Information correction and fusion using belief functions

Frédéric Pichon

Laboratory of Computer Engineering and Automation of Artois (LGI2A)
Université d'Artois, Béthune, France

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Information correction and fusion ...

- Problem: to extract truthful and precise knowledge about a quantity of interest, from information coming from various sources.
- Applications: computer vision, robotics, machine learning...
- Old problem: origin of probability theory, where formalizing and merging partially reliable testimonies was a concern.
- Requires meta-knowledge on the sources, i.e., knowledge about their quality (typically, their reliability).
- Called information correction when there is a single information source and information fusion when there are several sources.

... using belief functions

- Related to the issue of uncertainty modeling.
- Uncertainty theories: probability, possibility, belief function, imprecise probability theories.
- Central role in belief function theory (BFT):
  1. [Shafer, 1976]: BFT as an approach for representing and merging partially reliable and elementary testimonies;
  2. Numerous theoretical contributions on information fusion;
  3. BFT used in applications for merging information.

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  2. Numerous theoretical contributions on information fusion;
  3. BFT used in applications for merging information.

→ This lecture: some recent results in line with 1 – 3 , based on a modeling of source quality, reinforcing the relevance of BFT for information correction and fusion.

Contents of this lecture

- A general approach to information correction and fusion using belief functions
- A prism to understand some important belief function correction and fusion schemes
- An interpretation of belief functions ($\sim$ [Shafer, 1976] revisited)
- Means to tackle correction and fusion problems in practice

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- Means to tackle correction and fusion problems in practice

Not in this lecture:

- An exhaustive review of all combination rules
- A discussion on conflit measurement (see Anne-Laure's lecture)
- A discussion on rule properties (see, e.g., Sébastien's lecture at the 2015 BFAS school)
- Implementation aspects (see Arnaud's lecture)

Outline

Outline

Reliability

- Classically, to interpret information items provided by sources (sensor, human, ...), assumptions are made about their reliability (relevance), where a reliable source is a source providing useful information regarding the quantity of interest.
- Examples :
  - A broken watch is useless to try and find the time it is since there is no way to know whether the supplied information is correct or not: this source is not reliable for the time;
  - My six-year-old child is ignorant about the name of the latest Nobel Peace Prize laureate: he is not reliable for this question (in contrast to the source nobelprize.org).
- Basic idea : a piece of information received from a reliable source is considered valid, whereas it is useless if the source is not reliable.

Formalization

- Let X be a variable of interest taking values in a finite set $\mathcal{X} = \{x_1, \ldots, x_n\}$ (frame of discernment), and whose actual value is unknown
- Assume a source $\mathfrak{s}$ telling that $X \in A \subseteq \mathcal{X}$
  - If $\mathfrak{s}$ is not reliable, we replace $X \in A$ by $X \in \mathcal{X}$
  - If $\mathfrak{s}$ is reliable, we keep $X \in A$

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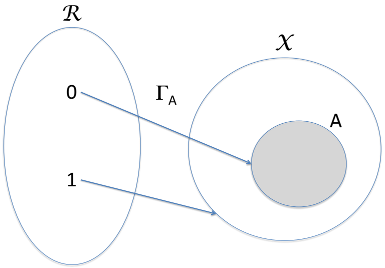
  - If $\mathfrak{s}$ is reliable, we keep $X \in A$
- Let R be the variable denoting its reliability, defined on $\mathcal{R} = \{0, 1\}$ where 0 means that $\mathfrak{s}$ is reliable and 1 means not reliable.
- The interpretation of the testimony according to the reliability may be encoded by $\Gamma_A : \mathcal{R} \rightarrow 2^{\mathcal{X}}$ such that

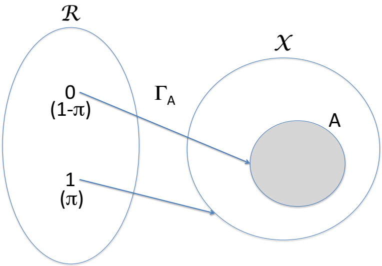
$$\Gamma_A(0) = A,$$
$$\Gamma_A(1) = \mathcal{X}.$$



Uncertain reliability

- Assume now $\mathfrak{s}$ is not reliable with probability $P^{\mathcal{R}}(R=1)=\pi$ (and reliable with probability $P^{\mathcal{R}}(R=0)=1-\pi$) with $\pi \in [0,1]$
- What can then be inferred about X?
- π should be transferred to $\Gamma_A(1)=\mathcal{X}$, $1-\pi$ to $\Gamma_A(0)=A$, and thus our knowledge about X is represented by a mass function (MF) on $\mathcal{X}$ such that

$$\begin{aligned} m(A) &= 1-\pi, \\ m(\mathcal{X}) &= \pi \end{aligned}$$

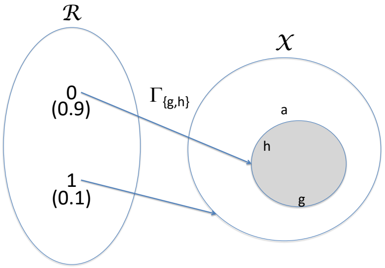


- $m(A)$: probability of knowing that $X \in A$ and nothing more, given the available evidence.
- m is a so-called simple mass function (SMF), since it has two focal sets including $\mathcal{X}$. It is more simply denoted by A^{π}.
- Other useful notation for m: $m[P^{\mathcal{R}}, A]$

Example

- Assume a sensor $\mathfrak{s}$ in charge of recognizing the type X of an aircraft which can be airplane (a), glider (g), or helicopter (h), i.e., $\mathcal{X} = \{a, g, h\}$.
- $\mathfrak{s}$ tells it is a glider or a helicopter, i.e., $X \in A = \{g, h\}$.
- The probability that the sensor is not reliable is 0.1, i.e., $\pi = 0.1$.
- Hence, our knowledge about X is represented by the SMF $\{g, h\}^{0.1}$

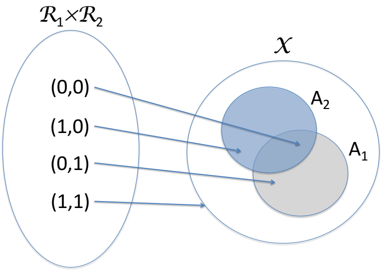
$$
\begin{aligned}
m(\{g, h\}) &= 0.9 \\
m(\mathcal{X}) &= 0.1
\end{aligned}
$$



Outline

Two information sources

- Assume now two sources $\mathfrak{s}_1$ and $\mathfrak{s}_2$ providing information $X \in A_1$ and $X \in A_2$, respectively.
- Let $\Gamma_{A_i} : \mathcal{R}_i \rightarrow 2^{\mathcal{X}}$ represent the interpretation of information A_i from $\mathfrak{s}_i$ given its reliability R_i defined on $\mathcal{R}_i = \{0, 1\}$.
- If they are in the state
  - $(R_1 = 0, R_2 = 0)$, then $X \in \Gamma_{A_1}(0) \cap \Gamma_{A_2}(0) = A_1 \cap A_2$
  - $(R_1 = 1, R_2 = 0)$, then $X \in \Gamma_{A_1}(1) \cap \Gamma_{A_2}(0) = \mathcal{X} \cap A_2 = A_2$
  - $(R_1 = 0, R_2 = 1)$, then $X \in \Gamma_{A_1}(0) \cap \Gamma_{A_2}(1) = A_1 \cap \mathcal{X} = A_1$
  - $(R_1 = 1, R_2 = 1)$, then $X \in \Gamma_{A_1}(1) \cap \Gamma_{A_2}(1) = \mathcal{X} \cap \mathcal{X} = \mathcal{X}$



Notation

- When the sources provide information $\mathbf{A} = (A_1, A_2)$ and are in the state $\mathbf{r} = (r_1, r_2) \in \boldsymbol{\mathcal{R}} := \mathcal{R}_1 \times \mathcal{R}_2$, we should deduce

$$X \in \Gamma_{\mathbf{A}}(\mathbf{r}) := \Gamma_{A_1}(r_1) \cap \Gamma_{A_2}(r_2)$$

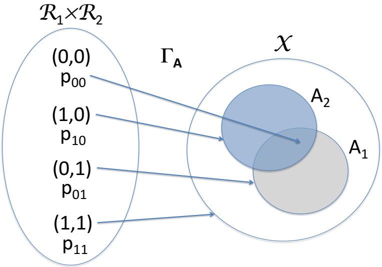
- $\Gamma_{\mathbf{A}} : \boldsymbol{\mathcal{R}} \to 2^{\mathcal{X}}$
- Example

$$\begin{aligned} \Gamma_{\mathbf{A}}(0,1) &= \Gamma_{A_1}(0) \cap \Gamma_{A_2}(1) \\ &= A_1 \cap \mathcal{X} \\ &= A_1 \end{aligned}$$

Uncertain reliabilities

- Assume now the sources are in state $\mathbf{r} = (r_1, r_2)$ with probability $P^{\mathcal{R}}(R_1 = r_1, R_2 = r_2) = p_{\mathbf{r}}$
- $p_{\mathbf{r}}$ should be transferred to $\Gamma_{\mathbf{A}}(\mathbf{r})$.
- Our knowledge about X can then be represented by

$$m(B) = \sum_{\mathbf{r}:\Gamma_{\mathbf{A}}(\mathbf{r})=B} p_{\mathbf{r}}.$$

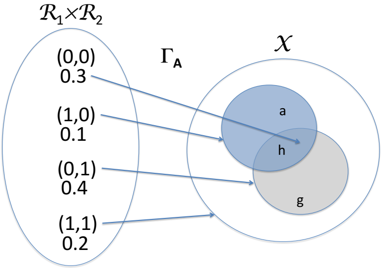


- Notation: $m[P^{\mathcal{R}}, \mathbf{A}]$

Example

- Two sensors $\mathfrak{s}_1$ and $\mathfrak{s}_2$ for the type X of an aircraft
- $\mathfrak{s}_1$ tells $X \in A_1 = \overline{\{a\}} = \{g, h\}$
- $\mathfrak{s}_2$ tells $X \in A_2 = \overline{\{g\}} = \{a, h\}$
- We have

$$\begin{aligned}
\Gamma_{\mathbf{A}}(0,0) &= A_1 \cap A_2 = \{h\} \\
\Gamma_{\mathbf{A}}(1,0) &= A_2 = \{a, h\} \\
\Gamma_{\mathbf{A}}(0,1) &= A_1 = \{g, h\} \\
\Gamma_{\mathbf{A}}(1,1) &= \mathcal{X}
\end{aligned}$$



- Induced knowledge about X:

$$m(\{h\}) = 0.3, m(\{a, h\}) = 0.1, m(\{g, h\}) = 0.4, m(\mathcal{X}) = 0.2$$

Decomposition of meta-knowledge

- $P^{\mathcal{R}}$ is a bivariate Bernoulli distribution
- It is characterized by

$$\begin{aligned}
\pi_i &:= \mathbb{E}[R_i] = P^{\mathcal{R}_i}(R_i = 1), \quad i = 1, 2, \\
\sigma &:= \mathbb{E}\left[(R_1 - \pi_1)(R_2 - \pi_2)\right] = \mathbb{E}\left[R_1 R_2\right] - \mathbb{E}[R_1]\mathbb{E}[R_2] \\
&= P^{\mathcal{R}}(R_1 = 1, R_2 = 1) - P^{\mathcal{R}_1}(R_1 = 1)P^{\mathcal{R}_2}(R_2 = 1)
\end{aligned}$$

- We have

$$\begin{aligned}
P^{\mathcal{R}}(R_1 = 0, R_2 = 0) &= \overline{\pi_1} \cdot \overline{\pi_2} + \sigma \\
P^{\mathcal{R}}(R_1 = 1, R_2 = 0) &= \pi_1 \cdot \overline{\pi_2} - \sigma \\
P^{\mathcal{R}}(R_1 = 0, R_2 = 1) &= \overline{\pi_1} \cdot \pi_2 - \sigma \\
P^{\mathcal{R}}(R_1 = 1, R_2 = 1) &= \pi_1 \cdot \pi_2 + \sigma
\end{aligned}$$

with $\overline{\pi_i} = 1 - \pi_i$

Example

- Knowledge on the reliabilities of the sensors $\mathfrak{s}_1$ and $\mathfrak{s}_2$:

$$\left.\begin{array}{l} P^{\mathcal{R}}(R_1 = 0, R_2 = 0) = 0.3 \\ P^{\mathcal{R}}(R_1 = 1, R_2 = 0) = 0.1 \\ P^{\mathcal{R}}(R_1 = 0, R_2 = 1) = 0.4 \\ P^{\mathcal{R}}(R_1 = 1, R_2 = 1) = 0.2 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \pi_1 = 0.3 \\ \pi_2 = 0.6 \\ \sigma = 0.02 \end{array}\right.$$

Independent reliabilities

SMF-based expression

- R_1 and R_2 independent $\Leftrightarrow \sigma = 0$
- In this case

$$\begin{aligned} m[P^{\mathcal{R}}, \mathbf{A}] &= m[P^{\mathcal{R}_1}, A_1] \textcircled{\cap} m[P^{\mathcal{R}_2}, A_2] \\ &= A_1^{\pi_1} \textcircled{\cap} A_2^{\pi_2} \end{aligned}$$

- Reminder: unnormalized Dempster's rule (conjunctive rule)

$$(m_1 \textcircled{\cap} m_2)(A) = \sum_{B \cap C = A} m_1(B) m_2(C), \quad \forall A \subseteq \mathcal{X}.$$

Dependent reliabilities

SMF-based expression

- More generally, i.e., for any dependency σ, $m[P^{\mathcal{R}}, \mathbf{A}]$ can always be expressed as a conjunctive combination of $A_1^{\pi_1}$ and $A_2^{\pi_2}$ having some dependency...
- "Reminder": conjunctive combination $m_\cap$ of m_1 and m_2 having some known dependency
  1. A joint MF $jm : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \to [0,1]$ is built, having m_1 and m_2 as marginals and encoding their mutual dependence
  2. Each joint mass $jm(B, C)$ is allocated to $B \cap C$:

$$m_\cap(A) = \sum_{B \cap C = A} jm(B, C)$$

Dependent reliabilities

SMF-based expression

- Let $m_i = A_i^{\pi_i}$. Any *jm* having $A_1^{\pi_1}$ and $A_2^{\pi_2}$ as marginals can always be written as

$$\begin{aligned} jm(A_1, A_2) &= \overline{\pi_1} \cdot \overline{\pi_2} + \sigma \\ jm(\mathcal{X}, A_2) &= \pi_1 \cdot \overline{\pi_2} - \sigma \\ jm(A_1, \mathcal{X}) &= \overline{\pi_1} \cdot \pi_2 - \sigma \\ jm(\mathcal{X}, \mathcal{X}) &= \pi_1 \cdot \pi_2 + \sigma \end{aligned}$$

  for some σ.

- Conjunctive combination of $A_1^{\pi_1}$ and $A_2^{\pi_2}$ with dependence structure represented by *jm*, is completely determined by σ.

→ Parameterized conjunctive rule $\circledcap_\sigma$ for two SMF, with parameter σ representing the dependence structure, such that

$$\circledcap_\sigma(A_1^{\pi_1}, A_2^{\pi_2}) := m_\cap$$

- For $\sigma = 0$, $\circledcap_\sigma \Leftrightarrow \circledcap$

Dependent reliabilities

SMF-based expression

- For any dependency σ between the source reliabilities, we have

$$\begin{aligned} m[P^{\mathcal{R}}, \mathbf{A}] &= \circledcap_{\sigma}(m[P^{\mathcal{R}_1}, A_1], m[P^{\mathcal{R}_2}, A_2]) \\ &= \circledcap_{\sigma}(A_1^{\pi_1}, A_2^{\pi_2}) \end{aligned}$$

- Example:
  - Sensor $\mathfrak{s}_1$ not reliable with probability $\pi_1 = 0.3$
  - Sensor $\mathfrak{s}_2$ not reliable with probability $\pi_2 = 0.6$
  - Dependence between their reliability: $\sigma = 0.02$
  - Induced knowledge on $\mathcal{X}$ from the information $\mathbf{A} = (\{g, h\}, \{a, h\})$ provided by the sensors satisfies

$$m[P^{\mathcal{R}}, \mathbf{A}] = \circledcap_{(0.02)} \left(\{g, h\}^{0.3}, \{a, h\}^{0.6}\right)$$

Cautious rule for SMF

- Let $A_1^{\pi_1}$ and $A_2^{\pi_2}$ be two non-independent SMF.
- How to combine them ?
- Cautious conjunctive combination $\owedge$: select the least committed (according to some informational ordering) MF among those that are at least as committed as $A_1^{\pi_1}$ and $A_2^{\pi_2}$.
- Solution based on the w-ordering yields

$$A_1^{\pi_1} \owedge A_2^{\pi_2} = \begin{cases} A_1^{\pi_1 \wedge \pi_2} & \text{if } A_1 = A_2 \\ A_1^{\pi_1} \textcircled{\scriptsize$\cap$} A_2^{\pi_2} & \text{if } A_1 \neq A_2 \end{cases}$$

Cautious rule revisited

- We have

$$A_1^{\pi_1} \owedge A_2^{\pi_2} = \Cap_\sigma(A_1^{\pi_1}, A_2^{\pi_2})$$

  with

$$\sigma = \begin{cases} \pi_1 \wedge \pi_2 - \pi_1\pi_2 & \text{if } A_1 = A_2 \\ 0 & \text{if } A_1 \neq A_2 \end{cases}$$

- Partially reliable sources analysis:
  - $\mathfrak{s}_i$ not reliable with probability π_i and telling $X \in A_i$
  - $\mathfrak{s}_1$ and $\mathfrak{s}_2$ have dependent reliabilities if they support the same subset (actually, perfect dependence between them not being reliable) and independent reliabilities otherwise.

Outline

K information sources

- Assume sources $\mathfrak{s}_i$, $i = 1, \ldots, K$, providing $\mathbf{A} = (A_1, \ldots, A_K)$.
- When the sources are in the state $\mathbf{r} = (r_1, \ldots, r_K) \in \boldsymbol{\mathcal{R}} := \times_{i=1}^{K} \mathcal{R}_i$, we should deduce

$$X \in \Gamma_{\mathbf{A}}(\mathbf{r}) := \bigcap_{i=1}^{K} \Gamma_{A_i}(r_i)$$

- Example: $K = 3$

$$\begin{aligned} \Gamma_{\mathbf{A}}(0,0,1) &= \Gamma_{A_1}(0) \cap \Gamma_{A_2}(0) \cap \Gamma_{A_3}(1) \\ &= A_1 \cap A_2 \cap \mathcal{X} \\ &= A_1 \cap A_2 \end{aligned}$$

Uncertain reliabilities

- Assume state $\mathbf{r}$ is allocated probability $p_{\mathbf{r}}$:

$$P^{\mathcal{R}}(R_1 = r_1, \ldots, R_K = r_K) = p_{\mathbf{r}}$$

- Knowledge about X is then represented by

$$m[P^{\mathcal{R}}, \mathbf{A}](B) = \sum_{\mathbf{r}:\Gamma_{\mathbf{A}}(\mathbf{r})=B} p_{\mathbf{r}}$$

$\rightarrow$ Any set of partially reliable and elementary testimonies is represented by a (unique) MF

Separable mass function

- Independence of all the R_i

$$\begin{aligned} m[P^{\mathcal{R}}, \mathbf{A}] &= \textcircled{\cap}_{i=1}^{K} m[P^{\mathcal{R}_i}, A_i] \\ &= \textcircled{\cap}_{i=1}^{K} A_i^{\pi_i} \end{aligned}$$

with $\pi_i := P^{\mathcal{R}_i}(R_i = 1)$.

$\rightarrow$ Choosing $\mathbf{A}$ s.t. $A_i \neq A_j$, $1 \leq i < j \leq K$, we obtain a partially reliable sources analysis of separable MF[1], which form an important class of MF (often encountered in practice)

[1] MF that can be written as a conjunctive combination of independent SMF supporting different subsets

Separable mass function

- Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $A^{\mathbf{r}}$ denote the subset of $\mathcal{X}$ such that $x_i \in A^{\mathbf{r}}$ if $r_i = 1$ and $x_i \notin A^{\mathbf{r}}$ if $r_i = 0$, for $\mathbf{r} = (r_1, \ldots, r_K)$.
- Example: $\mathcal{X} = \{x_1, x_2, x_3\}$ then $A^{001} = \{x_3\}$

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Proposition

Let m be a MF on $\mathcal{X}$, $K = |\mathcal{X}|$, $A_i = \overline{\{x_i\}}$, and $p_{\mathbf{r}} = m(A^{\mathbf{r}})$. We have

$$m[P^{\mathcal{R}}, \mathbf{A}] = m$$

Proof: Follows from $\Gamma_{\mathbf{A}}(\mathbf{r}) = A^{\mathbf{r}}$.

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$$m[P^{\mathcal{R}}, \mathbf{A}] = m$$

Proof: Follows from $\Gamma_{\mathbf{A}}(\mathbf{r}) = A^{\mathbf{r}}$.

- Recall: Any set of partially reliable and elementary testimonies is represented by a (unique) MF.

$\rightarrow$ Any mass function represents (at least) a set of partially reliable and elementary testimonies

Example

- Let m be the FM on $\mathcal{X} = \{a, g, h\}$ defined by

$$m(\{a, g\}) = 0.5, m(\{h\}) = 0.2, m(\{g, h\}) = 0.3$$

- Consider $K = |\mathcal{X}| = 3$ sources $\mathfrak{s}_1$, $\mathfrak{s}_2$ and $\mathfrak{s}_3$, providing respectively information

$$\begin{aligned} X \in A_1 &= \overline{\{a\}} = \{g, h\} \\ X \in A_2 &= \overline{\{g\}} = \{a, h\} \\ X \in A_3 &= \overline{\{h\}} = \{a, g\} \end{aligned}$$

Example

- Consider meta-knowledge $P^{\mathcal{R}}$ on the sources such that

$\mathbf{r}$	$A^{\mathbf{r}}$	$m(A^{\mathbf{r}}) = p_{\mathbf{r}}$
$(0,0,0)$	$\emptyset$	0
$(1,0,0)$	$\{a\}$	0
$(0,1,0)$	$\{g\}$	0
$(1,1,0)$	$\{a,g\}$	0.5
$(0,0,1)$	$\{h\}$	0.2
$(1,0,1)$	$\{a,h\}$	0
$(0,1,1)$	$\{g,h\}$	0.3
$(1,1,1)$	$\{a,g,h\}$	0

- Then, e.g., $p_{001} = P^{\mathcal{R}}(R_1 = 0, R_2 = 0, R_3 = 1) = m(A^{001}) = 0.2$ is allocated to $\Gamma_{\mathbf{A}}(0,0,1) = A_1 \cap A_2 \cap \mathcal{X} = \overline{\{a\}} \cap \overline{\{g\}} = \{h\} = A^{001}$.
- Since this happens for all $\mathbf{r}$, we have $m[P^{\mathcal{R}}, \mathbf{A}] = m$.

Decomposition of meta-knowledge

- $P^{\mathcal{R}}$ is a multivariate Bernoulli distribution
- It is characterized by

$$\pi_i = \mathbb{E}[R_i]$$

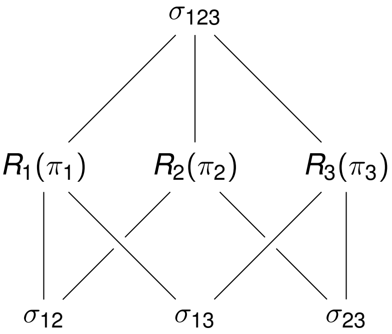
$$\sigma_{\mathbf{r}} = \mathbb{E}\left[\prod_{i=1}^{K}(R_i - \pi_i)^{r_i}\right]$$

with $\mathbf{r} = (r_1, \ldots, r_K)$ such that $\sum_{i=1}^{K} r_i > 1$

- There are $2^K - K - 1$ central moments $\sigma_{\mathbf{r}}$. They represent the dependencies between any subset (of at least two) of all the R_i.
- Notation: $\boldsymbol{\sigma}$ vector whose elements are the dependencies $\sigma_{\mathbf{r}}$

Example

$K = 3$



where

$$\sigma_{12} := \sigma_{110} = \mathbb{E}\left[(R_1 - \pi_1)(R_2 - \pi_2)\right]$$

and $\sigma_{13} := \sigma_{101}$, $\sigma_{23} := \sigma_{011}$, $\sigma_{123} := \sigma_{111}$

$$\boldsymbol{\sigma} = (\sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{123})$$

Example

- Knowledge on the reliabilities of the sources $\mathfrak{s}_1$, $\mathfrak{s}_2$ and $\mathfrak{s}_3$:

$$\left.\begin{array}{l} P^{\mathcal{R}}(R_1 = 0, R_2 = 0, R_3 = 0) = 0 \\ P^{\mathcal{R}}(R_1 = 1, R_2 = 0, R_3 = 0) = 0 \\ P^{\mathcal{R}}(R_1 = 0, R_2 = 1, R_3 = 0) = 0 \\ P^{\mathcal{R}}(R_1 = 1, R_2 = 1, R_3 = 0) = 0.5 \\ P^{\mathcal{R}}(R_1 = 0, R_2 = 0, R_3 = 1) = 0.2 \\ P^{\mathcal{R}}(R_1 = 1, R_2 = 0, R_3 = 1) = 0 \\ P^{\mathcal{R}}(R_1 = 0, R_2 = 1, R_3 = 1) = 0.3 \\ P^{\mathcal{R}}(R_1 = 1, R_2 = 1, R_3 = 1) = 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{r} \pi_1 = 0.5 \\ \pi_2 = 0.8 \\ \pi_3 = 0.5 \\ \sigma_{12} = 0.1 \\ \sigma_{13} = -0.25 \\ \sigma_{23} = -0.1 \\ \sigma_{123} = 0 \end{array}\right.$$

Example

- $P^{\mathcal{R}}$ is recovered from π_i and $\sigma_{\mathbf{r}}$ as follows:

$$
\begin{aligned}
P^{\mathcal{R}}(R_1=0,R_2=0,R_3=0) &= \overline{\pi_1}\,\overline{\pi_2}\,\overline{\pi_3}+\overline{\pi_3}\sigma_{12}+\overline{\pi_2}\sigma_{13}+\overline{\pi_1}\sigma_{23}-\sigma_{123}\\
P^{\mathcal{R}}(R_1=1,R_2=0,R_3=0) &= \pi_1\overline{\pi_2}\,\overline{\pi_3}-\overline{\pi_3}\sigma_{12}-\overline{\pi_2}\sigma_{13}+\pi_1\sigma_{23}+\sigma_{123}\\
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P^{\mathcal{R}}(R_1=1,R_2=1,R_3=1) &= \pi_1\pi_2\pi_3+\pi_3\sigma_{12}+\pi_2\sigma_{13}+\pi_1\sigma_{23}+\sigma_{123}
\end{aligned}
$$

- Remark: simple matrix-based expressions exist to switch from one representation to the other.

Decomposition of a mass function

Based on the previous proposition as well as the decomposition of meta-knowledge, any MF m on $\mathcal{X}$ is induced by the following basic components:

1. A set of $|\mathcal{X}|$ sources $\mathfrak{S} = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_{|\mathcal{X}|}\}$, with $\mathfrak{s}_i$ providing information $X \in \overline{\{x_i\}}$;
2. Probabilistic knowledge on their reliability characterized by :
   - For each $\mathfrak{s}_i$, a (marginal) probability π_i of being not reliable;
   - For each $S_\mathbf{r} \subseteq \mathfrak{S}$, knowledge about the dependency between their reliabilities in the form of the central moment $\sigma_\mathbf{r}$.

Remark: we have $\pi_i = pl(\{x_i\})$.

Example

- m defined by $m(\{a,g\}) = 0.5, m(\{h\}) = 0.2, m(\{g,h\}) = 0.3$.
- This MF is induced by
  1. Considering $|\mathcal{X}| = 3$ sources $\mathfrak{s}_1$, $\mathfrak{s}_2$ and $\mathfrak{s}_3$, providing information

$$\begin{aligned} X \in A_1 &= \overline{\{a\}} \\ X \in A_2 &= \overline{\{g\}} \\ X \in A_3 &= \overline{\{h\}} \end{aligned}$$

  2. And by assuming that
     - $\mathfrak{s}_1$, $\mathfrak{s}_2$ and $\mathfrak{s}_3$ are not reliable with respective probabilities $\pi_1 = 0.5$, $\pi_2 = 0.8$ and $\pi_3 = 0.5$
     - there is a dependence $\sigma_{12} = 0.1$ between the reliabilities of $\mathfrak{s}_1$ and $\mathfrak{s}_2$, $\sigma_{13} = -0.25$ between $\mathfrak{s}_1$ and $\mathfrak{s}_3$, $\sigma_{23} = -0.1$ between $\mathfrak{s}_2$ and $\mathfrak{s}_3$, and $\sigma_{123} = 0$ between $\mathfrak{s}_1$, $\mathfrak{s}_2$ and $\mathfrak{s}_3$.

SMF-based expression of $m[P^{\mathcal{R}}, \mathbf{A}]$

- Let $m_\cap$ denote the conjunctive combination of SMF $A_i^{\pi_i}$, $1 \leq i \leq K$, with dependence structure represented by *jm*
- *jm* can always be written in a particular (familiar) form such that the dependencies it encodes are completely determined by $2^K - K - 1$ parameters

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- jm can always be written in a particular (familiar) form such that the dependencies it encodes are completely determined by $2^K - K - 1$ parameters
- Example: $K = 3$, four parameters noted $\sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{123}$

$$
\begin{aligned}
jm(A_1, A_2, A_3) &= \overline{\pi_1}\,\overline{\pi_2}\,\overline{\pi_3} + \overline{\pi_3}\sigma_{12} + \overline{\pi_2}\sigma_{13} + \overline{\pi_1}\sigma_{23} - \sigma_{123}\\
jm(\mathcal{X}, A_2, A_3) &= \pi_1\overline{\pi_2}\,\overline{\pi_3} - \overline{\pi_3}\sigma_{12} - \overline{\pi_2}\sigma_{13} + \pi_1\sigma_{23} + \sigma_{123}\\
jm(A_1, A_2, A_3) &= \overline{\pi_1}\pi_2\overline{\pi_3} - \overline{\pi_3}\sigma_{12} + \pi_2\sigma_{13} - \overline{\pi_1}\sigma_{23} + \sigma_{123}\\
jm(\mathcal{X}, \mathcal{X}, A_3) &= \pi_1\pi_2\overline{\pi_3} + \overline{\pi_3}\sigma_{12} - \pi_2\sigma_{13} - \pi_1\sigma_{23} - \sigma_{123}\\
jm(A_1, A_2, \mathcal{X}) &= \overline{\pi_1}\,\overline{\pi_2}\pi_3 + \pi_3\sigma_{12} - \overline{\pi_2}\sigma_{13} - \overline{\pi_1}\sigma_{23} + \sigma_{123}\\
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jm(A_1, \mathcal{X}, \mathcal{X}) &= \overline{\pi_1}\pi_2\pi_3 - \pi_3\sigma_{12} - \pi_2\sigma_{13} + \overline{\pi_1}\sigma_{23} - \sigma_{123}\\
jm(\mathcal{X}, \mathcal{X}, \mathcal{X}) &= \pi_1\pi_2\pi_3 + \pi_3\sigma_{12} + \pi_2\sigma_{13} + \pi_1\sigma_{23} + \sigma_{123}
\end{aligned}
$$

SMF-based expression of $m[P^{\mathcal{R}}, \mathbf{A}]$

- Parameterized conjunctive rule $\circledcap_{\boldsymbol{\sigma}}$ for K SMF, with $\boldsymbol{\sigma}$ the vector whose elements are the dependency parameters $\sigma_{\mathbf{r}}$, such that

$$\circledcap_{\boldsymbol{\sigma}}(A_1^{\pi_1}, \ldots, A_K^{\pi_K}) := m_{\cap}$$

SMF-based expression of $m[P^{\mathcal{R}}, \mathbf{A}]$

- Parameterized conjunctive rule $\bigcirc\!\!\!\!\cap_{\boldsymbol{\sigma}}$ for K SMF, with $\boldsymbol{\sigma}$ the vector whose elements are the dependency parameters $\sigma_{\mathbf{r}}$, such that

$$\bigcirc\!\!\!\!\cap_{\boldsymbol{\sigma}}(A_1^{\pi_1}, \ldots, A_K^{\pi_K}) := m_\cap$$

Theorem

For any dependency $\boldsymbol{\sigma}$ between the source reliabilities, we have

$$\begin{aligned} m[P^{\mathcal{R}}, \mathbf{A}] &= \bigcirc\!\!\!\!\cap_{\boldsymbol{\sigma}}(m[P^{\mathcal{R}_1}, A_1], \ldots, m[P^{\mathcal{R}_K}, A_K]) \\ &= \bigcirc\!\!\!\!\cap_{\boldsymbol{\sigma}}(A_1^{\pi_1}, \ldots, A_K^{\pi_K}) \end{aligned}$$

Conjunctive canonical decomposition of a MF

Theorem

Any MF m satisfies

$$m = \circledcap_{\boldsymbol{\sigma}} \left(\overline{\{x_1\}}^{pl(x_1)}, \ldots, \overline{\{x_n\}}^{pl(x_n)} \right),$$

with $\boldsymbol{\sigma}$ the vector of dependencies between the reliabilities of the sources underlying m.

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- Example: m defined by

$$m(\{a,g\}) = 0.5, m(\{h\}) = 0.2, m(\{g,h\}) = 0.3$$

satisfies

$$m = \circledcap_{(0.1,-0.25,-0.1,0)} \left(\overline{\{a\}}^{0.5}, \overline{\{g\}}^{0.8}, \overline{\{h\}}^{0.5} \right)$$

Conclusions

- Any mass function can be seen as the result of the
  - interpretation of a set of partially reliable and elementary testimonies
  - conjunctive combination of SMF (focused on $\overline{\{x_i\}}$) having some dependencies.
- In the spirit of [Shafer, 1976] and [Smets, 1995] interpretations of belief functions (they considered only independent SMF to try and recover the entire space of mass functions, see Didier's lecture)
- Combining conjunctively SMF focused on $\overline{\{x_i\}}$ is found in some important results: generalized Bayesian theorem, BFT analysis of binary logistic regression [Denoeux, 2019].

Outline

The case of a single source

- Assume now the source $\mathfrak{s}$ provides an uncertain testimony about X in the form of a MF $m_{\mathfrak{s}}$.
- If $\mathfrak{s}$ is reliable, then each $m_{\mathfrak{s}}(A)$ should be transferred to $\Gamma_A(0) = A$.
- If $\mathfrak{s}$ is not reliable, then each $m_{\mathfrak{s}}(A)$ should be transferred to $\Gamma_A(1) = \mathcal{X}$.
- Assuming $\mathfrak{s}$ to be not reliable with probability $P^{\mathcal{R}}(R = 1) = \pi$ then yields the following MF on $\mathcal{X}$:

$$m[P^{\mathcal{R}}, m_{\mathfrak{s}}] = (1 - \pi) \cdot m_{\mathfrak{s}} + \pi \cdot m_{\mathcal{X}}$$

  with $m_{\mathcal{X}}$ the vacuous MF ($m_{\mathcal{X}}(\mathcal{X}) = 1$).
- This is known as the discounting of $m_{\mathfrak{s}}$ with discount rate π (basic information correction mechanism in BFT).

Example

- Uncertain testimony:

$$m_{\mathfrak{s}}(\{a,g\}) = 0.8, m_{\mathfrak{s}}(\{h\}) = 0.2$$

- Uncertain reliability:

$$P^{\mathcal{R}}(R=0) = 0.7, P^{\mathcal{R}}(R=1) = 0.3$$

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$m_{\mathfrak{s}} \setminus P^{\mathcal{R}}$	$R=0$ 0.7	$R=1$ 0.3
$\{a,g\}$ 0.8	$\{a,g\}$ 0.56	$\mathcal{X}$ 0.24
$\{h\}$ 0.2	$\{h\}$ 0.14	$\mathcal{X}$ 0.06

Example

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$m_{\mathfrak{s}} \setminus P^{\mathcal{R}}$	$R=0$ 0.7	$R=1$ 0.3
$\{a,g\}$ 0.8	$\{a,g\}$ 0.56	$\mathcal{X}$ 0.24
$\{h\}$ 0.2	$\{h\}$ 0.14	$\mathcal{X}$ 0.06

$\rightarrow$ $m[P^{\mathcal{R}}, m_{\mathfrak{s}}](\{a,g\}) = 0.56$, $m[P^{\mathcal{R}}, m_{\mathfrak{s}}](\{h\}) = 0.14$, $m[P^{\mathcal{R}}, m_{\mathfrak{s}}](\mathcal{X}) = 0.30$

The case of multiple sources

- Assume sources $\mathfrak{s}_1, \ldots, \mathfrak{s}_K$ provide uncertain testimonies $\mathbf{m} = (m_1, \ldots, m_K)$.
- Assume they are independent: interpreting $m_i(A_i)$ as the probability that $\mathfrak{s}_i$ supplies $X \in A_i$, then the probability, denoted $m(\mathbf{A})$, that they supply $\mathbf{A} = (A_1, \ldots, A_K)$ is $\prod_{i=1}^{K} m_i(A_i)$.
- If they are in state $\mathbf{r} \in \mathcal{R}$, then $m(\mathbf{A})$ should be transferred to $\Gamma_{\mathbf{A}}(\mathbf{r})$
- Assuming they are in state $\mathbf{r}$ with probability $p_{\mathbf{r}}$ then yields the following MF on $\mathcal{X}$:

$$m[P^{\mathcal{R}}, \mathbf{m}](B) = \sum_{\mathbf{r}, \mathbf{A} : \Gamma_{\mathbf{A}}(\mathbf{r}) = B} p_{\mathbf{r}} \cdot m(\mathbf{A})$$

Particular cases

$m[P^{\mathcal{R}}, \mathbf{m}]$ reduces to

- $\textcircled{\cap}_{i=1}^{K} m_i$ if $p_{\mathbf{r}} = 1$ for $\mathbf{r} = (0, 0, \ldots, 0, 0)$, i.e., all sources are reliable

$\rightarrow$ conjunctive rule

- $\textcircled{\cap}_{i=1}^{K} m[P^{\mathcal{R}_i}, m_i]$ if $p_{\mathbf{r}} = \prod_{i=1}^{K} P^{\mathcal{R}_i}(R_i = r_i)$, i.e., the sources have independent reliabilities

$\rightarrow$ discount and combine approach

- $\sum_{i=1}^{K} w_i \cdot m_i$ if $p_{\mathbf{r}} = w_i$ for $\mathbf{r}$ such that $r_i = 1$ and $r_j = 0$, for all $j \neq i$, i.e., source $\mathfrak{s}_i$ is reliable and the other are not with probability w_i

$\rightarrow$ weighted average

Outline

Truthfulness

- Reliability of a source includes another dimension besides its relevance: its truthfulness.
- A source is said truthful if it actually supplies the information it possesses.
- Lack of truthfulness can take several forms, and can be intentional or accidental.
- For instance, a sensor that has a systematic bias (such as a watch that has not been calibrated to winter time), is a kind of (unintentional) lack of truthfulness.
- Let us first consider the crudest form, where a source is non truthful if it tells the contrary of what it knows.

Outline

Formalization

- Assume a source $\mathfrak{s}$ supplying information item $X \in A$
  - If $\mathfrak{s}$ is not relevant, we replace $X \in A$ by $X \in \mathcal{X}$
  - If $\mathfrak{s}$ is relevant,
    - either it is truthful, in which case we keep $X \in A$
    - or it lies, in which case we replace $X \in A$ by $X \in \overline{A}$

Formalization

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  - If $\mathfrak{s}$ is relevant,
    - either it is truthful, in which case we keep $X \in A$
    - or it lies, in which case we replace $X \in A$ by $X \in \overline{A}$
- Relevance R defined on $\mathcal{R} = \{\text{rel}, \neg\text{rel}\}$
- Truthfulness T defined on $\mathcal{T} = \{\text{tru}, \neg\text{tru}\}$
- Let $\mathcal{RT} := \mathcal{R} \times \mathcal{T}$
- The interpretation of the testimony according to the relevance and truthfulness may be encoded by $\Lambda_A : \mathcal{RT} \to 2^{\mathcal{X}}$ such that

$$
\begin{aligned}
\Lambda_A(\text{rel}, \text{tru}) &= A \\
\Lambda_A(\text{rel}, \neg\text{tru}) &= \overline{A} \\
\Lambda_A(\neg\text{rel}, \text{tru}) &= \mathcal{X} \\
\Lambda_A(\neg\text{rel}, \neg\text{tru}) &= \mathcal{X}
\end{aligned}
$$

Example

- Sensor $\mathfrak{s}$ tells $X \in A = \overline{\{a\}} = \{g, h\}$.
- It is assumed to be relevant and non truthful, i.e., in state (rel, ¬tru)
- Knowledge about X:

$$X \in \Lambda_{\overline{\{a\}}}(\text{rel}, \neg\text{tru}) = \{a\}$$

Uncertain meta-knowledge and testimony

- Assume now $\mathfrak{s}$ is in state $(r, t) \in \mathcal{RT}$ with probability $p_{rt} = P^{\mathcal{RT}}(R = r, T = t)$ (and still $P^{\mathcal{R}}(R = \neg\text{rel}) = \pi$)
- In addition, $\mathfrak{s}$ provides the uncertain testimony $m_{\mathfrak{s}}$.
- This yields the following knowledge on $\mathcal{X}$

$$m[P^{\mathcal{RT}}, m_{\mathfrak{s}}] \quad = \quad p_{\text{rel,tru}} \cdot m_{\mathfrak{s}} + p_{\text{rel},\neg\text{tru}} \cdot \overline{m}_{\mathfrak{s}} + \pi m_{\mathcal{X}}$$

with $\overline{m}_{\mathfrak{s}}$ the negation of $m_{\mathfrak{s}}$ ($\overline{m}_{\mathfrak{s}}(A) = m_{\mathfrak{s}}(\overline{A})$, for all $A \subseteq \mathcal{X}$)

Particular cases

$m[P^{\mathcal{RT}}, m_{\mathfrak{s}}]$ reduces to

- $m[P^{\mathcal{R}}, m_{\mathfrak{s}}]$ if $P^{\mathcal{T}}(T = \text{tru}) = 1$, i.e., if $\mathfrak{s}$ is partially relevant and totally truthful

$\rightarrow$ discounting

- $P^{\mathcal{T}}(T = \text{tru}) \cdot m_{\mathfrak{s}} + P^{\mathcal{T}}(T = \neg\text{tru}) \cdot \overline{m}_{\mathfrak{s}}$ if $P^{\mathcal{R}}(R = \text{rel}) = 1$, i.e., if $\mathfrak{s}$ is totally relevant and partially truthful

$\rightarrow$ negating

The case of multiple sources

- Assume sources $\mathfrak{s}_i$, $i = 1\ldots, K$, supplying $\mathbf{A} = (A_1, \ldots, A_K)$.
- Let $\Lambda_{A_i} : \mathcal{RT}_i \rightarrow 2^{\mathcal{X}}$ represent the interpretation of $X \in A_i$ given the reliability R_i and truthfulness T_i of $\mathfrak{s}_i$
- When the sources are in the state

$$\mathbf{rt} = (rt_1, \ldots, rt_K) \in \boldsymbol{\mathcal{RT}} := \times_{i=1}^{K} \mathcal{RT}_i$$

we must conclude

$$X \in \Lambda_{\mathbf{A}}(\mathbf{rt}) := \bigcap_{i=1}^{K} \Lambda_{A_i}(rt_i)$$

- Example: K=2

$$\begin{aligned}\Lambda_{\mathbf{A}}(\mathrm{rel}_1, \mathrm{tru}_1, \mathrm{rel}_2, \neg\mathrm{tru}_2) &= \Lambda_{A_1}(\mathrm{rel}_1, \mathrm{tru}_1) \cap \Lambda_{A_2}(\mathrm{rel}_2, \neg\mathrm{tru}_2) \\ &= A_1 \cap \overline{A_2}\end{aligned}$$

Non-elementary behavior assumptions

- Non-elementary assumptions $\mathbf{RT} \subseteq \mathcal{RT}$ on the relevance and truthfulness of the sources can also be considered.
- We have

$$\Lambda_{\mathbf{A}}(\mathbf{RT}) = \bigcup_{\mathbf{rt} \in \mathbf{RT}} \Lambda_{\mathbf{A}}(\mathbf{rt})$$

- Example: $\mathbf{RT} = \{(\mathrm{rel}_1, \mathrm{tru}_1, \mathrm{rel}_2, \neg\mathrm{tru}_2), (\mathrm{rel}_1, \neg\mathrm{tru}_1, \mathrm{rel}_2, \mathrm{tru}_2)\}$ ($\mathfrak{s}_1$ and $\mathfrak{s}_2$ relevant and exactly one of them is truthful)

$$\begin{aligned} \Lambda_{\mathbf{A}}(\mathbf{RT}) &= \Lambda_{\mathbf{A}}(\mathrm{rel}_1, \mathrm{tru}_1, \mathrm{rel}_2, \neg\mathrm{tru}_2) \cup \Lambda_{\mathbf{A}}(\mathrm{rel}_1, \neg\mathrm{tru}_1, \mathrm{rel}_2, \mathrm{tru}_2) \\ &= (A_1 \cap \overline{A_2}) \cup (\overline{A_1} \cap A_2) \\ &= A_1 \Delta A_2 \text{ (exclusive or)} \end{aligned}$$

$\rightarrow$ All connectives of Boolean logic can be reinterpreted in terms of source behavior assumptions wrt relevance and truthfulness

- $\otimes_{\mathbf{RT}}$ Boolean connective associated to **RT**.
- Different assumptions may induce the same connective.

Uncertain meta-knowledge and testimonies

- Sources $\mathfrak{s}_1, \ldots, \mathfrak{s}_K$ provide $\mathbf{m} = (m_1, \ldots, m_K)$ and are assumed to be independent.
- Uncertain meta-knowledge in the form of a MF $m^{\mathcal{RT}}$:

$$
\begin{aligned}
m[m^{\mathcal{RT}}, \mathbf{m}](B) &= \sum_{\mathbf{RT}, \mathbf{A} : \Lambda_{\mathbf{A}}(\mathbf{RT}) = B} m^{\mathcal{RT}}(\mathbf{RT}) \cdot m(\mathbf{A}) \\
&= \sum_{\otimes, \mathbf{A} : \otimes(\mathbf{A}) = B} p(\otimes) \cdot m(\mathbf{A})
\end{aligned}
$$

with

$$
p(\otimes) = \sum_{\otimes_{\mathbf{RT}} = \otimes} m^{\mathcal{RT}}(\mathbf{RT})
$$

Particular cases

- Suppose $m^{\mathcal{RT}}$ is such that $m^{\mathcal{RT}}(\mathbf{RT}) = 1$ for some $\mathbf{RT} \subseteq \mathcal{RT}$ and $K = 2$. Then, $m[m^{\mathcal{RT}}, \mathbf{m}]$ reduces to
  - $m_1 \textcircled{\cap} m_2$ for **RT**= $\mathfrak{s}_1$ and $\mathfrak{s}_2$ relevant and truthful
  - $m_1 \textcircled{\cup} m_2$ for **RT**= $\mathfrak{s}_1$ and $\mathfrak{s}_2$ relevant and at least one of them is truthful (disjunctive rule, def. of $\textcircled{\cap}$ with $\cap$ replaced by $\cup$)
  - $m_1 \underline{\textcircled{\cup}} m_2$ for **RT**= $\mathfrak{s}_1$ and $\mathfrak{s}_2$ relevant and exaclty one of them is truthful (exclusive disjunctive rule, $\cap$ replaced by Δ)
  - $m_1 \underline{\textcircled{\cap}} m_2$ for **RT**= $\mathfrak{s}_1$ and $\mathfrak{s}_2$ relevant and $\mathfrak{s}_1$ is truthful if and only if $\mathfrak{s}_2$ is so too (equivalence rule, $\cap$ replaced by $\leftrightarrow$)
- More generally, all rules relying on Boolean connectives are particular cases. For instance, the rule extending q-relaxation from interval analysis is recovered for **RT**= $(K - q)$-out-of-K sources relevant and all truthful (ranges from $\textcircled{\cap}$ to $\textcircled{\cup}$).

Outline

Contextual and polarized lack of truthfulness

- $\neg$tru : one must deduce the contrary of what $\mathfrak{s}$ tells for each $x_i \in \mathcal{X}$ and whatever the polarity of the clause used by $\mathfrak{s}$ regarding x_i.

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- $\mathfrak{s}$ non truthful only for some $x_i \in \mathcal{X}$, and maybe even only for the positive or negative clauses regarding x_i.
- Example : Sensor $\mathfrak{s}$ is
  - non truthful when it tells that a is not a possible value for X,
  - non truthful when it tells that g is a possible value for X,
  - and truthful in all other cases, e.g., truthful when it tells that a is a possible value for X
- Sensor $\mathfrak{s}$ tells $X \in A = \{g, h\}$, i.e., a is not a possible value and g and h are possible values for X
- We deduce (assuming $\mathfrak{s}$ relevant): $X \in \{a, h\}$

Contextual and polarized lack of truthfulness

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$\rightarrow$ By considering source states based on this refined form of lack of truthfulness, we can recover contextual discounting and the α-junctions, and contextualize negating (see appendix)

Outline

Beyond relevance and truthfulness

- Knowledge about the source quality may be different from knowing their relevance and truthfulness
- The provided information by a source may also bear on another variable Y, related to X.

$\rightarrow$ An approach to account for general source quality (behaviour) assumptions

$$\begin{array}{rcl} \mathcal{R}, \mathcal{RT} & \rightsquigarrow & \mathcal{H} = \{h^1, \ldots, h^N\} \\ X \in A \subseteq \mathcal{X} & \rightsquigarrow & Y \in A \subseteq \mathcal{Y} \end{array}$$

- If the source is in state $h \in \mathcal{H}$, we should deduce $X \in B \subseteq \mathcal{X}$ from information item $Y \in A \subseteq \mathcal{Y}$.
- For all $A \subseteq \mathcal{Y}$, $\Pi_A : \mathcal{H} \rightarrow 2^{\mathcal{X}}$ such that

$$\Pi_A(h) = B$$

Example

- X with possible values in $\mathcal{X} = \{a, g, h\}$
- Sensor $\mathfrak{s}$ does not know the type airplane, i.e., $\mathcal{Y} = \{g, h\}$.
- It uses either the shape or the material of the aircraft
  - If $\mathfrak{s}$ uses the shape, then when it tells
    - glider, we can deduce airplane or glider
    - helicopter, we keep this piece of information
  - If $\mathfrak{s}$ uses the material, then when it tells
    - glider, we keep this piece of information
    - helicopter, we replace by helicopter or airplane

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    - glider, we can deduce airplane or glider
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  - If $\mathfrak{s}$ uses the material, then when it tells
    - glider, we keep this piece of information
    - helicopter, we replace by helicopter or airplane
- $\mathcal{H} = \{\text{shape}, \text{material}\}$

$$\begin{aligned} \Pi_g(\text{shape}) &= \{a, g\} \\ \Pi_h(\text{shape}) &= \{h\} \\ \Pi_g(\text{material}) &= \{g\} \\ \Pi_h(\text{material}) &= \{a, h\} \end{aligned}$$

Example

- We are interested by the number $X \in \mathcal{X} = \{x_1, \ldots, x_n\}$=$\{1, \ldots, n\}$ of aircrafts in a particular area.
- Information about X comes from a source $\mathfrak{s}$, which can be reliable, approximately reliable or non reliable.
- If $\mathfrak{s}$ is approximately reliable, the information item it supplies must be expanded using the lowest and highest closest values.

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- We are interested by the number $X \in \mathcal{X} = \{x_1, \ldots, x_n\}$=$\{1, \ldots, n\}$ of aircrafts in a particular area.
- Information about X comes from a source $\mathfrak{s}$, which can be reliable, approximately reliable or non reliable.
- If $\mathfrak{s}$ is approximately reliable, the information item it supplies must be expanded using the lowest and highest closest values.
- $\mathcal{H} = \{\text{rel}, \text{ap-rel}, \neg\text{rel}\}$
- For any $A_{i,j} \subseteq \mathcal{X}$, with $A_{i,j} = \{x_i, \ldots, x_j\}$, $1 \leq i \leq j \leq n$

$$
\begin{aligned}
\Pi_{A_{i,j}}(\text{rel}) &= A_{i,j} \\
\Pi_{A_{i,j}}(\text{ap-rel}) &= \{x_{i-1}\} \cup A_{i,j} \cup \{x_{j+1}\} \\
\Pi_{A_{i,j}}(\neg\text{rel}) &= \mathcal{X}
\end{aligned}
$$

with $x_0 = x_{n+1} = \emptyset$.

Uncertain meta-knowledge and testimonies

- Single information source

$$m[m^{\mathcal{H}}, m^{\mathcal{Y}}_{\mathfrak{S}}](B) = \sum_{H,A:\Pi_A(H)=B} m^{\mathcal{H}}(H) \cdot m^{\mathcal{Y}}_{\mathfrak{S}}(A)$$

Behaviour-based correction (BBC)

Uncertain meta-knowledge and testimonies

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$$m[m^{\mathcal{H}}, m^{\mathcal{Y}}_{\mathfrak{S}}](B) \;=\; \sum_{H,A:\Pi_A(H)=B} m^{\mathcal{H}}(H) \cdot m^{\mathcal{Y}}_{\mathfrak{S}}(A)$$

Behaviour-based correction (BBC)

- Multiple information sources: $\boldsymbol{\mathcal{H}} := \times_{i=1}^{K} \mathcal{H}_i$

$$m[m^{\boldsymbol{\mathcal{H}}}, \mathbf{m}](B) \;=\; \sum_{\mathbf{H},\mathbf{A}:\Pi_{\mathbf{A}}(\mathbf{H})=B} m^{\boldsymbol{\mathcal{H}}}(\mathbf{H}) \cdot m(\mathbf{A})$$

with $m(\mathbf{A}) = \prod_{i=1}^{K} m_i^{\mathcal{Y}}(A_i)$

Behaviour-based fusion (BBF)

Operations on product spaces

BBC and BBF can be recovered using the following standard operations of BFT :

- Marginalization $\downarrow$

$$m^{\mathcal{X}\times\mathcal{Y}\downarrow\mathcal{X}}(A) = \sum_{\{B\subseteq\mathcal{X}\times\mathcal{Y}, (B\downarrow X)=A\}} m^{\mathcal{X}\times\mathcal{Y}}(B), \quad \forall A \subseteq \mathcal{X},$$

- Conjunctive rule on product spaces

$$m_1^{\mathcal{X}} \textcircled{\cap} m_2^{\mathcal{Y}} = m_1^{\mathcal{X}\uparrow\mathcal{X}\times\mathcal{Y}} \textcircled{\cap} m_2^{\mathcal{Y}\uparrow\mathcal{X}\times\mathcal{Y}}.$$

with $\uparrow$ (vacuous extension) defined as

$$m^{\mathcal{X}\uparrow\mathcal{X}\times\mathcal{Y}}(B) = \begin{cases} m^{\mathcal{X}}(A) & \text{if } B = A\times\mathcal{Y} \text{ for some } A\subseteq\mathcal{X}, \\ 0 & \text{otherwise.} \end{cases}$$

BBC

- Mappings Π_A, $A \subseteq \mathcal{Y}$, define a relation between spaces $\mathcal{H}$, $2^{\mathcal{Y}}$ and $\mathcal{X}$, which can be represented by MF m_Π on $\mathcal{H} \times 2^{\mathcal{Y}} \times \mathcal{X}$ s.t.

$$m_\Pi \left[\bigcup_{h \in \mathcal{H}, A \in 2^{\mathcal{Y}}} (\{h\} \times \{A\} \times \Pi_A(h)) \right] = 1$$

BBC

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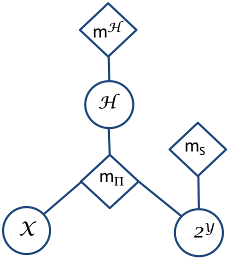
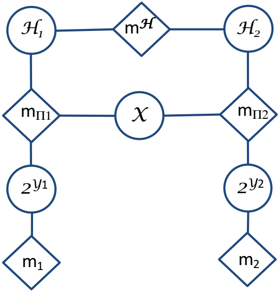


Lemma

$$m[m^{\mathcal{H}}, m_{\mathfrak{S}}^{\mathcal{Y}}] = (m_{\mathfrak{S}} \circledcap m_\Pi \circledcap m^{\mathcal{H}})^{\downarrow \mathcal{X}}$$

with $m_{\mathfrak{S}}$ on $2^{\mathcal{Y}}$ s.t. $m_{\mathfrak{S}}(\{A\}) = m_{\mathfrak{S}}^{\mathcal{Y}}(A)$

BBF



Lemma

$$m[m^{\mathcal{H}}, \mathbf{m}] \quad = \quad \left(\bigcap_{i=1}^{K} (m_i \bigcap m_{\Pi i}) \bigcap m^{\mathcal{H}} \right)^{\downarrow \mathcal{X}}$$

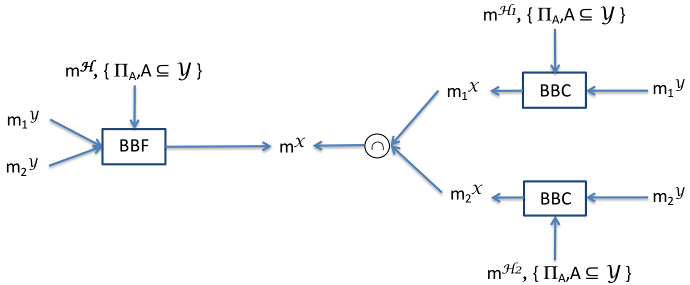
Independent behaviours (meta-independence)

Theorem

If $m^{\mathcal{H}} = \textcircled{\cap}_{i=1}^{K} m^{\mathcal{H}_i}$ *then*

$$m[m^{\mathcal{H}}, \mathbf{m}] = \textcircled{\cap}_{i=1}^{K} m[m^{\mathcal{H}_i}, m_i]$$

Proof: Uses local computation (see Prakash's lecture).



Outline

Typology of approaches

- The model allows to interpret pieces of information given meta-knowledge on the emitting sources.
- It does not however indicate which meta-knowledge to use.

$\rightarrow$ Means to select meta-knowledge

- Two possible situations:
  1. One has some prior information (learning data, expert knowledge) on the sources
  2. The only available information are the pieces of information received
- Typically, in both cases, a set $\mathcal{S}$ of candidate assumptions (meta-knowledge) is considered, and some sensible strategy is used to pick an assumption in this set.

Outline

Consistency and specificity

- Only $\mathbf{m} = (m_1, \ldots, m_K)$ available.

→ Selection of meta-knowledge based on the two primary features sought regarding knowledge about X: consistency and specificity

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- 3 sources about $X \in \mathcal{X} = \{a, g, h\}$ supplying $\mathbf{A} = (A_1, A_2, A_3)$ s.t.

$$A_1 = \{a\}, A_2 = \{a, g\}, A_3 = \{g, h\}$$

- Assumption $\mathbf{R}_1$ = "all sources are reliable" yields

$$X \in \Gamma_{\mathbf{A}}(\mathbf{R}_1) = A_1 \cap A_2 \cap A_3 = \emptyset$$

  i.e. an inconsistent result, and thus cannot hold.

- In contrast, the assumption $\mathbf{R}_3$ = "at least one of the sources is reliable" yields

$$X \in \Gamma_{\mathbf{A}}(\mathbf{R}_3) = A_1 \cup A_2 \cup A_3 = \mathcal{X}$$

  and is thus plausible (it does not yield a contradiction). However, it is useless as it is not informative at all.

Meta-knowledge selection strategy

- The intermediate assumption $\mathbf{R}_2 =$ "at least two of the sources are reliable" yields

$$X \in \Gamma_{\mathbf{A}}(\mathbf{R}_2) = (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3) = \{a, g\}$$

  $\mathbf{R}_2$ is plausible (the result is consistent) and informative (or, at least, more informative than $\mathbf{R}_3$).

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  $\mathbf{R}_2$ is plausible (the result is consistent) and informative (or, at least, more informative than $\mathbf{R}_3$).
- Here, $\mathbf{R}_2$ is preferable, but for other $\mathbf{A}$, it could be $\mathbf{R}_1$ or $\mathbf{R}_3$ due to

$$\Gamma_{\mathbf{A}}(\mathbf{R}_1) \subseteq \Gamma_{\mathbf{A}}(\mathbf{R}_2) \subseteq \Gamma_{\mathbf{A}}(\mathbf{R}_3), \quad \forall \mathbf{A}$$

  $\mathbf{R}_{i+1}$ will always yield a result that is on the hand at least as consistent as that of $\mathbf{R}_i$, but also on the other hand as most as specific as that of $\mathbf{R}_i$.

$\rightarrow$ Consistency and specificity are antagonists goals

- Sensible strategy for a given $\mathbf{A}$: test iteratively each $\mathbf{R}_i$ and select the first one which yields a consistent result (it will then be the most specific and consistent possible result).

Extension to uncertain meta-knowledge and testimonies

- In general, meta-knowledge and supplied information are uncertain, i.e., we have $\mathbf{m}^{\mathcal{H}}$ and $\mathbf{m} = (m_1, \ldots, m_K)$, and thus their interpretation is the MF $m[m^{\mathcal{H}}, \mathbf{m}]$ (assuming independent sources).
- Need extensions to MF of consistency and specificity in order to compare pieces of meta-knowledge:
  - consistency of a MF m: $\phi(m) = \max_{x \in \mathcal{X}} pl(x)$.
  - specificity: $m_1 \sqsubseteq m_2$ with $\sqsubseteq$ the specialization

Proposition

Let $m_1^{\mathcal{H}}$ and $m_2^{\mathcal{H}}$ be two assumptions.

$$m[m_1^{\mathcal{H}}, \mathbf{m}] \sqsubseteq m[m_2^{\mathcal{H}}, \mathbf{m}], \forall \mathbf{m} \Rightarrow \phi(m[m_1^{\mathcal{H}}, \mathbf{m}]) \leq \phi(m[m_2^{\mathcal{H}}, \mathbf{m}]), \forall \mathbf{m}$$

$\rightarrow$ Consistency and specificity are at odds !

General meta-knowledge selection strategy

Strategy

1. Define a set $\mathcal{S} = \{m_1^{\mathcal{H}}, ..., m_M^{\mathcal{H}}\}$:
   - $m[m_j^{\mathcal{H}}, \mathbf{m}] \sqsubseteq m[m_{j+1}^{\mathcal{H}}, \mathbf{m}], \forall \mathbf{m}$;
   - $m_1^{\mathcal{H}}$ corresponds to the conjunctive rule.
2. Test iteratively each $m_j^{\mathcal{H}}$ until $\phi(\mathbf{m}[m_j^{\mathcal{H}}, \mathbf{m}]) \geq \tau$.

- Practical instances of $\mathcal{S}$:
  - $m_j^{\mathcal{H}}$: $K - j + 1$ out of K reliable sources.
  - $m_j^{\mathcal{H}}$: sources with independent reliabilities, source i reliable with probability p_i^j such that $p_i^j \geq p_i^{j+1}$ (increasing discount and combine, often used for conflict management)
  - $m_j^{\mathcal{H}}$: meta-knowledge corresponding to the α-conjunctions for some $\alpha = \alpha_j$ such that $\alpha_j \geq \alpha_{j+1}$.

Application

Nuclear reactor safety

- Project BEMUSE of the Nuclear Energy Agency.
- $K = 10$ sources (CEA, IRSN,...) providing uncertain estimates of parameter values of a nuclear power plant.
- Costly data and complex phenomena involved $\rightarrow$ no reliable means to know the source reliabilities.
- Chose $\mathcal{S}$ with $m_j^{\mathcal{H}} = K - j + 1$ out of K reliable sources.
- PCT2 parameter with domain $\mathcal{X} = \{x_1, \ldots, x_6\}$, $\mathbf{m} := (m_1, \ldots, m_{10})$.
  - $\phi(m[m_1^{\mathcal{H}}, \mathbf{m}]) = 0.19$ (all sources reliable)
  - $\phi(m[m_2^{\mathcal{H}}, \mathbf{m}]) = 0.81$ (9 out of 10 reliable)
  - $\phi(m[m_3^{\mathcal{H}}, \mathbf{m}]) = 1$ (8 out of 10 reliable)
  - Values x_4 and x_5 are definitely more plausible.

$\rightarrow$ Results that are consistent, informative and readable by the end-user.

Outline

General setting

- Consider a system which outputs for a given object o, a guess about the actual value x^* of some feature $X \in \mathcal{X}$ of o.
- To produce this output, the system uses internally some information correction or fusion, characterized by some $m^{\mathcal{H}} \in \mathcal{S}$.
- Output for object o may thus be noted $f(o; m^{\mathcal{H}})$.

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- Output for object o may thus be noted $f(o; m^{\mathcal{H}})$.
- Assume a set of ℓ objects for which the true value of X is known, i.e., $\{x_i^*\}_{i=1}^{\ell}$ is available.
- Assume outputs $\{f(o_i; m^{\mathcal{H}})\}_{i=1}^{\ell}$ may be obtained for any $m^{\mathcal{H}} \in \mathcal{S}$.

Loss minimization

- The $\hat{m}^{\mathcal{H}}$ to be used to produce the output for a new object may then be the chosen as the one in $\mathcal{S}$ minimizing the average loss

$$J(m^{\mathcal{H}}) = \frac{1}{n}\sum_{i=1}^{\ell} \mathcal{L}(f(o_i; m^{\mathcal{H}}), x_i^*)$$

  for some loss function $\mathcal{L}(f(o; m^{\mathcal{H}}), x^*)$
- Typically, $f(o; m^{\mathcal{H}})$ is a MF on $\mathcal{X}$, which is transformed into a probability measure $P_o^{\mathcal{X}}$, and the squared error (SE) or cross-entropy (CE) loss is used:

$$\begin{aligned}\mathcal{L}_{SE}(f(o; m^{\mathcal{H}}), x^*) &= \sum_{x\in\mathcal{X}} (1_{x^*}(x) - p_o(x))^2 \\ \mathcal{L}_{CE}(f(o; m^{\mathcal{H}}), x^*) &= -\sum_{x\in\mathcal{X}} 1_{x^*}(x) \log p_o(x)\end{aligned}$$

- Remark: more or less complex optimisation problem to solve depending on $\mathcal{S}$ and $\mathcal{L}$

Application

Classifier correction [Elouedi et al., 2004]

- X is the class of an object.
- The system is a classifier whose outputs are corrected with meta-knowledge $m^{\mathcal{H}} = P^{\mathcal{R}}$ (discounting) with

$$P^{\mathcal{R}} \in \mathcal{S} = \{P^{\mathcal{R}} | \pi \in [0, 1]\}$$

- The classifier output for a given object o is a mass function m_o.
- The system output is thus

$$f(o; m^{\mathcal{H}}) = m[P^{\mathcal{R}}, m_o]$$

- Loss function : pignistic probability transformation with SE.

Application

Illustrative example

- Classifier outputs m_{o_i} for 4 objects with actual values x_i^* in $\mathcal{X} = \{a, g, h\}$.

	g	h	$\{a,h\}$	$\{g,h\}$	$\mathcal{X}$	x_i^*
m_{o_1}	0	0.5	0	0.3	0.2	a
m_{o_2}	0.5	0.2	0	0	0.3	g
m_{o_3}	0.4	0	0.6	0	0	a
m_{o_4}	0	0	0.6	0.4	0	h

- Meta-knowledge minimizing the average loss: $\hat{\pi} = 0.66$

Summary

- Interpretation of BFT as a theory of partially reliable and elementary pieces of information
  - Any set of such pieces of information is represented by a unique MF
  - To any MF can be associated uniquely such a set.
- Beyond reliability, information correction and fusion given knowledge on other aspects of source quality, such as truthfulness.
- Numerous and important correction and fusion approaches can be read using this prism.
- Means to determine knowledge on source quality in practice, with and without prior information on the sources.

Open topics of interest

- Exploitation of the $\bigcirc\!\!\!\!\cap_\sigma$ rule for SMF and the associated decomposition of a MF into (in)dependent SMF
  - Cautious combination
  - Refining of approaches based on conjunctive combination of independent SMF, such as GBT, E-KNN, DS analysis of GLR classifiers, contextual reinforcement.
- Interpretation of other correction and fusion approaches.
- Selection of meta-knowledge: refine arguments for the
  - Choice of $\mathcal{S}$ (include dependence between the sources)
  - Choice of $\mathcal{L}$ (including for the case of partially known true values)
- Conflict measurement: decomposition, measure selection for a given situation (properties, learning), refine with measures from logic, links with distances

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- Some other relevant BFT-based references on modeling and selecting assumptions on sources
- Some other interesting references, and in particular some more application-oriented papers, where correction/fusion is not the main topic but plays an important part.

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Contextual and polarized lack of truthfulness

- $\neg$tru : one must deduce the contrary of what $\mathfrak{s}$ tells for each $x_i \in \mathcal{X}$ and whatever the polarity of the clause used by $\mathfrak{s}$ regarding x_i.
- $\mathfrak{s}$ non truthful only for some $x_i \in \mathcal{X}$, and maybe even only for the positive or negative clauses regarding x_i.
- Example : Sensor $\mathfrak{s}$ is
  - non truthful when it tells that a is not a possible value for X (negatively non truthful for a)
  - and non truthful when it tells that g is a possible value for X (positively non truthful for g)
  - and truthful in all other cases, e.g., truthful when it tells that a is a possible value for X (positively truthful for a).
- Sensor $\mathfrak{s}$ tells $X \in A = \{g, h\}$, i.e., a is not a possible value and g and h are possible values for X
- We deduce (assuming $\mathfrak{s}$ relevant): $X \in \{a, h\}$

Contextual and polarized lack of truthfulness

- Three interesting states (contextual lies):
  - n_B : negatively non truthfull for $x_i \in B$;
  - p_B : positively non truthful for $x_i \in \overline{B}$;
  - ℓ_B : non truthful for $x_i \in \overline{B}$.
- Let $\tilde{\mathcal{T}} = \{n_B, p_B, \ell_B | B \subseteq \mathcal{X}\}$ and $\tilde{\Lambda}_A : \mathcal{R}\tilde{\mathcal{T}} \rightarrow 2^X$ represent the interpretations of testimony $X \in A$ given the possible states in $\mathcal{R}\tilde{\mathcal{T}}$ of the source
- $\tilde{\Lambda}_A$ extends Λ_A, e.g., $\tilde{\Lambda}_A(\ell_\emptyset) = \Lambda_A(\neg \text{tru})$ and $\tilde{\Lambda}_A(\ell_\mathcal{X}) = \Lambda_A(\text{tru})$ (assuming relevance)

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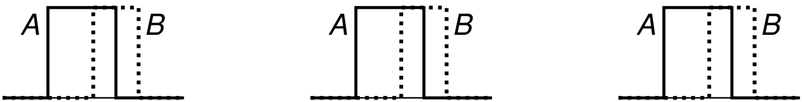

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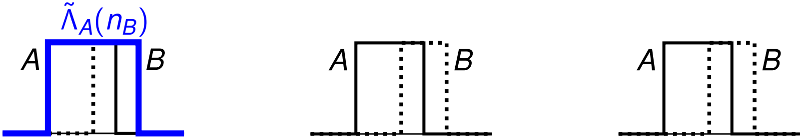
- Let $\tilde{\mathcal{T}} = \{n_B, p_B, \ell_B | B \subseteq \mathcal{X}\}$ and $\tilde{\Lambda}_A : \mathcal{R}\tilde{\mathcal{T}} \rightarrow 2^X$ represent the interpretations of testimony $X \in A$ given the possible states in $\mathcal{R}\tilde{\mathcal{T}}$ of the source
- $\tilde{\Lambda}_A$ extends Λ_A, e.g., $\tilde{\Lambda}_A(\ell_\emptyset) = \Lambda_A(\neg\text{tru})$ and $\tilde{\Lambda}_A(\ell_\mathcal{X}) = \Lambda_A(\text{tru})$ (assuming relevance)



Contextual and polarized lack of truthfulness

- Three interesting states (contextual lies):
  - n_B : negatively non truthfull for $x_i \in B$;
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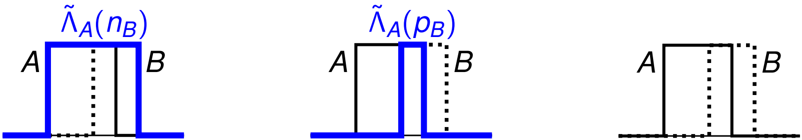

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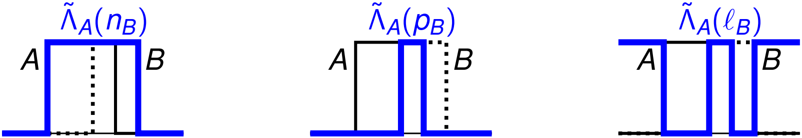

Uncertain meta-knowledge and testimonies

- Single information source

$$m[m^{\mathcal{R}\tilde{\mathcal{T}}}, m_{\mathfrak{s}}](B) = \sum_{R\tilde{T}, A : \tilde{\Lambda}_A(R\tilde{T}) = B} m^{\mathcal{R}\tilde{\mathcal{T}}}(R\tilde{T}) \cdot m_{\mathfrak{s}}(A)$$

- Multiple information sources

$$m[m^{\boldsymbol{\mathcal{R}}\tilde{\boldsymbol{\mathcal{T}}}}, \mathbf{m}](B) = \sum_{\mathbf{R}\tilde{\mathbf{T}}, \mathbf{A} : \tilde{\Lambda}_{\mathbf{A}}(\mathbf{R}\tilde{\mathbf{T}}) = B} m^{\boldsymbol{\mathcal{R}}\tilde{\boldsymbol{\mathcal{T}}}}(\mathbf{R}\tilde{\mathbf{T}}) \cdot m(\mathbf{A})$$

Particular cases

- Let $\mathcal{B} = \{B_1, \ldots, B_N\} \subseteq 2^X$. Consider iterative corrections (series of agents) of testimony $m_{\mathfrak{S}}$ provided by agent 1 with respective assumptions "preceding agent i is relevant, and is truthful with probability β_{B_i} and with probability $1 - \beta_{B_i}$ commits lie
  - n_{B_i}": $m_{\mathfrak{S}} \textcircled{\cup}_{B_i \in \mathcal{B}} m_{B_i}$ with $m_{B_i}(\emptyset) = \beta_{B_i}, m_{B_i}(B_i) = 1 - \beta_{B_i}$, called contextual discounting (it can also be obtained as a single correction $m[m^{\mathcal{R}\tilde{\mathcal{T}}}, m_{\mathfrak{S}}]$ with $m^{\mathcal{R}\tilde{\mathcal{T}}}$ the $\textcircled{\cup}$-combination of the preceding assumptions)
  - ℓ_{B_i}": $m_{\mathfrak{S}} \textcircled{\underline{\cap}}_{B_i \in \mathcal{B}} B_i^{\beta_{B_i}}$, contextual negating
  - p_{B_i}": $m_{\mathfrak{S}} \textcircled{\cap}_{B_i \in \mathcal{B}} B_i^{\beta_{B_i}}$, contextual reinforcement
- Remarks:
  - These correction mechanisms generalize their non-contextual versions for specific $\mathcal{B}$ such that $|\mathcal{B}| = 1$, hence their names.
  - An alternative interpretation exists for contextual discounting when $\mathcal{B}$ is a partition of $\mathcal{X}$ (see Thierry's lecture).

Example

Contextual discounting

- Suppose a sensor $\mathfrak{s}$ supplies information $X \in A = \{g\}$
- We know that $\mathfrak{s}$ is relevant and that at least one of the following independent pieces of meta-knowledge holds:
  - $\mathfrak{s}$ commits lie $n_{\{a,g\}}$ with probability 0.2
  - $\mathfrak{s}$ commits lie $n_{\{g,h\}}$ with probability 0.3
- Our knowledge on $\mathcal{X}$ is then obtained by

$$m_{\mathfrak{s}}(\{g\}) = 1 \ \text{\textcircled{$\cup$}} \left\{ \begin{array}{r} m_{\{a,g\}}(\{a,g\}) = 0.2 \\ m_{\{a,g\}}(\emptyset) = 0.8 \end{array} \right\} \text{\textcircled{$\cup$}} \left\{ \begin{array}{r} m_{\{g,h\}}(\{g,h\}) = 0.3 \\ m_{\{g,h\}}(\emptyset) = 0.7 \end{array} \right\}$$

which yields

$$m(\{g\}) = 0.56, m(\{a,g\}) = 0.14, m(\{g,h\}) = 0.24, m(\mathcal{X}) = 0.06$$

Particular cases

- Consider the following meta-knowledge about two sources $\mathfrak{s}_1$ and $\mathfrak{s}_2$ supplying information m_1 and m_2:
  - They are both relevant
  - And they are either both truthful or commit the same contextual lie ℓ_B with probability $\alpha^{|B|}(1-\alpha)^{|\overline{B}|}$, for some $\alpha \in [0,1]$
- Then

$$m[m^{\mathcal{R}\tilde{\mathcal{T}}}, \mathbf{m}](A) = \sum_{(A_1 \cap A_2) \cup (\overline{A_1} \cap \overline{A_2} \cap B) = A} m_1(A_1)\, m_2(A_2)\, m_\alpha(B)$$

where $m_\alpha(B) = \alpha^{|\overline{B}|}(1-\alpha)^{|B|}$

$\rightarrow$ α-conjunctions $\textcircled{\cap}^\alpha$ [Smets, 1997]: family of the associative, commutative and linear combination rules having the vacuous mass function as neutral element (family depending on a parameter $\alpha \in [0,1]$, such that $\textcircled{\cap}^1 = \textcircled{\cap}$ and $\textcircled{\cap}^0 = \underline{\textcircled{\cap}}$).