot Encyclopedia. Recently, Cholvy [2] also investigated this issue. Eventually one may consider the case of series-parallel networks of more or less reliable sources with uncertain information flows.

Another interesting perspective is the possibility to learn the behavior of sources by comparing the pieces of information provided by those sources with the ground truth, as done in a simple framework for discounting [10] and contextual discounting [18].

Acknowledgments

Frédéric Pichon was supported in part by a grant from the French national research agency through the CSOSG research program (project CAHORS).

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