Information correction and fusion using belief functions

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Problem: to extract truthful and precise knowledge about a quantity of interest, from information coming from various sources.

Applications: computer vision, robotics, machine learning...

Old problem: origin of probability theory, where formalizing and merging partially reliable testimonies was a concern.

Requires meta-knowledge on the sources, i.e., knowledge about their quality (typically, their reliability).

Called information correction when there is a single information source and information fusion when there are several sources.
... using belief functions

- Related to the issue of uncertainty modeling.
- Uncertainty theories: probability, possibility, belief function, imprecise probability theories.
- **Central role in belief function theory (BFT):**
  1. [Shafer, 1976]: BFT as an approach for representing and merging partially reliable and elementary testimonies;
  2. Numerous theoretical contributions on information fusion;
  3. BFT used in applications for merging information.
... using belief functions

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- **Central role in belief function theory (BFT):**
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  2. Numerous theoretical contributions on information fusion;
  3. BFT used in applications for merging information.

→ This lecture: some recent results in line with 1 – 3, based on a modeling of source quality, reinforcing the relevance of BFT for information correction and fusion.
Contents of this lecture

- A general approach to information correction and fusion using belief functions
- A prism to understand some important belief function correction and fusion schemes
- An interpretation of belief functions (≈ [Shafer, 1976] revisited)
- Means to tackle correction and fusion problems in practice

Not in this lecture:
- An exhaustive review of all combination rules
- A discussion on conflict measurement (see Anne-Laure’s lecture)
- A discussion on rule properties (see, e.g., Sébastien’s lecture at the 2015 BFAS school)
- Implementation aspects (see Arnaud’s lecture)
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Outline

1 Reliability
   - One source
   - Two sources
   - $K$ sources
   - Uncertain testimonies

2 Truthfulness and beyond
   - Crudest form
   - Refined form
   - General model

3 Selecting meta-knowledge
   - Absence of prior information
   - Learning data
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Reliability

Classically, to interpret information items provided by sources (sensor, human, ...), assumptions are made about their reliability (relevance), where a **reliable source is a source providing useful information** regarding the quantity of interest.

Examples:

- A broken watch is useless to try and find the time it is since there is no way to know whether the supplied information is correct or not: this source is not reliable for the time;
- My six-year-old child is ignorant about the name of the latest Nobel Peace Prize laureate: he is not reliable for this question (in contrast to the source nobelprize.org).

Basic idea: a piece of information received from a reliable source is considered valid, whereas it is useless if the source is not reliable.
Formalization

- Let $X$ be a variable of interest taking values in a finite set $\mathcal{X} = \{x_1, \ldots, x_n\}$ (frame of discernment), and whose actual value is unknown.
- Assume a source $s$ telling that $X \in A \subseteq \mathcal{X}$.
  - If $s$ is not reliable, we replace $X \in A$ by $X \in \mathcal{X}$.
  - If $s$ is reliable, we keep $X \in A$.
Formalization

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- Assume a source $s$ telling that $X \in A \subseteq \mathcal{X}$
  - If $s$ is not reliable, we replace $X \in A$ by $X \in \mathcal{X}$
  - If $s$ is reliable, we keep $X \in A$

- Let $R$ be the variable denoting its reliability, defined on $\mathcal{R} = \{0, 1\}$ where 0 means that $s$ is reliable and 1 means not reliable.

- The interpretation of the testimony according to the reliability may be encoded by $\Gamma_A : \mathcal{R} \rightarrow 2^{\mathcal{X}}$ such that
  
  \[
  \Gamma_A(0) = A, \\
  \Gamma_A(1) = \mathcal{X}.
  \]
Uncertain reliability

- Assume now $s$ is not reliable with probability $P^R(R = 1) = \pi$ (and reliable with probability $P^R(R = 0) = 1 - \pi$) with $\pi \in [0, 1]$.
- What can then be inferred about $X$?
- $\pi$ should be transferred to $\Gamma_A(1) = \mathcal{X}$, $1 - \pi$ to $\Gamma_A(0) = A$, and thus our knowledge about $X$ is represented by a mass function (MF) on $\mathcal{X}$ such that
  \[
  m(A) = 1 - \pi, \\
  m(\mathcal{X}) = \pi
  \]
- $m(A)$: probability of knowing that $X \in A$ and nothing more, given the available evidence.
- $m$ is a so-called simple mass function (SMF), since it has two focal sets including $\mathcal{X}$. It is more simply denoted by $A^\pi$.
- Other useful notation for $m$: $m[P^R, A]$
Example

- Assume a sensor $s$ in charge of recognizing the type $X$ of an aircraft which can be airplane (a), glider (g), or helicopter (h), i.e., $\mathcal{X} = \{a, g, h\}$.
- $s$ tells it is a glider or a helicopter, i.e., $X \in A = \{g, h\}$.
- The probability that the sensor is not reliable is 0.1, i.e., $\pi = 0.1$.
- Hence, our knowledge about $X$ is represented by the SMF $\{g, h\}^{0.1}$.

\[
m(\{g, h\}) = 0.9 \\
m(\mathcal{X}) = 0.1
\]
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Two information sources

- Assume now two sources $s_1$ and $s_2$ providing information $X \in A_1$ and $X \in A_2$, respectively.
- Let $\Gamma_{A_i} : R_i \to 2^X$ represent the interpretation of information $A_i$ from $s_i$ given its reliability $R_i$ defined on $\mathcal{R}_i = \{0, 1\}$.
- If they are in the state
  
  - $(R_1 = 0, R_2 = 0)$, then $X \in \Gamma_{A_1}(0) \cap \Gamma_{A_2}(0) = A_1 \cap A_2$
  - $(R_1 = 1, R_2 = 0)$, then $X \in \Gamma_{A_1}(1) \cap \Gamma_{A_2}(0) = X \cap A_2 = A_2$
  - $(R_1 = 0, R_2 = 1)$, then $X \in \Gamma_{A_1}(0) \cap \Gamma_{A_2}(1) = A_1 \cap X = A_1$
  - $(R_1 = 1, R_2 = 1)$, then $X \in \Gamma_{A_1}(1) \cap \Gamma_{A_2}(1) = X \cap X = X$
Notation

- When the sources provide information $\mathbf{A} = (A_1, A_2)$ and are in the state $\mathbf{r} = (r_1, r_2) \in \mathcal{R} := \mathcal{R}_1 \times \mathcal{R}_2$, we should deduce

  $$X \in \Gamma_{\mathbf{A}}(\mathbf{r}) := \Gamma_{A_1}(r_1) \cap \Gamma_{A_2}(r_2)$$

- $\Gamma_{\mathbf{A}} : \mathcal{R} \rightarrow 2^X$

- Example

  $$\Gamma_{\mathbf{A}}(0, 1) = \Gamma_{A_1}(0) \cap \Gamma_{A_2}(1) = A_1 \cap X = A_1$$
Uncertain reliabilities

- Assume now the sources are in state \( r = (r_1, r_2) \) with probability \( P^R(R_1 = r_1, R_2 = r_2) = p_r \)
- \( p_r \) should be transferred to \( \Gamma_A(r) \).
- Our knowledge about \( X \) can then be represented by

\[
m(B) = \sum_{r: \Gamma_A(r) = B} p_r.
\]

- Notation: \( m[P^R, A] \)
Example

- Two sensors \(s_1\) and \(s_2\) for the type \(X\) of an aircraft
- \(s_1\) tells \(X \in A_1 = \{a\} = \{g, h\}\)
- \(s_2\) tells \(X \in A_2 = \{g\} = \{a, h\}\)

We have

\[
\begin{align*}
\Gamma_A(0, 0) &= A_1 \cap A_2 = \{h\} \\
\Gamma_A(1, 0) &= A_2 = \{a, h\} \\
\Gamma_A(0, 1) &= A_1 = \{g, h\} \\
\Gamma_A(1, 1) &= X
\end{align*}
\]

- Induced knowledge about \(X\):

\[
m(\{h\}) = 0.3, m(\{a, h\}) = 0.1, m(\{g, h\}) = 0.4, m(X) = 0.2
\]
Decomposition of meta-knowledge

- $P^\mathcal{R}$ is a bivariate Bernoulli distribution
- It is characterized by

\[
\pi_i := \mathbb{E}[R_i] = P^\mathcal{R}_i(R_i = 1), \quad i = 1, 2,
\]

\[
\sigma := \mathbb{E}[(R_1 - \pi_1)(R_2 - \pi_2)] = \mathbb{E}[R_1R_2] - \mathbb{E}[R_1]\mathbb{E}[R_2] = P^\mathcal{R}(R_1 = 1, R_2 = 1) - P^\mathcal{R}_1(R_1 = 1)P^\mathcal{R}_2(R_2 = 1)
\]

- We have

\[
P^\mathcal{R}(R_1 = 0, R_2 = 0) = \pi_1 \cdot \pi_2 + \sigma
\]

\[
P^\mathcal{R}(R_1 = 1, R_2 = 0) = \pi_1 \cdot \pi_2 - \sigma
\]

\[
P^\mathcal{R}(R_1 = 0, R_2 = 1) = \overline{\pi_1} \cdot \pi_2 - \sigma
\]

\[
P^\mathcal{R}(R_1 = 1, R_2 = 1) = \pi_1 \cdot \pi_2 + \sigma
\]

with $\overline{\pi_i} = 1 - \pi_i$
Knowledge on the reliabilities of the sensors $s_1$ and $s_2$: 

\[
\begin{align*}
P^R(R_1 = 0, R_2 = 0) &= 0.3 \\
P^R(R_1 = 1, R_2 = 0) &= 0.1 \\
P^R(R_1 = 0, R_2 = 1) &= 0.4 \\
P^R(R_1 = 1, R_2 = 1) &= 0.2
\end{align*}
\]

\[\iff \begin{cases} 
\pi_1 = 0.3 \\
\pi_2 = 0.6 \\
\sigma = 0.02
\end{cases} \]
Independent reliabilities

SMF-based expression

- $R_1$ and $R_2$ independent $\iff \sigma = 0$
- In this case

$$m[P^R, A] = m[P^{R_1}, A_1] \Delta m[P^{R_2}, A_2] = A_1^{\pi_1} \cap A_2^{\pi_2}$$

- Reminder: unnormalized Dempster’s rule (conjunctive rule)

$$(m_1 \cap m_2)(A) = \sum_{B \cap C = A} m_1(B)m_2(C), \quad \forall A \subseteq \mathcal{X}.$$
Dependent reliabilities

SMF-based expression

More generally, i.e., for any dependency $\sigma$, $m_{P^R, A}$ can always be expressed as a conjunctive combination of $A_{1}^{\pi_1}$ and $A_{2}^{\pi_2}$ having some dependency...

“Reminder”: conjunctive combination $m_\cap$ of $m_1$ and $m_2$ having some known dependency

1. A joint MF $jm : 2^X \times 2^X \rightarrow [0, 1]$ is built, having $m_1$ and $m_2$ as marginals and encoding their mutual dependence

2. Each joint mass $jm(B, C)$ is allocated to $B \cap C$:

$$m_\cap(A) = \sum_{B \cap C = A} jm(B, C)$$
Dependent reliabilities

SMF-based expression

- Let \( m_i = A_i^{\pi_i} \). Any \( jm \) having \( A_1^{\pi_1} \) and \( A_2^{\pi_2} \) as marginals can always be written as

\[
\begin{align*}
jm(A_1, A_2) &= \overline{\pi_1} \cdot \overline{\pi_2} + \sigma \\
jm(x, A_2) &= \pi_1 \cdot \overline{\pi_2} - \sigma \\
jm(A_1, x') &= \overline{\pi_1} \cdot \pi_2 - \sigma \\
jm(x', x') &= \pi_1 \cdot \pi_2 + \sigma
\end{align*}
\]

for some \( \sigma \).

- Conjunctive combination of \( A_1^{\pi_1} \) and \( A_2^{\pi_2} \) with dependence structure represented by \( jm \), is completely determined by \( \sigma \).

\[ \rightarrow \text{Parameterized conjunctive rule } \sqcap_\sigma \text{ for two SMF, with parameter } \sigma \text{ representing the dependence structure, such that} \]

\[ \sqcap_\sigma (A_1^{\pi_1}, A_2^{\pi_2}) := m_\cap \]

- For \( \sigma = 0 \), \( \sqcap_\sigma \Leftrightarrow \sqcap \)
Dependent reliabilities

SMF-based expression

For any dependency $\sigma$ between the source reliabilities, we have

$$m[P^R, A] = \ominus_\sigma(m[P^{R_1}, A_1], m[P^{R_2}, A_2])$$

$$= \ominus_\sigma(A_1^{\pi_1}, A_2^{\pi_2})$$

Example:

- Sensor $s_1$ not reliable with probability $\pi_1 = 0.3$
- Sensor $s_2$ not reliable with probability $\pi_2 = 0.6$
- Dependence between their reliability: $\sigma = 0.02$
- Induced knowledge on $\mathcal{X}$ from the information $A = (\{g, h\}, \{a, h\})$

provided by the sensors satisfies

$$m[P^R, A] = \ominus_{(0.02)} (\{g, h\}^{0.3}, \{a, h\}^{0.6})$$
Cautious rule for SMF

- Let $A_1^{\pi_1}$ and $A_2^{\pi_2}$ be two non-independent SMF.
- How to combine them?
- Cautious conjunctive combination $\land$: select the least committed (according to some informational ordering) MF among those that are at least as committed as $A_1^{\pi_1}$ and $A_2^{\pi_2}$.
- Solution based on the $w$-ordering yields

$$A_1^{\pi_1} \land A_2^{\pi_2} = \begin{cases} A_1^{\pi_1} \land \pi_2 & \text{if } A_1 = A_2 \\ A_1^{\pi_1} \circ A_2^{\pi_2} & \text{if } A_1 \neq A_2 \end{cases}$$
Cautious rule revisited

- We have

\[ A_1^{\pi_1} \land A_2^{\pi_2} = \cap_{\sigma} (A_1^{\pi_1}, A_2^{\pi_2}) \]

with

\[ \sigma = \begin{cases} 
\pi_1 \land \pi_2 - \pi_1 \pi_2 & \text{if } A_1 = A_2 \\
0 & \text{if } A_1 \neq A_2 
\end{cases} \]

- Partially reliable sources analysis:
  - \( s_i \) not reliable with probability \( \pi_i \) and telling \( X \in A_i \)
  - \( s_1 \) and \( s_2 \) have dependent reliabilities if they support the same subset (actually, perfect dependence between them not being reliable) and independent reliabilities otherwise.
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Assume sources \( s_i, \ i = 1, \ldots, K \), providing \( A = (A_1, \ldots, A_K) \).

When the sources are in the state \( r = (r_1, \ldots, r_K) \in \mathcal{R} := \prod_{i=1}^{K} \mathcal{R}_i \), we should deduce

\[
X \in \Gamma_A(r) := \bigcap_{i=1}^{K} \Gamma_{A_i}(r_i)
\]

Example: \( K = 3 \)

\[
\Gamma_A(0, 0, 1) = \Gamma_{A_1}(0) \cap \Gamma_{A_2}(0) \cap \Gamma_{A_3}(1) = A_1 \cap A_2 \cap X = A_1 \cap A_2
\]
Uncertain reliabilities

- Assume state $r$ is allocated probability $p_r$:

$$P^R(R_1 = r_1, \ldots, R_K = r_K) = p_r$$

- Knowledge about $X$ is then represented by

$$m[P^R, A](B) = \sum_{r: \Gamma_A(r) = B} p_r$$

→ Any set of partially reliable and elementary testimonies is represented by a (unique) MF
Separable mass function

- Independence of all the $R_i$

\[ m[P^R, A] = \bigcap_{i=1}^{K} m[P^{R_i}, A_i] = \bigcap_{i=1}^{K} A_i^{\pi_i} \]

with $\pi_i := P^{R_i}(R_i = 1)$.

→ Choosing $A$ s.t. $A_i \neq A_j$, $1 \leq i < j \leq K$, we obtain a partially reliable sources analysis of separable MF\(^1\), which form an important class of MF (often encountered in practice)

\(^1\)MF that can be written as a conjunctive combination of independent SMF supporting different subsets
Separable mass function

- Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $A^r$ denote the subset of $\mathcal{X}$ such that $x_i \in A^r$ if $r_i = 1$ and $x_i \not\in A^r$ if $r_i = 0$, for $r = (r_1, \ldots, r_K)$.

- Example: $\mathcal{X} = \{x_1, x_2, x_3\}$ then $A^{001} = \{x_3\}$
Separable mass function

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- Example: $\mathcal{X} = \{x_1, x_2, x_3\}$ then $A^{001} = \{x_3\}$

**Proposition**

Let $m$ be a MF on $\mathcal{X}$, $K = |\mathcal{X}|$, $A_i = \{x_i\}$, and $p_r = m(A^r)$. We have

$$m[P^R, \mathcal{A}] = m$$

**Proof:** Follows from $\Gamma_A(r) = A^r$. 
Separable mass function

- Let \( X = \{x_1, \ldots, x_n\} \) and \( A^r \) denote the subset of \( X \) such that \( x_i \in A^r \) if \( r_i = 1 \) and \( x_i \notin A^r \) if \( r_i = 0 \), for \( r = (r_1, \ldots, r_K) \).
- Example: \( X = \{x_1, x_2, x_3\} \) then \( A^{001} = \{x_3\} \)

Proposition

Let \( m \) be a MF on \( X \), \( K = |X| \), \( A_i = \{x_i\} \), and \( p_r = m(A^r) \). We have

\[
m[P^R, A] = m
\]

Proof: Follows from \( \Gamma_A(r) = A^r \).

- Recall: Any set of partially reliable and elementary testimonies is represented by a (unique) MF.
Separable mass function

- Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $A^r$ denote the subset of $\mathcal{X}$ such that $x_i \in A^r$ if $r_i = 1$ and $x_i \notin A^r$ if $r_i = 0$, for $r = (r_1, \ldots, r_K)$.
- Example: $\mathcal{X} = \{x_1, x_2, x_3\}$ then $A^{001} = \{x_3\}$

Proposition

Let $m$ be a MF on $\mathcal{X}$, $K = |\mathcal{X}|$, $A_i = \overline{\{x_i\}}$, and $p_r = m(A^r)$. We have

$$m[\mathcal{P}^\mathcal{R}, \mathcal{A}] = m$$

Proof: Follows from $\Gamma_{\mathcal{A}}(r) = A^r$.

- Recall: Any set of partially reliable and elementary testimonies is represented by a (unique) MF.

$\rightarrow$ Any mass function represents (at least) a set of partially reliable and elementary testimonies
Example

- Let $m$ be the FM on $\mathcal{X} = \{a, g, h\}$ defined by

$$m(\{a, g\}) = 0.5, m(\{h\}) = 0.2, m(\{g, h\}) = 0.3$$

- Consider $K = |\mathcal{X}| = 3$ sources $s_1, s_2$ and $s_3$, providing respectively information

$$X \in A_1 = \{a\} = \{g, h\}$$
$$X \in A_2 = \{g\} = \{a, h\}$$
$$X \in A_3 = \{h\} = \{a, g\}$$
Example

Consider meta-knowledge $P^R$ on the sources such that

<table>
<thead>
<tr>
<th>$r$</th>
<th>$A^r$</th>
<th>$m(A^r) = p_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0, 0)</td>
<td>$\emptyset$</td>
<td>0</td>
</tr>
<tr>
<td>(1, 0, 0)</td>
<td>{a}</td>
<td>0</td>
</tr>
<tr>
<td>(0, 1, 0)</td>
<td>{g}</td>
<td>0</td>
</tr>
<tr>
<td>(1, 1, 0)</td>
<td>{a, g}</td>
<td>0.5</td>
</tr>
<tr>
<td>(0, 0, 1)</td>
<td>{h}</td>
<td>0.2</td>
</tr>
<tr>
<td>(1, 0, 1)</td>
<td>{a, h}</td>
<td>0</td>
</tr>
<tr>
<td>(0, 1, 1)</td>
<td>{g, h}</td>
<td>0.3</td>
</tr>
<tr>
<td>(1, 1, 1)</td>
<td>{a, g, h}</td>
<td>0</td>
</tr>
</tbody>
</table>

Then, e.g., $p_{001} = P^R(R_1 = 0, R_2 = 0, R_3 = 1) = m(A^{001}) = 0.2$ is allocated to $\Gamma_A(0, 0, 1) = A_1 \cap A_2 \cap \mathcal{X} = \overline{\{a\}} \cap \overline{\{g\}} = \overline{\{h\}} = A^{001}$.

Since this happens for all $r$, we have $m[P^R, A] = m$. 
Decomposition of meta-knowledge

- $P^R$ is a multivariate Bernoulli distribution
- It is characterized by
  $$\pi_i = \mathbb{E}[R_i]$$
- $\sigma_r = \mathbb{E} \left[ \prod_{i=1}^{K} (R_i - \pi_i)^{r_i} \right]$ with $r = (r_1, \ldots, r_K)$ such that $\sum_{i=1}^{K} r_i > 1$
- There are $2^K - K - 1$ central moments $\sigma_r$. They represent the dependencies between any subset (of at least two) of all the $R_i$.
- Notation: $\sigma$ vector whose elements are the dependencies $\sigma_r$
Example

\( K = 3 \)

\[
\begin{align*}
\sigma_{123} & \quad | \\
R_1(\pi_1) & \quad | \quad R_2(\pi_2) & \quad | \quad R_3(\pi_3) \\
\sigma_{12} & \quad | \quad \sigma_{13} & \quad | \quad \sigma_{23} \\
\end{align*}
\]

where

\[
\sigma_{12} := \sigma_{110} = \mathbb{E} [(R_1 - \pi_1)(R_2 - \pi_2)]
\]

and \( \sigma_{13} := \sigma_{101}, \sigma_{23} := \sigma_{011}, \sigma_{123} := \sigma_{111} \)

\[
\sigma = (\sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{123})
\]
Example

Knowledge on the reliabilities of the sources $s_1$, $s_2$ and $s_3$:

\[
\begin{align*}
P_{R}(R_1 = 0, R_2 = 0, R_3 = 0) &= 0 \\
P_{R}(R_1 = 1, R_2 = 0, R_3 = 0) &= 0 \\
P_{R}(R_1 = 0, R_2 = 1, R_3 = 0) &= 0 \\
P_{R}(R_1 = 1, R_2 = 1, R_3 = 0) &= 0.5 \\
P_{R}(R_1 = 0, R_2 = 0, R_3 = 1) &= 0.2 \\
P_{R}(R_1 = 1, R_2 = 0, R_3 = 1) &= 0 \\
P_{R}(R_1 = 0, R_2 = 1, R_3 = 1) &= 0.3 \\
P_{R}(R_1 = 1, R_2 = 1, R_3 = 1) &= 0
\end{align*}
\]

\[
\begin{align*}
\pi_1 &= 0.5 \\
\pi_2 &= 0.8 \\
\pi_3 &= 0.5 \\
\sigma_{12} &= 0.1 \\
\sigma_{13} &= -0.25 \\
\sigma_{23} &= -0.1 \\
\sigma_{123} &= 0
\end{align*}
\]
Example

- $P^R$ is recovered from $\pi_i$ and $\sigma_r$ as follows:

  $P^R(R_1 = 0, R_2 = 0, R_3 = 0) = \pi_1 \pi_2 \pi_3 + \pi_3 \sigma_{12} + \pi_2 \sigma_{13} + \pi_1 \sigma_{23} - \sigma_{123}$

  $P^R(R_1 = 1, R_2 = 0, R_3 = 0) = \pi_1 \pi_2 \pi_3 - \pi_3 \sigma_{12} - \pi_2 \sigma_{13} + \pi_1 \sigma_{23} + \sigma_{123}$

  $P^R(R_1 = 0, R_2 = 1, R_3 = 0) = \pi_1 \pi_2 \pi_3 - \pi_3 \sigma_{12} + \pi_2 \sigma_{13} - \pi_1 \sigma_{23} + \sigma_{123}$

  $P^R(R_1 = 1, R_2 = 1, R_3 = 0) = \pi_1 \pi_2 \pi_3 + \pi_3 \sigma_{12} - \pi_2 \sigma_{13} - \pi_1 \sigma_{23} - \sigma_{123}$

  $P^R(R_1 = 0, R_2 = 0, R_3 = 1) = \pi_1 \pi_2 \pi_3 + \pi_3 \sigma_{12} - \pi_2 \sigma_{13} - \pi_1 \sigma_{23} + \sigma_{123}$

  $P^R(R_1 = 1, R_2 = 0, R_3 = 1) = \pi_1 \pi_2 \pi_3 - \pi_3 \sigma_{12} + \pi_2 \sigma_{13} - \pi_1 \sigma_{23} - \sigma_{123}$

  $P^R(R_1 = 0, R_2 = 1, R_3 = 1) = \pi_1 \pi_2 \pi_3 - \pi_3 \sigma_{12} - \pi_2 \sigma_{13} + \pi_1 \sigma_{23} - \sigma_{123}$

  $P^R(R_1 = 1, R_2 = 1, R_3 = 1) = \pi_1 \pi_2 \pi_3 + \pi_3 \sigma_{12} + \pi_2 \sigma_{13} + \pi_1 \sigma_{23} + \sigma_{123}$

- Remark: simple matrix-based expressions exist to switch from one representation to the other.
Decomposition of a mass function

Based on the previous proposition as well as the decomposition of meta-knowledge, any MF $m$ on $\mathcal{X}$ is induced by the following basic components:

1. A set of $|\mathcal{X}|$ sources $\mathcal{S} = \{s_1, \ldots, s_{|\mathcal{X}|}\}$, with $s_i$ providing information $X \in \{x_i\}$;

2. Probabilistic knowledge on their reliability characterized by:
   - For each $s_i$, a (marginal) probability $\pi_i$ of being not reliable;
   - For each $S_r \subseteq \mathcal{S}$, knowledge about the dependency between their reliabilities in the form of the central moment $\sigma_r$.

Remark: we have $\pi_i = pl(\{x_i\})$. 
Example

- $m$ defined by $m(\{a, g\}) = 0.5$, $m(\{h\}) = 0.2$, $m(\{g, h\}) = 0.3$.
- This MF is induced by
  1. Considering $|\mathcal{X}| = 3$ sources $s_1$, $s_2$, and $s_3$, providing information:

      $X \in A_1 = \{a\}$
      $X \in A_2 = \{g\}$
      $X \in A_3 = \{h\}$

  2. And by assuming that
     - $s_1$, $s_2$, and $s_3$ are not reliable with respective probabilities $\pi_1 = 0.5$, $\pi_2 = 0.8$, and $\pi_3 = 0.5$
     - there is a dependence $\sigma_{12} = 0.1$ between the reliabilities of $s_1$ and $s_2$, $\sigma_{13} = -0.25$ between $s_1$ and $s_3$, $\sigma_{23} = -0.1$ between $s_2$ and $s_3$, and $\sigma_{123} = 0$ between $s_1$, $s_2$, and $s_3$. 
SMF-based expression of $m[\mathcal{P}^R, \mathbf{A}]$

- Let $m_\cap$ denote the conjunctive combination of SMF $A_i^{\pi_i}$, $1 \leq i \leq K$, with dependence structure represented by $jm$
- $jm$ can always be written in a particular (familiar) form such that the dependencies it encodes are completely determined by $2^K - K - 1$ parameters
SMF-based expression of \( m[\mathcal{P}^R, \mathbf{A}] \)

- Let \( m_\cap \) denote the conjunctive combination of SMF \( A_i^{\pi_i}, 1 \leq i \leq K \), with dependence structure represented by \( jm \)
- \( jm \) can always be written in a particular (familiar) form such that the dependencies it encodes are completely determined by \( 2^K - K - 1 \) parameters
- Example: \( K = 3 \), four parameters noted \( \sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{123} \)

\[
\begin{align*}
jm(A_1, A_2, A_3) &= \bar{\pi}_1 \bar{\pi}_2 \bar{\pi}_3 + \bar{\pi}_3 \sigma_{12} + \bar{\pi}_2 \sigma_{13} + \bar{\pi}_1 \sigma_{23} - \sigma_{123} \\
jm(\mathcal{X}, A_2, A_3) &= \pi_1 \bar{\pi}_2 \bar{\pi}_3 - \bar{\pi}_3 \sigma_{12} - \bar{\pi}_2 \sigma_{13} + \pi_1 \sigma_{23} + \sigma_{123} \\
jm(A_1, A_2, A_3) &= \bar{\pi}_1 \pi_2 \bar{\pi}_3 - \bar{\pi}_3 \sigma_{12} + \pi_2 \sigma_{13} - \bar{\pi}_1 \sigma_{23} + \sigma_{123} \\
jm(\mathcal{X}, \mathcal{X}, A_3) &= \pi_1 \pi_2 \bar{\pi}_3 + \bar{\pi}_3 \sigma_{12} - \pi_2 \sigma_{13} - \pi_1 \sigma_{23} - \sigma_{123} \\
jm(A_1, A_2, \mathcal{X}) &= \bar{\pi}_1 \bar{\pi}_2 \bar{\pi}_3 + \pi_3 \sigma_{12} - \bar{\pi}_2 \sigma_{13} - \bar{\pi}_1 \sigma_{23} + \sigma_{123} \\
jm(\mathcal{X}, A_2, \mathcal{X}) &= \pi_1 \pi_2 \bar{\pi}_3 - \pi_3 \sigma_{12} + \bar{\pi}_2 \sigma_{13} - \pi_1 \sigma_{23} - \sigma_{123} \\
jm(A_1, \mathcal{X}, \mathcal{X}) &= \bar{\pi}_1 \pi_2 \bar{\pi}_3 - \pi_3 \sigma_{12} - \pi_2 \sigma_{13} + \bar{\pi}_1 \sigma_{23} - \sigma_{123} \\
jm(\mathcal{X}, \mathcal{X}, \mathcal{X}) &= \pi_1 \pi_2 \bar{\pi}_3 + \pi_3 \sigma_{12} + \pi_2 \sigma_{13} + \pi_1 \sigma_{23} + \sigma_{123}
\end{align*}
\]
SMF-based expression of $m[P^R, A]$

- Parameterized conjunctive rule $\cap_\sigma$ for $K$ SMF, with $\sigma$ the vector whose elements are the dependency parameters $\sigma_r$, such that

$$\cap_\sigma(A_1^{\pi_1}, \ldots, A_K^{\pi_K}) := m_\cap$$
SMF-based expression of $m[P^R, A]$

- Parameterized conjunctive rule $\cap_\sigma$ for $K$ SMF, with $\sigma$ the vector whose elements are the dependency parameters $\sigma_r$, such that

$$\cap_\sigma(A^{\pi_1}_1, \ldots, A^{\pi_K}_K) := m_\cap$$

**Theorem**

*For any dependency $\sigma$ between the source reliabilities, we have*

$$m[P^R, A] = \cap_\sigma(m[P^{R_1}, A_1], \ldots, m[P^{R_K}, A_K]) = \cap_\sigma(A^{\pi_1}_1, \ldots, A^{\pi_K}_K)$$
Conjunctive canonical decomposition of a MF

**Theorem**

Any MF \( m \) satisfies

\[
m = \bigodot_{\sigma} \left( \{x_1\}^{pl}(x_1), \ldots, \{x_n\}^{pl}(x_n) \right),
\]

with \( \sigma \) the vector of dependencies between the reliabilities of the sources underlying \( m \).
Conjunctive canonical decomposition of a MF

**Theorem**

Any MF $m$ satisfies

$$m = \Box_{\sigma} \left( \{x_1\}^{pl(x_1)}, \ldots, \{x_n\}^{pl(x_n)} \right),$$

with $\sigma$ the vector of dependencies between the reliabilities of the sources underlying $m$.

- **Example:** $m$ defined by

  $$m(\{a, g\}) = 0.5, m(\{h\}) = 0.2, m(\{g, h\}) = 0.3$$

  satisfies

  $$m = \Box_{(0.1, -0.25, -0.1, 0)} \left( \{a\}^{0.5}, \{g\}^{0.8}, \{h\}^{0.5} \right)$$
Conclusions

- Any mass function can be seen as the result of the
  - interpretation of a set of partially reliable and elementary testimonies
  - conjunctive combination of SMF (focused on $\{x_i\}$) having some dependencies.

- In the spirit of [Shafer, 1976] and [Smets, 1995] interpretations of belief functions (they considered only independent SMF to try and recover the entire space of mass functions, see Didier’s lecture)

- Combining conjunctively SMF focused on $\{x_i\}$ is found in some important results: generalized Bayesian theorem, BFT analysis of binary logistic regression [Denoeux, 2019].
Outline

1. Reliability
   - One source
   - Two sources
   - \( K \) sources
   - Uncertain testimonies

2. Truthfulness and beyond
   - Crudest form
   - Refined form
   - General model

3. Selecting meta-knowledge
   - Absence of prior information
   - Learning data
The case of a single source

- Assume now the source $s$ provides an uncertain testimony about $X$ in the form of a MF $m_s$.
- If $s$ is reliable, then each $m_s(A)$ should be transferred to $\Gamma_A(0) = A$.
- If $s$ is not reliable, then each $m_s(A)$ should be transferred to $\Gamma_A(1) = \mathcal{X}$.
- Assuming $s$ to be not reliable with probability $P^R(R = 1) = \pi$ then yields the following MF on $\mathcal{X}$:

$$m[P^R, m_s] = (1 - \pi) \cdot m_s + \pi \cdot m_{\mathcal{X}}$$

with $m_{\mathcal{X}}$ the vacuous MF ($m_{\mathcal{X}}(\mathcal{X}) = 1$).

- This is known as the discounting of $m_s$ with discount rate $\pi$ (basic information correction mechanism in BFT).
Example

- Uncertain testimony:

\[ m_5(\{a, g\}) = 0.8, \ m_5(\{h\}) = 0.2 \]

- Uncertain reliability:

\[ P^R(R = 0) = 0.7, \ P^R(R = 1) = 0.3 \]
Example

- Uncertain testimony:
  \[ m_5(\{a, g\}) = 0.8, \ m_5(\{h\}) = 0.2 \]

- Uncertain reliability:
  \[ P^R(R = 0) = 0.7, \ P^R(R = 1) = 0.3 \]

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<th>(P^R)</th>
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<th>(R = 1)</th>
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<tr>
<td>({a, g})</td>
<td>0.8</td>
<td>0.7</td>
<td>0.3</td>
</tr>
<tr>
<td>({h})</td>
<td>0.2</td>
<td>0.14</td>
<td>0.06</td>
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Example

- Uncertain testimony:
  \[ m_5(\{a, g\}) = 0.8, m_5(\{h\}) = 0.2 \]

- Uncertain reliability:
  \[ P^R(R = 0) = 0.7, P^R(R = 1) = 0.3 \]

<table>
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<th>( R = 0 )</th>
<th>( R = 1 )</th>
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<tbody>
<tr>
<td>{a, g}</td>
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<td>0.3</td>
</tr>
<tr>
<td>0.8</td>
<td>0.56</td>
<td>0.24</td>
</tr>
<tr>
<td>{h}</td>
<td>0.2</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>0.14</td>
<td>0.06</td>
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</tbody>
</table>

\[ \rightarrow m[P^R, m_5](\{a, g\}) = 0.56, m[P^R, m_5](\{h\}) = 0.14, m[P^R, m_5](\lambda) = 0.30 \]
The case of multiple sources

- Assume sources $s_1, \ldots, s_K$ provide uncertain testimonies $m = (m_1, \ldots, m_K)$.
- Assume they are independent: interpreting $m_i(A_i)$ as the probability that $s_i$ supplies $X \in A_i$, then the probability, denoted $m(A)$, that they supply $A = (A_1, \ldots, A_K)$ is $\prod_{i=1}^{K} m_i(A_i)$.
- If they are in state $r \in \mathcal{R}$, then $m(A)$ should be transferred to $\Gamma_A(r)$.
- Assuming they are in state $r$ with probability $p_r$ then yields the following MF on $\mathcal{X}$:

$$m[P^\mathcal{R}, m](B) = \sum_{r, A: \Gamma_A(r) = B} p_r \cdot m(A)$$
Particular cases

$m[P^{R}, m]$ reduces to

- $\cap_{i=1}^{K} m_{i}$ if $p_{r} = 1$ for $r = (0, 0, \ldots, 0, 0)$, i.e., all sources are reliable

→ conjunctive rule

- $\cap_{i=1}^{K} m[P^{R_{i}}, m_{i}]$ if $p_{r} = \prod_{i=1}^{K} P^{R_{i}}(R_{i} = r_{i})$, i.e., the sources have independent reliabilities

→ discount and combine approach

- $\sum_{i=1}^{K} w_{i} \cdot m_{i}$ if $p_{r} = w_{i}$ for $r$ such that $r_{i} = 1$ and $r_{j} = 0$, for all $j \neq i$, i.e., source $s_{i}$ is reliable and the other are not with probability $w_{i}$

→ weighted average
Outline

1. Reliability
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   - Two sources
   - $K$ sources
   - Uncertain testimonies

2. Truthfulness and beyond
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   - Refined form
   - General model

3. Selecting meta-knowledge
   - Absence of prior information
   - Learning data
Truthfulness

- Reliability of a source includes another dimension besides its relevance: its truthfulness.
- A source is said truthful if it actually supplies the information it possesses.
- Lack of truthfulness can take several forms, and can be intentional or accidental.
- For instance, a sensor that has a systematic bias (such as a watch that has not been calibrated to winter time), is a kind of (unintentional) lack of truthfulness.
- Let us first consider the crudest form, where a source is nontruthful if it tells the contrary of what it knows.
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Formalization

Assume a source \( s \) supplying information item \( X \in A \)

- If \( s \) is not relevant, we replace \( X \in A \) by \( X \in \mathcal{X} \)
- If \( s \) is relevant,
  - either it is truthful, in which case we keep \( X \in A \)
  - or it lies, in which case we replace \( X \in A \) by \( X \in \overline{A} \)
Formalization

- Assume a source $s$ supplying information item $X \in A$
  - If $s$ is not relevant, we replace $X \in A$ by $X \in \mathcal{X}$
  - If $s$ is relevant,
    - either it is truthful, in which case we keep $X \in A$
    - or it lies, in which case we replace $X \in A$ by $X \in \overline{A}$

- Relevance $R$ defined on $\mathcal{R} = \{\text{rel}, \neg\text{rel}\}$
- Truthfulness $T$ defined on $\mathcal{T} = \{\text{tru}, \neg\text{tru}\}$
- Let $\mathcal{RT} := \mathcal{R} \times \mathcal{T}$
- The interpretation of the testimony according to the relevance and truthfulness may be encoded by $\Lambda_A : \mathcal{RT} \to 2^\mathcal{X}$ such that
  
  \[
  \begin{align*}
  \Lambda_A(\text{rel}, \text{tru}) &= A \\
  \Lambda_A(\text{rel}, \neg\text{tru}) &= \overline{A} \\
  \Lambda_A(\neg\text{rel}, \text{tru}) &= \mathcal{X} \\
  \Lambda_A(\neg\text{rel}, \neg\text{tru}) &= \mathcal{X}
  \end{align*}
  \]
Example

- Sensor $s$ tells $X \in A = \{a\} = \{g, h\}$.
- It is assumed to be relevant and non truthful, i.e., in state $(\text{rel}, \neg \text{tru})$.
- Knowledge about $X$:

$$X \in \Lambda_{\{a\}}(\text{rel}, \neg \text{tru}) = \{a\}$$
Uncertain meta-knowledge and testimony

- Assume now $s$ is in state $(r, t) \in RT$ with probability $\rho_{rt} = P_{RT}^{RT}(R = r, T = t)$ (and still $P_{RT}^R(R = \neg \text{rel}) = \pi$)
- In addition, $s$ provides the uncertain testimony $m_s$.
- This yields the following knowledge on $X$

$$m[P_{RT}^{RT}, m_s] = \rho_{\text{rel,tru}} \cdot m_s + \rho_{\text{rel,\neg tru}} \cdot \overline{m_s} + \pi m_X$$

with $\overline{m_s}$ the negation of $m_s$ ($\overline{m_s}(A) = m_s(\overline{A})$, for all $A \subseteq X$)
Particular cases

\( m[P^R, m_5] \) reduces to

- \( m[P^R, m_5] \) if \( P^T(T = \text{tru}) = 1 \), i.e., if \( s \) is partially relevant and totally truthful
  \( \rightarrow \) discounting

- \( P^T(T = \text{tru}) \cdot m_5 + P^T(T = \neg \text{tru}) \cdot \overline{m_5} \) if \( P^R(R = \text{rel}) = 1 \), i.e., if \( s \) is totally relevant and partially truthful
  \( \rightarrow \) negating
The case of multiple sources

- Assume sources $s_i$, $i = 1 \ldots, K$, supplying $A = (A_1, \ldots, A_K)$.
- Let $\Lambda_{A_i} : \mathcal{RT}_i \rightarrow 2^X$ represent the interpretation of $X \in A_i$ given the reliability $R_i$ and truthfulness $T_i$ of $s_i$.
- When the sources are in the state

$$rt = (rt_1, \ldots, rt_K) \in \mathcal{RT} := \times_{i=1}^{K} \mathcal{RT}_i$$

we must conclude

$$X \in \Lambda_A(rt) := \bigcap_{i=1}^{K} \Lambda_{A_i}(rt_i)$$

- Example: $K=2$

$$\Lambda_A(\text{rel}_1, \text{tru}_1, \text{rel}_2, \neg \text{tru}_2) = \Lambda_{A_1}(\text{rel}_1, \text{tru}_1) \cap \Lambda_{A_2}(\text{rel}_2, \neg \text{tru}_2)$$

$$= A_1 \cap \overline{A_2}$$
Non-elementary behavior assumptions

- Non-elementary assumptions $RT \subseteq \mathcal{RT}$ on the relevance and truthfulness of the sources can also be considered.
- We have
  $$\Lambda_A(RT) = \bigcup_{rt \in RT} \Lambda_A(rt)$$
- Example: $RT = \{(rel_1, tru_1, rel_2, \neg tru_2), (rel_1, \neg tru_1, rel_2, tru_2)\}$ ($s_1$ and $s_2$ relevant and exactly one of them is truthful)
  $$\Lambda_A(RT) = \Lambda_A(rel_1, tru_1, rel_2, \neg tru_2) \cup \Lambda_A(rel_1, \neg tru_1, rel_2, tru_2) = (A_1 \cap \overline{A_2}) \cup (\overline{A_1} \cap A_2) = A_1 \Delta A_2$$ (exclusive or)

$\rightarrow$ All connectives of Boolean logic can be reinterpreted in terms of source behavior assumptions wrt relevance and truthfulness

- $\otimes_{RT}$ Boolean connective associated to $RT$.
- Different assumptions may induce the same connective.
Uncertain meta-knowledge and testimonies

- Sources $s_1, \ldots, s_K$ provide $m = (m_1, \ldots, m_K)$ and are assumed to be independent.

- Uncertain meta-knowledge in the form of a MF $m^{RT}$:

\[
m[\cdot] = \sum_{RT, A: \text{RT}(A) = B} m^{RT}(RT) \cdot m(A)
\]

\[
m[\cdot] = \sum_{\otimes, A: \text{RT}(A) = B} \rho(\otimes) \cdot m(A)
\]

with

\[
\rho(\otimes) = \sum_{\otimes RT = \otimes} m^{RT}(RT)
\]
Particular cases

Suppose $m^{RT}$ is such that $m^{RT}(RT) = 1$ for some $RT \subseteq RT$ and $K = 2$. Then, $m[m^{RT}, m]$ reduces to

- $m_1 \cap m_2$ for $RT = s_1$ and $s_2$ relevant and truthful
- $m_1 \cup m_2$ for $RT = s_1$ and $s_2$ relevant and at least one of them is truthful (disjunctive rule, def. of $\cap$ with $\cap$ replaced by $\cup$)
- $m_1 \cup m_2$ for $RT = s_1$ and $s_2$ relevant and exactly one of them is truthful (exclusive disjunctive rule, $\cap$ replaced by $\Delta$)
- $m_1 \cap m_2$ for $RT = s_1$ and $s_2$ relevant and $s_1$ is truthful if and only if $s_2$ is so too (equivalence rule, $\cap$ replaced by $\leftrightarrow$)

More generally, all rules relying on Boolean connectives are particular cases. For instance, the rule extending q-relaxation from interval analysis is recovered for $RT = (K - q)$-out-of-$K$ sources relevant and all truthful (ranges from $\cap$ to $\cup$).
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Contextual and polarized lack of truthfulness

\[ \neg \text{tru} : \text{one must deduce the contrary of what } s \text{ tells for each } x_i \in \mathcal{X} \]
and whatever the polarity of the clause used by \( s \) regarding \( x_i \).
Contextual and polarized lack of truthfulness

- ¬tru: one must deduce the contrary of what $s$ tells for each $x_i \in X$ and whatever the polarity of the clause used by $s$ regarding $x_i$.
- $s$ non truthful only for some $x_i \in X$, and maybe even only for the positive or negative clauses regarding $x_i$. 

Example: Sensor $s$ is non truthful when it tells that $a$ is not a possible value for $X$, non truthful when it tells that $g$ is a possible value for $X$, and truthful in all other cases, e.g., truthful when it tells that $a$ is a possible value for $X$. Sensor $s$ tells $X \in A = \{g, h\}$, i.e., $a$ is not a possible value and $g$ and $h$ are possible values for $X$.

By considering source states based on this refined form of lack of truthfulness, we can recover contextual discounting and the $\alpha$-junctions, and contextualize negating (see appendix).
Contextual and polarized lack of truthfulness

- tru : one must deduce the contrary of what \( s \) tells for each \( x_i \in X \) and whatever the polarity of the clause used by \( s \) regarding \( x_i \).
- \( s \) non truthful only for some \( x_i \in X \), and maybe even only for the positive or negative clauses regarding \( x_i \).
- Example : Sensor \( s \) is
  - non truthful when it tells that \( a \) is not a possible value for \( X \),
  - non truthful when it tells that \( g \) is a possible value for \( X \),
  - and truthful in all other cases, e.g., truthful when it tells that \( a \) is a possible value for \( X \)
- Sensor \( s \) tells \( X \in A = \{ g, h \} \), i.e., \( a \) is not a possible value and \( g \) and \( h \) are possible values for \( X \)
- We deduce (assuming \( s \) relevant): \( X \in \{ a, h \} \)
Contextual and polarized lack of truthfulness

- ¬tru : one must deduce the contrary of what $s$ tells for each $x_i \in X$ and whatever the polarity of the clause used by $s$ regarding $x_i$.
- $s$ non truthful only for some $x_i \in X$, and maybe even only for the positive or negative clauses regarding $x_i$.
- Example : Sensor $s$ is
  - non truthful when it tells that $a$ is not a possible value for $X$,
  - non truthful when it tells that $g$ is a possible value for $X$,
  - and truthful in all other cases, e.g., truthful when it tells that $a$ is a possible value for $X$
- Sensor $s$ tells $X \in A = \{g, h\}$, i.e., $a$ is not a possible value and $g$ and $h$ are possible values for $X$
- We deduce (assuming $s$ relevant): $X \in \{a, h\}$

→ By considering source states based on this refined form of lack of truthfulness, we can recover contextual discounting and the $\alpha$-junctions, and contextualize negating (see appendix)
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Beyond relevance and truthfulness

- Knowledge about the source quality may be different from knowing their relevance and truthfulness.
- The provided information by a source may also bear on another variable $Y$, related to $X$.

$\rightarrow$ An approach to account for general source quality (behaviour) assumptions

$$\mathcal{R}, \mathcal{RT} \leadsto \mathcal{H} = \{h^1, \ldots, h^N\}$$

$$X \in A \subseteq \mathcal{X} \leadsto Y \in A \subseteq \mathcal{Y}$$

- If the source is in state $h \in \mathcal{H}$, we should deduce $X \in B \subseteq \mathcal{X}$ from information item $Y \in A \subseteq \mathcal{Y}$.

- For all $A \subseteq \mathcal{Y}$, $\Pi_A : \mathcal{H} \rightarrow 2^\mathcal{X}$ such that

$$\Pi_A(h) = B$$
Example

- $X$ with possible values in $\mathcal{X} = \{a, g, h\}$
- Sensor $s$ does not know the type airplane, i.e., $\mathcal{Y} = \{g, h\}$.
- It uses either the shape or the material of the aircraft
  - If $s$ uses the shape, then when it tells
    - glider, we can deduce airplane or glider
    - helicopter, we keep this piece of information
  - If $s$ uses the material, then when it tells
    - glider, we keep this piece of information
    - helicopter, we replace by helicopter or airplane
Example

- $X$ with possible values in $\mathcal{X} = \{a, g, h\}$
- Sensor $s$ does not know the type airplane, i.e., $\mathcal{Y} = \{g, h\}$.
- It uses either the shape or the material of the aircraft
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    - helicopter, we keep this piece of information
  - If $s$ uses the material, then when it tells
    - glider, we keep this piece of information
    - helicopter, we replace by helicopter or airplane

- $\mathcal{H} = \{\text{shape, material}\}$

\[
\begin{align*}
\Pi_g(\text{shape}) & = \{a, g\} \\
\Pi_h(\text{shape}) & = \{h\} \\
\Pi_g(\text{material}) & = \{g\} \\
\Pi_h(\text{material}) & = \{a, h\}
\end{align*}
\]
Example

- We are interested by the number $X \in \mathcal{X} = \{x_1, \ldots, x_n\} = \{1, \ldots, n\}$ of aircrafts in a particular area.
- Information about $X$ comes from a source $s$, which can be reliable, approximately reliable or non reliable.
- If $s$ is approximately reliable, the information item it supplies must be expanded using the lowest and highest closest values.
Example

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$\mathcal{H} = \{\text{rel}, \text{ap-rel}, \neg\text{rel}\}$

- For any $A_{i,j} \subseteq \mathcal{X}$, with $A_{i,j} = \{x_i, \ldots, x_j\}$, $1 \leq i \leq j \leq n$

$$
\begin{align*}
\Pi_{A_{i,j}}(\text{rel}) &= A_{i,j} \\
\Pi_{A_{i,j}}(\text{ap-rel}) &= \{x_{i-1}\} \cup A_{i,j} \cup \{x_{j+1}\} \\
\Pi_{A_{i,j}}(\neg\text{rel}) &= \mathcal{X}
\end{align*}
$$

with $x_0 = x_{n+1} = \emptyset$. 

Uncertain meta-knowledge and testimonies

- Single information source

\[
m[m^H, m^\gamma](B) = \sum_{H,A:\Pi_A(H)=B} m^H(H) \cdot m^\gamma(A)
\]

Behaviour-based correction (BBC)
Uncertain meta-knowledge and testimonies

- Single information source

\[ m[m^H, m^Y](B) = \sum_{H,A: \Pi_A(H) = B} m^H(H) \cdot m^Y(A) \]

**Behaviour-based correction (BBC)**

- Multiple information sources: \( H := \times_{i=1}^K \mathcal{H}_i \)

\[ m[m^H, m](B) = \sum_{H,A: \Pi_A(H) = B} m^H(H) \cdot m(A) \]

with \( m(A) = \prod_{i=1}^K m_i^Y(A_i) \)

**Behaviour-based fusion (BBF)**
Operations on product spaces

BBC and BBF can be recovered using the following standard operations of BFT:

- **Marginalization** ↓
  \[
  m^{X \times Y \downarrow X}(A) = \sum_{\{B \subseteq X \times Y, (B \downarrow X) = A\}} m^{X \times Y}(B), \quad \forall A \subseteq X,
  \]

- **Conjunctive rule on product spaces**
  \[
  m_1^X \bigotimes m_2^Y = m_1^{X \uparrow X \times Y} \bigotimes m_2^{Y \uparrow X \times Y}.
  \]

With \(\uparrow\) (vacuous extension) defined as

\[
  m^{X \uparrow X \times Y}(B) = \begin{cases} 
  m^X(A) & \text{if } B = A \times Y \text{ for some } A \subseteq X, \\
  0 & \text{otherwise}.
  \end{cases}
\]
BBC

- Mappings \( \Pi_A, A \subseteq Y \), define a relation between spaces \( \mathcal{H}, 2^Y \) and \( X \), which can be represented by MF \( m_\Pi \) on \( \mathcal{H} \times 2^Y \times X \) s.t.

\[
m_\Pi \left[ \bigcup_{h \in \mathcal{H}, A \in 2^Y} \{h\} \times \{A\} \times \Pi_A(h) \right] = 1
\]
**BBB**

- Mappings $\Pi_A$, $A \subseteq \mathcal{Y}$, define a relation between spaces $\mathcal{H}$, $2^\mathcal{Y}$ and $\mathcal{X}$, which can be represented by MF $m_\Pi$ on $\mathcal{H} \times 2^\mathcal{Y} \times \mathcal{X}$ s.t.

$$m_\Pi \left[ \bigcup_{h \in \mathcal{H}, A \in 2^\mathcal{Y}} (\{h\} \times \{A\} \times \Pi_A(h)) \right] = 1$$

**Lemma**

$$m[m_\mathcal{H}, m_\mathcal{Y}_5] = (m_5 \odot m_\Pi \odot m_\mathcal{H}) \downarrow \mathcal{X}$$

with $m_5$ on $2^\mathcal{Y}$ s.t. $m_5(\{A\}) = m_5^\mathcal{Y}(A)$
Lemma

\[ m[m^\mathcal{H}, m] = \left( \mathbin{\bigcap}_{i=1}^{K} (m_i \cup m_{\Pi i}) \cap m^\mathcal{H} \right) \downarrow \chi \]
Independent behaviours (meta-independence)

**Theorem**

If $m^\mathcal{H} = \bigcap_{i=1}^{K} m^{\mathcal{H}_i}$ then

$$m[m^\mathcal{H}, m] = \bigcap_{i=1}^{K} m[m^{\mathcal{H}_i}, m_i]$$

**Proof**: Uses local computation (see Prakash’s lecture).
Outline

1 Reliability
   - One source
   - Two sources
   - $K$ sources
   - Uncertain testimonies

2 Truthfulness and beyond
   - Crudest form
   - Refined form
   - General model

3 Selecting meta-knowledge
   - Absence of prior information
   - Learning data
Typology of approaches

- The model allows to interpret pieces of information given meta-knowledge on the emitting sources.
- It does not however indicate which meta-knowledge to use.

→ Means to select meta-knowledge

- Two possible situations:
  1. One has some prior information (learning data, expert knowledge) on the sources
  2. The only available information are the pieces of information received

- Typically, in both cases, a set $S$ of candidate assumptions (meta-knowledge) is considered, and some sensible strategy is used to pick an assumption in this set.
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Consistency and specificity

- Only $\mathbf{m} = (m_1, \ldots, m_K)$ available.

→ Selection of meta-knowledge based on the two primary features sought regarding knowledge about $X$: consistency and specificity
Consistency and specificity

- Only $m = (m_1, \ldots, m_K)$ available.

→ Selection of meta-knowledge based on the two primary features sought regarding knowledge about $X$: consistency and specificity

- 3 sources about $X \in \mathcal{X} = \{a, g, h\}$ supplying $A = (A_1, A_2, A_3)$ s.t.
  
  
  $$A_1 = \{a\}, A_2 = \{a, g\}, A_3 = \{g, h\}$$

- Assumption $R_1 = \text{“all sources are reliable”}$ yields
  
  $$X \in \Gamma_A(R_1) = A_1 \cap A_2 \cap A_3 = \emptyset$$
  
  i.e. an inconsistent result, and thus cannot hold.

- In contrast, the assumption $R_3 = \text{“at least one of the sources is reliable”}$ yields
  
  $$X \in \Gamma_A(R_3) = A_1 \cup A_2 \cup A_3 = \mathcal{X}$$
  
  and is thus plausible (it does not yield a contradiction). However, it is useless as it is not informative at all.
Meta-knowledge selection strategy

- The intermediate assumption $R_2 = \text{“at least two of the sources are reliable”}$ yields

$$X \in \Gamma_A(R_2) = (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3) = \{a, g\}$$

$R_2$ is plausible (the result is consistent) and informative (or, at least, more informative than $R_3$).
Meta-knowledge selection strategy

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$$X \in \Gamma_A(R_2) = (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3) = \{a, g\}$$

$R_2$ is plausible (the result is consistent) and informative (or, at least, more informative than $R_3$).

- Here, $R_2$ is preferable, but for other $A$, it could be $R_1$ or $R_3$ due to

$$\Gamma_A(R_1) \subseteq \Gamma_A(R_2) \subseteq \Gamma_A(R_3), \quad \forall A$$

$R_{i+1}$ will always yield a result that is on the hand at least as consistent as that of $R_i$, but also on the other hand as most as specific as that of $R_i$.

$\rightarrow$ Consistency and specificity are antagonists goals

- Sensible strategy for a given $A$: test iteratively each $R_i$ and select the first one which yields a consistent result (it will then be the most specific and consistent possible result).
Extension to uncertain meta-knowledge and testimonies

- In general, meta-knowledge and supplied information are uncertain, i.e., we have $m^H$ and $m = (m_1, \ldots, m_K)$, and thus their interpretation is the MF $m[m^H, m]$ (assuming independent sources).
- Need extensions to MF of consistency and specificity in order to compare pieces of meta-knowledge:
  - consistency of a MF $m$: $\phi(m) = \max_{x \in \mathcal{X}} p_l(x)$.
  - specificity: $m_1 \sqsubseteq m_2$ with $\sqsubseteq$ the specialization

**Proposition**

Let $m^H_1$ and $m^H_2$ be two assumptions.

$$m[m^H_1, m] \sqsubseteq m[m^H_2, m], \forall m \Rightarrow \phi(m[m^H_1, m]) \leq \phi(m[m^H_2, m]), \forall m$$

→ Consistency and specificity are at odds!
General meta-knowledge selection strategy

**Strategy**

1. Define a set $S = \{ m_1^\mathcal{H}, \ldots, m_M^\mathcal{H} \}$:
   - $m[m_j^\mathcal{H}, m] \subseteq m[m_{j+1}^\mathcal{H}, m], \forall m$;
   - $m_1^\mathcal{H}$ corresponds to the conjunctive rule.
2. Test iteratively each $m_j^\mathcal{H}$ until $\phi(m[m_j^\mathcal{H}, m]) \geq \tau$.

**Practical instances of $S$:**

- $m_j^\mathcal{H}$: $K - j + 1$ out of $K$ reliable sources.
- $m_j^\mathcal{H}$: sources with independent reliabilities, source $i$ reliable with probability $p_i^j$ such that $p_i^j \geq p_i^{j+1}$ (increasing discount and combine, often used for conflict management)
- $m_j^\mathcal{H}$: meta-knowledge corresponding to the $\alpha$-conjunctions for some $\alpha = \alpha_j$ such that $\alpha_j \geq \alpha_{j+1}$.
Application

Nuclear reactor safety

- Project BEMUSE of the Nuclear Energy Agency.
- $K = 10$ sources (CEA, IRSN,...) providing uncertain estimates of parameter values of a nuclear power plant.
- Costly data and complex phenomena involved → no reliable means to know the source reliabilities.
- Chose $S$ with $m_j^H = K - j + 1$ out of $K$ reliable sources.
- PCT2 parameter with domain $\mathcal{X} = \{x_1, \ldots, x_6\}$, $\mathbf{m} := (m_1, \ldots, m_{10})$.
  - $\phi(m[m_1^H, \mathbf{m}]) = 0.19$ (all sources reliable)
  - $\phi(m[m_2^H, \mathbf{m}]) = 0.81$ (9 out of 10 reliable)
  - $\phi(m[m_3^H, \mathbf{m}]) = 1$ (8 out of 10 reliable)
- Values $x_4$ and $x_5$ are definitely more plausible.

→ Results that are consistent, informative and readable by the end-user.
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   - Learning data
**General setting**

- Consider a system which outputs for a given object $o$, a guess about the actual value $x^*$ of some feature $X \in \mathcal{X}$ of $o$.
- To produce this output, the system uses internally some information correction or fusion, characterized by some $m^H \in S$.
- Output for object $o$ may thus be noted $f(o; m^H)$.
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To produce this output, the system uses internally some information correction or fusion, characterized by some $m^\mathcal{H} \in S$.

Output for object $o$ may thus be noted $f(o; m^\mathcal{H})$.

Assume a set of $\ell$ objects for which the true value of $X$ is known, i.e., $\{x^*_i\}_{i=1}^\ell$ is available.

Assume outputs $\{f(o_i; m^\mathcal{H})\}_{i=1}^\ell$ may be obtained for any $m^\mathcal{H} \in S$. 
Loss minimization

- The $\hat{m}^H$ to be used to produce the output for a new object may then be chosen as the one in $S$ minimizing the average loss

$$J(m^H) = \frac{1}{n} \sum_{i=1}^{\ell} \mathcal{L}(f(o_i; m^H), x_i^*)$$

for some loss function $\mathcal{L}(f(o; m^H), x^*)$

- Typically, $f(o; m^H)$ is a MF on $\mathcal{X}$, which is transformed into a probability measure $P_o^X$, and the squared error (SE) or cross-entropy (CE) loss is used:

$$\mathcal{L}_{SE}(f(o; m^H), x^*) = \sum_{x \in \mathcal{X}} (1_{x^*}(x) - p_o(x))^2$$

$$\mathcal{L}_{CE}(f(o; m^H), x^*) = - \sum_{x \in \mathcal{X}} 1_{x^*}(x) \log p_o(x)$$

- Remark: more or less complex optimisation problem to solve depending on $S$ and $\mathcal{L}$
Application

Classifier correction [Elouedi et al., 2004]

- $X$ is the class of an object.
- The system is a classifier whose outputs are corrected with meta-knowledge $m^H = P^R$ (discounting) with

$$P^R \in S = \{ P^R | \pi \in [0, 1] \}$$

- The classifier output for a given object $o$ is a mass function $m_o$.
- The system output is thus

$$f(o; m^H) = m[P^R, m_o]$$

- Loss function: pignistic probability transformation with SE.
Application

Illustrative example

- Classifier outputs $m_{o_i}$ for 4 objects with actual values $x_i^*$ in $\mathcal{X} = \{a, g, h\}$.

<table>
<thead>
<tr>
<th></th>
<th>$g$</th>
<th>$h$</th>
<th>${a, h}$</th>
<th>${g, h}$</th>
<th>$\mathcal{X}$</th>
<th>$x_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{o1}$</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>0.3</td>
<td>0.2</td>
<td>$a$</td>
</tr>
<tr>
<td>$m_{o2}$</td>
<td>0.5</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0.3</td>
<td>$g$</td>
</tr>
<tr>
<td>$m_{o3}$</td>
<td>0.4</td>
<td>0</td>
<td>0.6</td>
<td>0</td>
<td>0</td>
<td>$a$</td>
</tr>
<tr>
<td>$m_{o4}$</td>
<td>0</td>
<td>0</td>
<td>0.6</td>
<td>0.4</td>
<td>0</td>
<td>$h$</td>
</tr>
</tbody>
</table>

- Meta-knowledge minimizing the average loss: $\hat{\pi} = 0.66$
Summary

- Interpretation of BFT as a theory of partially reliable and elementary pieces of information
  - Any set of such pieces of information is represented by a unique MF
  - To any MF can be associated uniquely such a set.

- Beyond reliability, information correction and fusion given knowledge on other aspects of source quality, such as truthfulness.

- Numerous and important correction and fusion approaches can be read using this prism.

- Means to determine knowledge on source quality in practice, with and without prior information on the sources.
Open topics of interest

- **Exploitation of the** $\sqcap_\sigma$ **rule** for SMF and the associated decomposition of a MF into (in)dependent SMF
  - Cautious combination
  - Refining of approaches based on conjunctive combination of independent SMF, such as GBT, E-KNN, DS analysis of GLR classifiers, contextual reinforcement.

- **Interpretation** of other correction and fusion approaches.

- **Selection of meta-knowledge**: refine arguments for the
  - Choice of $S$ (include dependence between the sources)
  - Choice of $\mathcal{L}$ (including for the case of partially known true values)

- **Conflict measurement**: decomposition, measure selection for a given situation (properties, learning), refine with measures from logic, links with distances
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The following bibliography contains:

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- Some other interesting references, and in particular some more application-oriented papers, where correction/fusion is not the main topic but plays an important part.
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T. Denœux, O. Kanjanatarakul, and S. Sriboonchitta.
Contextual and polarized lack of truthfulness

- ¬tru : one must deduce the contrary of what $s$ tells for each $x_i \in \mathcal{X}$ and whatever the polarity of the clause used by $s$ regarding $x_i$.
- $s$ non truthful only for some $x_i \in \mathcal{X}$, and maybe even only for the positive or negative clauses regarding $x_i$.
- Example : Sensor $s$ is
  - non truthful when it tells that $a$ is not a possible value for $X$
    (negatively non truthful for $a$)
  - and non truthful when it tells that $g$ is a possible value for $X$
    (positively non truthful for $g$)
  - and truthful in all other cases, e.g., truthful when it tells that $a$ is a possible value for $X$ (positively truthful for $a$).

- Sensor $s$ tells $X \in A = \{g, h\}$, i.e., $a$ is not a possible value and $g$ and $h$ are possible values for $X$
- We deduce (assuming $s$ relevant): $X \in \{a, h\}$
Contextual and polarized lack of truthfulness

- Three interesting states (contextual lies):
  - $n_B$: negatively non truthful for $x_i \in B$;
  - $p_B$: positively non truthful for $x_i \in B$;
  - $\ell_B$: non truthful for $x_i \in B$.

- Let $\tilde{T} = \{n_B, p_B, \ell_B | B \subseteq X\}$ and $\tilde{\Lambda}_A : R\tilde{T} \rightarrow 2^X$ represent the interpretations of testimony $X \in A$ given the possible states in $R\tilde{T}$ of the source

- $\tilde{\Lambda}_A$ extends $\Lambda_A$, e.g., $\tilde{\Lambda}_A(\ell_\emptyset) = \Lambda_A(\neg\text{tru})$ and $\tilde{\Lambda}_A(\ell_X) = \Lambda_A(\text{tru})$ (assuming relevance)
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F. Pichon (LGI2A)  
Information correction and fusion  
BFAS School 92
Contextual and polarized lack of truthfulness

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  - \( p_B \): positively non truthful for \( x_i \in B \);
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Let \( \tilde{T} = \{ n_B, p_B, \ell_B \mid B \subseteq \mathcal{X} \} \) and \( \tilde{\Lambda}_A : \mathcal{R}\tilde{T} \rightarrow 2^\mathcal{X} \) represent the interpretations of testimony \( X \in A \) given the possible states in \( \mathcal{R}\tilde{T} \) of the source.

- \( \tilde{\Lambda}_A \) extends \( \Lambda_A \), e.g., \( \tilde{\Lambda}_A(\ell_{\emptyset}) = \Lambda_A(\neg \text{tru}) \) and \( \tilde{\Lambda}_A(\ell_{\mathcal{X}}) = \Lambda_A(\text{tru}) \) (assuming relevance).

\[ \begin{align*}
\tilde{\Lambda}_A(n_B) & \quad \tilde{\Lambda}_A(p_B) & \quad \tilde{\Lambda}_A(\ell_B)
\end{align*} \]
Uncertain meta-knowledge and testimonies

- Single information source
  
  \[ m[m^R\tilde{T}, m_s](B) = \sum_{R\tilde{T},A:\tilde{\Lambda}_A(R\tilde{T})=B} m^R\tilde{T}(R\tilde{T}) \cdot m_s(A) \]

- Multiple information sources

  \[ m[m^R\tilde{T}, \mathbf{m}](B) = \sum_{R\tilde{T},A:\tilde{\Lambda}_A(R\tilde{T})=B} m^R\tilde{T}(R\tilde{T}) \cdot m(A) \]
Particular cases

Let $\mathcal{B} = \{ B_1, \ldots, B_N \} \subseteq 2^X$. Consider iterative corrections (series of agents) of testimony $m_\mathcal{S}$ provided by agent 1 with respective assumptions “preceding agent $i$ is relevant, and is truthful with probability $\beta_{B_i}$ and with probability $1 - \beta_{B_i}$ commits lie $\nabla \beta_{B_i}$:

- $n_{B_i}$: $m_\mathcal{S} \uplus_{B_i \in \mathcal{B}} m_{B_i}$ with $m_{B_i}(\emptyset) = \beta_{B_i}$, $m_{B_i}(B_i) = 1 - \beta_{B_i}$, called contextual discounting (it can also be obtained as a single correction $m[m^\mathcal{R}^\mathcal{T}, m_\mathcal{S}]$ with $m^\mathcal{R}^\mathcal{T}$ the $\uplus$-combination of the preceding assumptions)
- $\ell_{B_i}$: $m_\mathcal{S} \uplus_{B_i \in \mathcal{B}} B_i^{\beta_{B_i}}$, contextual negating
- $p_{B_i}$: $m_\mathcal{S} \uplus_{B_i \in \mathcal{B}} B_i^{\beta_{B_i}}$, contextual reinforcement

Remarks:

- These correction mechanisms generalize their non-contextual versions for specific $\mathcal{B}$ such that $|\mathcal{B}| = 1$, hence their names.
- An alternative interpretation exists for contextual discounting when $\mathcal{B}$ is a partition of $X$ (see Thierry’s lecture).
Example

Contextual discounting

- Suppose a sensor \( s \) supplies information \( X \in A = \{g\} \)

- We know that \( s \) is relevant and that at least one of the following independent pieces of meta-knowledge holds:
  - \( s \) commits lie \( n_{\{a,g\}} \) with probability 0.2
  - \( s \) commits lie \( n_{\{g,h\}} \) with probability 0.3

- Our knowledge on \( X \) is then obtained by

\[
m_5(\{g\}) = 1 \quad \bigcirc \quad \left\{ \begin{array}{l}
m_{\{a,g\}}(\{a, g\}) = 0.2 \\
m_{\{a,g\}}(\emptyset) = 0.8 \\
m_{\{g,h\}}(\emptyset) = 0.7 \\
m_{\{g,h\}}(\{g, h\}) = 0.3 \end{array} \right\}
\]

which yields

\[
m(\{g\}) = 0.56, \quad m(\{a, g\}) = 0.14, \quad m(\{g, h\}) = 0.24, \quad m(X) = 0.06
\]
Particular cases

- Consider the following meta-knowledge about two sources $s_1$ and $s_2$ supplying information $m_1$ and $m_2$:
  - They are both relevant
  - And they are either both truthful or commit the same contextual lie $\ell_B$ with probability $\alpha^{|B|} (1 - \alpha)^{|\overline{B}|}$, for some $\alpha \in [0, 1]$

- Then

$$m[m^R \tilde{T}, m](A) = \sum_{(A_1 \cap A_2) \cup (\overline{A_1} \cap \overline{A_2} \cap B) = A} m_1(A_1) m_2(A_2) m_\alpha(B)$$

where $m_\alpha(B) = \alpha^{|\overline{B}|} (1 - \alpha)^{|B|}$

- $\alpha$-conjunctions $\ominus^\alpha$ [Smets, 1997]: family of the associative, commutative and linear combination rules having the vacuous mass function as neutral element (family depending on a parameter $\alpha \in [0, 1]$, such that $\ominus^1 = \ominus$ and $\ominus^0 = \ominus$).