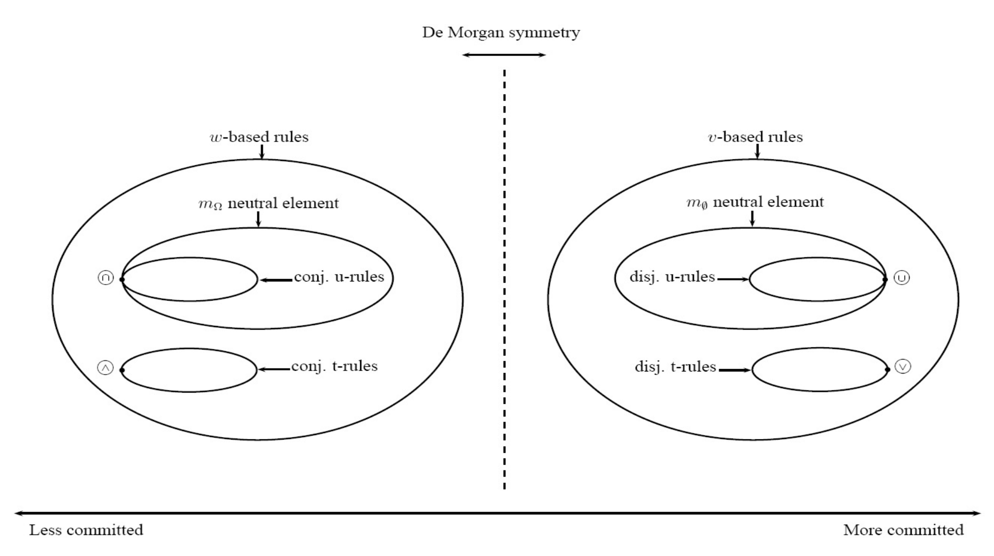
par Frédéric Pichon

Fonctions de Croyance: Décompositions Canoniques et Règles de Combinaison

Belief Functions: Canonical Decompositions and Combination Rules

Thèse présentée pour l'obtention du grade de Docteur de l'UTC.



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Frédéric Pichon

Thèse soutenue le 24 mars 2009 devant le jury composé de :

Mme	COUSO Inés	Rapporteur
M.	DUBOIS Didier	Rapporteur
M.	SCHÖN Walter	Président
Mme	MATTIOLI Juliette	Examinateur
M.	JOSSET François-Xavier	Examinateur
M.	DENŒUX Thierry	Directeur de thèse

À Marie.

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Introduction

Problem Statement

Information fusion problems, such as heterogeneous sensor fusion, classification, and expert opinion pooling, have gained considerable interest in the last decades. Such problems can be summarized as follows. Consider a variable, whose true value is unknown. Suppose a set of sources, e.g., sensors or experts, providing information on the value taken by this variable. The problem consists then in combining this information in order to determine the most plausible values of the variable.

The fusion process faces various difficulties. First, the information delivered by the sources is generally imperfect, that is, imprecise and uncertain. Second, knowledge about the sources, such as their truthfulness, must be accounted for. In order to deal with all these aspects of the information fusion problem, formal models are needed.

In the last thirty years, it has been acknowledged that the most classical ones, probability theory and set theory, cannot handle all facets of the imperfection of the information due to an "unreasonable requirement for precision" [100] in the case of probability theory and a poor expressivity of uncertainty in the case of set theory. Nowadays, the most researched alternatives to probability theory and set theory seem to be possibility theory [27], imprecise probability theory [100], and the Dempster–Shafer theory of belief functions [11, 77, 80]. These theories are related to one another (see, e.g., [60, 61]) and may be used in a complementary fashion, as illustrated in [3].

Different interpretations of the Dempster–Shafer theory of belief functions have been proposed [84] and, in particular, the Transferable Belief Model (TBM), which views belief functions as representing beliefs held by rational agents. In contrast to other interpretations (based, e.g., on random sets or imprecise probabilities), the TBM does not assume any underlying probability concepts [88, 96]. It has been successfully applied to the information fusion problems mentioned above (see, e.g., [2, 4, 72] for sensor fusion applications in the military domain, [14, 20, 58, 69] for classification applications, and [23, 40] for expert opinion pooling applications). All of these applications involve fusing belief functions and rely thus critically on belief functions aggregation operators, called combination rules in the TBM.

Despite the aforementioned successes, it may be argued that the TBM suffers from a lack of flexibility in terms of combination rules, especially as compared to possibility theory. Indeed, it appears that belief functions are usually combined in the TBM using either the unnormalized version of Dempster's rule [11], referred to as the TBM conjunctive rule throughout this thesis, or the TBM disjunctive rule

[26, 83]. Both rules fit with the particular situation where the sources are believed to be independent. Furthermore, the TBM conjunctive rule is appropriate when all sources are assumed to tell the truth, whereas the TBM disjunctive rule should be used when at least one of the sources is known to tell the truth. In possibility theory, the situation is quite different since there exist combination rules to deal with many more cases. In particular, one finds in possibility theory a conjunctive rule and a disjunctive rule suitable to the combination of information coming from nonindependent sources. More generally, possibility theory enjoys an infinity of conjunctive rules and an infinity of disjunctive rules, as well as parameterized and more refined fusion modes that fit with more elaborated knowledge about the sources (see, e.g., [29, 30]). This multiplicity of rules is particularly useful from an applicative point of view, since it allows the choice of a combination rule based on the application's characteristics.

A general and important problem that needs to be addressed in the TBM is thus the one of introducing some flexibility for the combination of belief functions. In particular, it seems interesting to have more choice in terms of conjunctive and disjunctive operators, as is the case in possibility theory. It seems also relevant to find operators allowing us to deal with situations other than the conjunctive and disjunctive ones, i.e., other than when all the sources are assumed to tell the truth or when at least one of the sources is assumed to tell the truth.

Let us remark that the flexibility issue is not new. Indeed, in [25, 28], Dubois and Prade already defended the idea that no single combination rule is suitable to all information fusion situations. As a matter of fact, alternatives to the TBM conjunctive and TBM disjunctive rules have been proposed in the literature (see, e.g., [76, 95] for recent surveys), but they do not seem to have enjoyed the success of the TBM conjunctive and TBM disjunctive rules and they never matched the extent of combination rules that are available in possibility theory. This thesis reports yet another attempt at discovering useful alternative combination rules for belief functions, in the context of the TBM.

Contributions

Two main contributions are exposed in this thesis. The first one consists in the introduction of infinite families of conjunctive and disjunctive combination rules, mimicking thus the situation in possibility theory. The second one is a set of results making a purely formal and infinite family of rules, discovered by Smets in 1997 and called the α-junctions [87], of practical interest. The next two sections introduce these contributions in more details.

Infinite Families of Rules Based on Weight Functions

A limitation, which applies to both Dempster's rule and the TBM conjunctive rule, is the requirement that the items of evidence combined be distinct, or in other words, that the information sources be independent. Some authors [7, 21, 36, 56] have attempted to address this issue. However, those proposals are either restricted to particular classes of belief functions or do not possess desirable properties such as associativity. Recently, Denœux [16, 18] proposed a rule, called the cautious conjunctive rule (or cautious rule for short), for the combination of nondistinct bodies of evidence. The term cautious is reminiscent of the derivation of the rule, which is based on the least commitment principle (LCP) [83]. The LCP stipulates that one should never give more beliefs than justified by the available information, hence it promotes a cautious attitude. The cautious rule is based on the conjunctive weight function [85], which is an equivalent representation of a nondogmatic belief function arising from its conjunctive canonical decomposition. The TBM conjunctive rule can also be expressed using the conjunctive weight function, which makes it interesting to study rules based on this rarely exploited function.

One of the main differences between the cautious rule and the TBM conjunctive rule is that the former has no neutral element, whereas the latter admits the vacuous belief function as neutral element. This last property is quite natural for a conjunctive operator, as the vacuous belief function encodes ignorance. Hence, rules based on the conjunctive weight function and that admit the vacuous belief function as neutral element are of particular interest. The first important result presented in this thesis is that, among those rules, the TBM conjunctive rule is the least committed one. This can be seen as a new formal justification of the TBM conjunctive rule as a rule that respects a central principle of the TBM. A counterpart to this result is also obtained for the TBM disjunctive rule using the disjunctive canonical decomposition of a belief function and its associated disjunctive weight function [18].

In addition to offering a new justification for the TBM conjunctive rule, this result is essential for the problem of introducing flexibility for the combination of belief functions. Indeed, it is useful to define a natural generalization of the TBM conjunctive rule: this rule can be seen as a member of an infinite family of conjunctive combination rules based on uninorms [108] on $(0, +\infty]$ having one as neutral element. As will be seen, the TBM conjunctive rule has also a special position in this family: it is the least committed element. Interestingly, similar facts are shown to hold for the cautious rule: it belongs to an infinite family of conjunctive combination rules based on t-norms [49] on $(0, +\infty]$, and it is the least committed element of this family. Counterparts to these results are also obtained for disjunctive combinations. The introduction of those infinite families of combination rules is interesting since it allows us to shed some new light on the fundamentally different behaviors of the cautious and TBM conjunctive rules, by putting them in a broader perspective. However, most importantly, it shows that the TBM is not poorer than possibility theory in terms of conjunctive and disjunctive operations. Besides the theoretical importance of this discovery of infinite families of combination rules, it is demonstrated that these rules have at least one practical use: in some classification applications, they lead to improved performances.

Computational aspects of the uninorm-based conjunctive combination rules in problems involving multiple variables, are also investigated, by studying whether these rules fit the valuation algebra framework [51]. This abstract framework is useful for many different AI formalisms. In particular, it can be used to manage efficiently information represented by belief functions defined on product spaces, if the belief functions are combined using the TBM conjunctive rule. It is shown that, despite the numerous properties shared by the TBM conjunctive rule and the family of conjunctive combination rules based on uninorms, the TBM conjunctive rule is the only rule in this family that satisfies an axiom of the valuation algebra framework [51]. The consequences of this result are twofold. On the one hand, this singular property of the TBM conjunctive rule strengthens the fact that this rule has a special position in this family of combination rules and may thus be seen as yet another argument in favor of this rule. On the other hand, it may be seen as a restriction to the breadth of problems that can be tackled by the combination rules based on uninorms, since these rules will be difficult to use in problems involving many variables. Finally, it is also shown that the cautious rule does not satisfy an axiom of the valuation algebra framework. Hence, it will also be difficult to use the cautious rule in applications involving a large number of variables.

α-Junctions

In [87], Smets introduced an infinite family of combination rules for belief functions, the so-called α-junctions or α-junctive rules. This family basically represents the set of associative, commutative and linear operators for belief functions with a neutral element. It includes as particular cases the TBM conjunctive rule, the TBM disjunctive rule, as well as the exclusive disjunctive rule and its negation [26, 87]. The exclusive disjunctive rule fits with the situation where exactly one source is assumed to tell the truth, without knowing which one, and the negation of the exclusive disjunctive rule is suitable when all or none of the sources are assumed to tell the truth or, equivalently, to be truthful [87]. The behavior of an α-junction is determined by a parameter α and the four special cases are recovered for particular values of α. For other values of this parameter, the α-junctions did not have an interpretation.

To our knowledge, this family of rules has never been exploited. This can be explained, at least in part, by the fact that these rules did not have an interpretation. However, the lack of interpretation of a rule is not a definite argument for not using it. Indeed, a rule with no interpretation may be useful for, e.g., a classification application, if this rule yields lower classification error rates than the other combination rules. This potential use of the α-junctions has not been considered in the literature. A possible explanation is that the mathematics involved in the computation of the combination by an α-junctive rule are rather hard to handle and difficult to implement.

In our search for alternatives to the TBM conjunctive and TBM disjunctive rules, we carefully reexamine in this thesis this never exploited, yet important contribution of the late Professor Philippe Smets to the theory of belief functions, and propose solutions to the interpretation and computation problems. Smets' mention of the existence of an α-junctive canonical decomposition [91] is also a motivation for this study since the results on combination rules mentioned in the previous section, rely on the (conjunctive and disjunctive) canonical decompositions of a belief functions. The three main findings reported in this thesis and related to the α-junctions are the following.

First, an interpretation for the α-junctions is proposed. It is shown that they correspond to a particular form of knowledge, determined by the parameter α, about the truthfulness of the sources. The α-junctions become thus suitable as flexible combination rules that allow one to take into account some particular knowledge about the sources. Second, several efficient and simple ways of computing a combination by an α-junction are laid bare, making the practical use of the α-junctions in applications possible. These new means are based on generalizations of mechanisms that can be used to compute the combinations by the TBM conjunctive and TBM disjunctive rules. In particular, the conditioning operation and the matrices that permit the easy computation of the commonality and implicability functions associated to a belief function [91], are generalized in the context of the α-junctions. Third, it is shown that an α-junctive canonical decomposition indeed exists. Although Smets already mentioned such a decomposition, we believe that the exposition of this result is worthwhile since finding this decomposition was not trivial. Indeed, it relies on the conjunctive and disjunctive canonical decompositions of a signed belief function [53], which is another result of this thesis. It is also proved that an α-junction actually amounts to the combination by the TBM conjunctive rule of signed belief functions. This last result is important since it sees this rule surfacing once again and because it leads to an expression of an α-junction in terms of so-called α-junctive weights that arise from the α-junctive canonical decomposition. As will be seen, this expression generalizes the expressions based on conjunctive and disjunctive weights of, respectively, the TBM conjunctive and TBM disjunctive rules.

Organization

This report is structured in three parts. The first part is dedicated to the presentation of the TBM and is divided into two chapters. Chapter 1 recalls fundamental concepts of the TBM such as belief functions, combination rules and the LCP. Chapter 2 focuses on the conjunctive and disjunctive weight functions that originate from the conjunctive and disjunctive canonical decompositions of a belief function. It also summarizes material on the cautious rule and its dual, the bold rule. The second part deals with new results related to rules based on the weight functions. This part is broken up into three chapters. Chapter 3 presents a new justification of the TBM conjunctive rule based on the LCP. Chapter 4 introduces four infinite families of rules based on generalized t-norms and uninorms. A singular property of the TBM conjunctive rule among rules based on uninorms is shown in Chapter 5. Eventually, Part III is concerned with the α-junctions. It is made of two chapters. Chapter 6 details new results on the interpretation and the computation of the α-junctions. The α-junctive decomposition of a belief function is unveiled by Chapter 7. The report ends with a general conclusion and some directions for future work.

Part I

The Transferable Belief Model

Fundamental Concepts

Summary

In this chapter, the main notions of the Transferable Belief Model (TBM) – a nonprobabilistic interpretation of Dempster-Shafer theory – are presented. This model is divided into two parts: the credal level, whose concern is reasoning under uncertainty, and the pignistic level, which deals with decision-making.

Our presentation starts with the credal level. At this level, beliefs are quantified using belief functions and combined using aggregation operators, called combination rules in the TBM. The most often encountered combination rules are recalled in this chapter. It is also explained how belief functions can be informationally compared, thus leading to partial orderings for belief functions. Those partial orderings are useful to make operational the least commitment principle of the TBM, which postulates that, given a set of belief functions compatible with a set of constraints, the most appropriate belief function is the least informative. The matrix notation, which is useful to greatly simplify the mathematics of belief function theory, is also described in this chapter.

We proceed then to the pignistic level. This level requires the transformation of a belief function to a probability measure. The pignistic transformation advocated by Smets is recalled. The plausibility transformation defended by Cobb and Shenoy is also presented.

Résumé

Dans ce chapitre, les notions principales du Modèle des Croyances Transférables (MCT) – une interprétation non probabiliste de la théorie de Dempster-Shafer – sont présentées. Ce modèle est divisé en deux parties : le niveau crédal, qui s'intéresse au raisonnement dans l'incertain, et le niveau pignistique, qui permet la prise de décision.

Notre présentation commence avec le niveau crédal. A ce niveau, les croyances sont quantifiées par des fonctions de croyance et combinées par des opérateurs d'agrégation, appelés règles de combinaison dans le MCT. Les règles de combinaison les plus fréquemment rencontrées sont rappelées dans ce chapitre. Il est également expliqué comment des fonctions de croyance peuvent être comparées par rapport à leur contenu informationnel, amenant ainsi à des ordres partiels sur l'ensemble des fonctions de croyance. Ces ordres partiels sont utiles car ils permettent d'appliquer le principe d'engagement minimal du MCT qui stipule que, étant donné un

ensemble de fonctions de croyance compatibles avec un ensemble de contraintes, la plus appropriée est la moins informative. La notation matricielle, qui est utile pour simplifier considérablement les mathématiques de la théorie des fonctions de croyance, est également décrite dans ce chapitre.

Le niveau pignistique est ensuite abordé. Ce niveau requiert la transformation d'une fonction de croyance en une mesure de probabilité. La transformation pignistique préconisée par Smets est rappelée. La transformation basée sur les plausibilités défendue par Cobb et Shenoy est aussi présentée.

1.1 Introduction

A core hypothesis underlying the TBM is that reasoning under uncertainty and decision-making are two different cognitive tasks, which can be handled at two distinct levels:

- the credal level, where beliefs are entertained and combined using belief functions;
- the pignistic level, where it is accepted that decisions are made according to the principle of maximization of expected utility, hence requiring the transformation of a belief function to a probability measure in order to compute those expectations.

The part of the credal level dealing with the representation of beliefs is called the *static* part, in opposition to the *dynamic* part, which handles the revision of beliefs in light of new information.

This chapter is organized as follows. First, the concepts pertaining to the static part of the credal level are detailed in Section 1.2. In particular, different representations of a belief function are defined. Then, Section 1.3 presents the dynamic part of the model: notions such as combination rules and the informational comparison of belief functions are reviewed in this section. A notation that helps to simplify the mathematics of belief function theory is introduced in Section 1.4. Eventually, the pignistic level is summarized in Section 1.5.

1.2 Credal Level - Static Part

Let $\Omega = \{\omega_1, ..., \omega_K\}$ denote a finite set of possible values of a variable $\boldsymbol{\omega}$; Ω is called the frame of discernment of $\boldsymbol{\omega}$. In the TBM, the state of belief of a rational agent Ag regarding the actual value ω_0 taken by $\boldsymbol{\omega}$ is represented by a basic belief assignment (BBA) m defined as a mapping from 2^Ω to $[0, 1]$ verifying $\sum_{A \subseteq \Omega} m(A) = 1$. The quantity $m(A)$, called *mass* of A, is interpreted as a fraction of a unit mass of belief that supports A (i.e., the hypothesis $\omega_0 \in A$) and that, due to a lack of information, cannot be allocated to any strict subset of A. Total ignorance is thus represented by the so called *vacuous* BBA, noted m_Ω and defined by $m(\Omega) = 1$, whereas full knowledge correspond to the case $m(\{\omega\}) = 1$, for some $\omega \in \Omega$. Subsets A of Ω such that $m(A) > 0$ are called focal sets of m. A BBA m is said to be:

- normal if $\emptyset$ is not a focal set;
- subnormal if $\emptyset$ is a focal set;
- dogmatic if Ω is not a focal set;
- categorical if it has only one focal set;
- Bayesian if its focal sets are singletons;
- simple if it has at most two focal sets and, if it has two, Ω is one of those.

The normality condition $m(\emptyset) = 0$ is not required by the TBM. Indeed, under the so-called open-world assumption [82], one may interpret the mass $m(\emptyset)$ as quantifying the agent's belief that $\omega_0 \notin \Omega$. Let us note that the quantity $m(\emptyset)$ may also be interpreted as the amount of conflict after combining several information sources; this interpretation will be discussed in the next section. A subnormal BBA m can be transformed into a normal BBA m^* by the normalization operation defined as follows:

$$m^*(A) = \begin{cases} k \cdot m(A) & \text{if } A \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

for all $A \subseteq \Omega$, with $k = (1 - m(\emptyset))^{-1}$.

Several set functions, which are in one-to-one correspondence with m, can be defined [77]. Two such functions are the belief function *bel* and plausibility function *pl* defined, respectively, as:

$$bel\,(A) = \sum_{\emptyset \neq B \subseteq A} m\,(B)\,,$$

and

$$pl\,(A) = \sum_{B \cap A \neq \emptyset} m\,(B) = bel\,(\Omega) - bel\,(\bar{A})\,,$$

for all $A \subseteq \Omega$. The quantity $bel(A)$ represents the total amount of justified and specific support committed by the agent to the proposition $\omega_0 \in A$ [96]: *justified* because only masses allocated to subsets of A are taken into consideration, and *specific* because $m(\emptyset)$ is not included, as $\emptyset$ is a subset of both A and $\overline{A}$, where $\overline{A}$ denotes the complement of A. The quantity $pl(A)$ measures to what extent one fails to believe in $\bar{A}$, i.e., to doubt A. Let us note that another interpretation can be given to $pl(A)$: it will be presented later since it requires some concepts not yet defined. We may end the presentation of the functions *bel* and *pl* with a remark related to their formal nature (Remark 1.1 below), which places those functions in a more general context. Material on this interesting theoretical aspect of belief function theory may be found in [50] and [39].

Remark 1.1. *A (normal) belief function bel is a Choquet capacity monotone of infinite order [8]. Its associated BBA m and plausibility function pl are, respectively, its Möbius transform [73] and its conjugate [39].*

Two other useful representations of a BBA m are the implicability and commonality functions, which are defined, respectively, as:

$$b\,(A) = \sum_{B \subseteq A} m\,(B)\,,$$

and

$$q\,(A) = \sum_{B \supseteq A} m\,(B)\,,$$

for all $A \subseteq \Omega$. The BBA m can be recovered from any of the functions bel, pl, b and q. In particular, we have:

$$m(A) = \sum_{B \supseteq A} (-1)^{|B|-|A|} q(B), \tag{1.2}$$

for all $A \subseteq \Omega$ and where $|A|$ denotes the cardinality of A. The functions q and b are mainly used in this thesis as mathematical tools. An interpretation of the commonality function will nonetheless be provided in the next section. Let us eventually note that since a BBA and its associated belief, plausibility, commonality and implicability functions are all in one-to-one correspondence, by abuse of language any one of them may sometimes be referred to by the term "belief function".

The negation (or complement) $\overline{m}$ of a BBA m is defined as the BBA verifying $\overline{m}(A) = m(\overline{A})$, $\forall A \subseteq \Omega$ [26]. $\overline{m}$ represents the BBA that would be induced if the agent knows that the source providing a BBA m is not telling the truth, i.e., is telling the false or, equivalently, is lying [87]. It can be shown that the implicability function $\overline{b}$ associated to $\overline{m}$ and the commonality function q associated to m are linked by the following relation:

$$\overline{b}(A) = q(\overline{A}), \quad \forall A \subseteq \Omega.$$

Knowledge about the reliability of a source of information is taken into account in the TBM through the discounting operation as follows. Suppose a source providing a BBA m. Let $1-\beta$, with $\beta \in [0,1]$, be the agent's degree of belief that the source is reliable. The agent's belief ${}^{\beta}m$ on Ω is then equal to [77, 83]:

$${}^{\beta}m(A) = \begin{cases} (1-\beta)m(A) & \text{if } A \neq \Omega, \\ \beta + (1-\beta)m(\Omega) & \text{if } A = \Omega. \end{cases} \tag{1.3}$$

We have presented at the beginning of this section some particular cases of BBAs. There exists another interesting situation, which allows one to build a bridge between possibility theory and the TBM. When its focal sets are nested, a BBA m is said to be consonant, and its associated plausibility function is a possibility measure [109]: it verifies $pl\,(A \cup B) = pl\,(A) \vee pl\,(B)$, for all $A, B \subseteq \Omega$, where $\vee$ denotes the maximum operator [77]. Consequently, a consonant BBA uniquely defines a possibility measure. The corresponding possibility distribution π is then given by

$$\pi(\omega) = pl(\{\omega\}) = q(\{\omega\}), \quad \forall \omega \in \Omega.$$

Noticing that we have $pl(A) = \max_{\omega \in A} \pi(\omega)$ for a consonant BBA m, one can reconstruct m from π as follows. Let us note $\pi_k = \pi(\omega_k)$ and let us assume that the elements of $\Omega = \{\omega_1, ..., \omega_K\}$ have been arranged in decreasing order of plausibility, i.e., we have $1 \geq \pi_1 \geq \pi_2 \geq ... \geq \pi_K \geq \pi_{K+1} = 0$. m can then be computed as [24]

$$m(A) = \begin{cases} 1-\pi_1 & \text{if } A = \emptyset, \\ \pi_k - \pi_{k+1} & \text{if } A = \{\omega_1, ..., \omega_k\}, 1 \leq k < K, \\ \pi_K & \text{if } A = \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

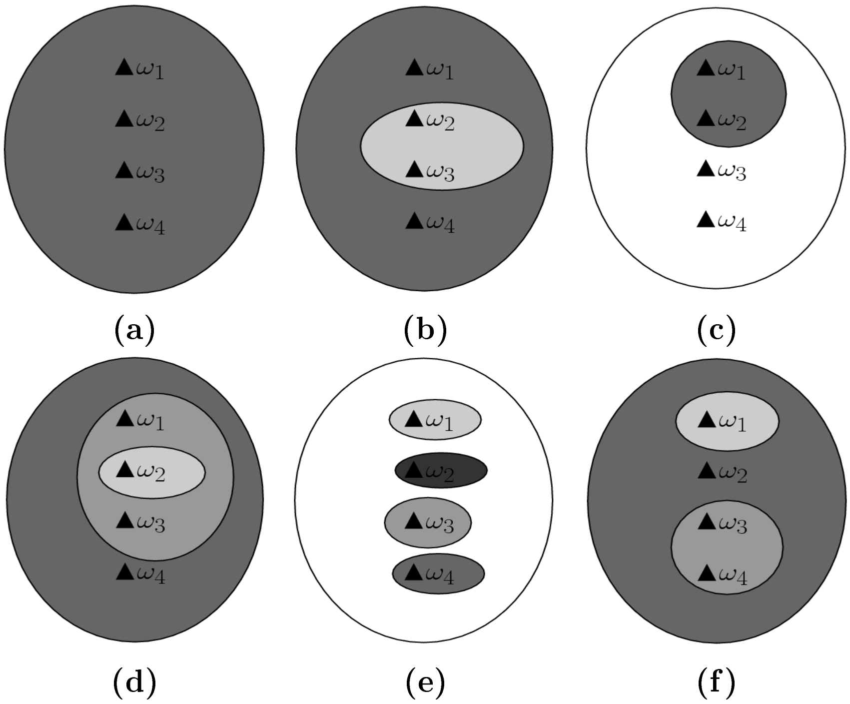

Figure 1.1: Different types of BBA
(a) vacuous, (b) simple, (c) categorical, (d) consonant, (e) Bayesian, (f) arbitrary. A white area indicates a null mass while a shaded area indicates a non-null mass.

Figure 1.1 illustrates the shapes of the different special cases of BBA that we have defined.

To conclude this section, we may note that, although not considered in this thesis, the TBM has been extended to continuous frames of discernment, fuzzy focal sets and fuzzy-valued masses. The reader is referred to [1, 15, 92] for clear presentations of those topics and for further references.

1.3 Credal Level - Dynamic Part

1.3.1 Combination rules

The beliefs represented by BBAs can be aggregated, at the credal level, using appropriate operators. Those operators are called combination rules in the TBM. Several combination rules have been proposed in the literature (see [76, 95] for recent surveys). We review in this section the most often encountered ones for the combination of heterogeneous[1] sources.

[1] In some problems, the set of sources can be viewed as a single source producing different inputs, in which case averaging operations are justified [31]. Such a situation is not investigated in this thesis and thus averaging operations are not covered in this section.

The TBM conjunctive rule

The TBM conjunctive rule is noted $\textcircled{\cap}$. It is defined as follows. Let m_1 and m_2 be two BBAs, and let $m_1 \textcircled{\cap} m_2 = m_{1\textcircled{\cap}2}$ denote the result[2] of their combination by $\textcircled{\cap}$. We have:

$$m_{1\textcircled{\cap}2}(A) = \sum_{B\cap C=A} m_1(B)\, m_2(C), \quad \forall A \subseteq \Omega. \tag{1.4}$$

This rule is appropriate when the sources that have induced m_1 and m_2, are known to tell the truth and to be independent, meaning that those sources are assumed to provide distinct, non overlapping pieces of evidence. This rule can be justified in several ways, which are presented in Chapter 3.

The TBM conjunctive rule is commutative, associative and admit a unique neutral element: the vacuous BBA. Let $\mathcal{M}$ be the set of BBAs, $(\mathcal{M}, \textcircled{\cap})$ is thus a commutative monoid. We may remark that reference [43] studies further properties of this monoid, called Dempster semigroup in [43], when $\mathcal{M}$ is the set of BBAs defined on binary frames of discernment.

The TBM conjunctive rule has a simple expression in terms of commonality functions. We have:

$$q_{1\textcircled{\cap}2}(A) = q_1(A) \cdot q_2(A), \quad \forall A \subseteq \Omega. \tag{1.5}$$

The combination by $\textcircled{\cap}$ may yield $m_{1\textcircled{\cap}2}(\emptyset) > 0$, even if m_1 and m_2 are normal. The mass $m_{1\textcircled{\cap}2}(\emptyset)$ is then interpreted in the TBM as representing the amount of conflict between the pieces of evidence that have induced m_1 and m_2. If a high conflict is observed after combination, it is important to understand why it has occurred. In the TBM, $m_{1\textcircled{\cap}2}(\emptyset)$ has at least three origins [95]:

- either the frame Ω is not exhaustive, in which case $m_{1\textcircled{\cap}2}(\emptyset)$ is interpreted as quantifying the belief that $\omega_0 \notin \Omega$;
- or $m_{1\textcircled{\cap}2}(\emptyset)$ represents a belief that the sources do not report on the same object – such an information may be useful to cluster sources according to which object they report about (see, e.g., [2, 75]);
- or the assumption that the sources are telling the truth, is wrong, in which case combination rules that do not make such an assumption should be used.

Other rules

When it cannot be assumed that all the sources tell the truth, it may be assumed that at least one of them tells the truth, without knowing which one. In such a situation, and provided that the sources are independent, the TBM disjunctive rule [26, 83] is appropriate. The TBM disjunctive rule is noted $\textcircled{\cup}$. Let m_1 and m_2 be two distinct BBAs, and let $m_{1\textcircled{\cup}2}$ be the result of their combination by $\textcircled{\cup}$. We have:

$$m_{1\textcircled{\cup}2}(A) = \sum_{B\cup C=A} m_1(B)\, m_2(C), \quad \forall A \subseteq \Omega.$$

[2] In this thesis, the result of the combination of two BBAs m_1 and m_2 by a combination operator $\otimes$ will be denoted interchangeably by $m_1 \otimes m_2$ or $m_{1\otimes 2}$.

This rule has a simple expression in terms of implicability functions, which is the counterpart of (1.5):

$$b_{1\textcircled{\cup}2}(A) = b_1(A) \cdot b_2(A), \quad \forall A \subseteq \Omega.$$

The TBM disjunctive rule is commutative, associative and admits a unique neutral element: the BBA which assigns the total mass of belief to the empty set, i.e., $m(\emptyset) = 1$. This BBA, which we note $m_\emptyset$, is the negation of the neutral BBA m_Ω of the TBM conjunctive rule and is sometimes called the or-vacuous BBA [87]. Eventually, we can remark that $(\mathcal{M}, \textcircled{\cup})$ is a commutative monoid.

The dual nature of $\textcircled{\cap}$ and $\textcircled{\cup}$ becomes apparent when one notices that these operators are linked by De Morgan's laws [26]:

$$\begin{aligned} \overline{m_1 \textcircled{\cup} m_2} &= \overline{m_1} \textcircled{\cap} \overline{m_2} \\ \overline{m_1 \textcircled{\cap} m_2} &= \overline{m_1} \textcircled{\cup} \overline{m_2}. \end{aligned} \tag{1.6}$$

The TBM disjunctive rule is suitable to situations where it is known that at least one of the sources tells the truth, which may occur for instance when combining beliefs held on climate sensitivity by groups of experts [40]. We may note that in addition to such situations, the TBM disjunctive rule finds at least one other use: it allows a simple expression of the (contextual) discounting operation [59].

Besides the TBM disjunctive rule, other notable proposals are Yager's rule [107] and Dubois and Prade's rule [28] defined as follows. Let m_1 and m_2 be two BBAs. Further, let m_{12}^{Y} and m_{12}^{DP} denote the result of the applications of, respectively, Yager's rule and Dubois and Prade's rule. We have

$$m_{12}^{Y}(A) = \begin{cases} m_{1\textcircled{\cap}2}(A) & \forall A \subset \Omega, A \neq \emptyset, \\ m_{1\textcircled{\cap}2}(\Omega) + m_{1\textcircled{\cap}2}(\emptyset) & \text{if } A = \Omega, \\ 0 & \text{if } A = \emptyset, \end{cases}$$

and

$$m_{12}^{DP}(A) = \begin{cases} m_{1\textcircled{\cap}2}(A) + \sum_{B \cap C = \emptyset, B \cup C = A} m_1(B)\, m_2(C) & \forall A \subseteq \Omega, A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

Dubois and Prade's rule is a mix of a conjunctive and a disjunctive behavior: it fits with the situation where the sources are assumed to tell the truth when they are not in conflict, and at least one of the sources is right when a conflict occurs [28]. Yager's rule is similar in that it also assumes the sources to tell the truth when they are not in conflict. However, when a conflict occurs, Yager's rule is different because it assumes that the sources are not reliable. Let us eventually note that, in contrast to the TBM conjunctive and TBM disjunctive rules, Yager's rule and Dubois and Prade's rule are not associative.

The TBM conjunctive rule and its related concepts

When it is safe to assume that the sources are independent, telling the truth and reporting on the same object, and if it can be defended that the frame of discernment is exhaustive, then the normalization operation (1.1) can be used after the combination by the TBM conjunctive rule, to eliminate the conflict [95]. This amounts to combining the belief functions by *Dempster's rule* [11] – the rule proposed in Shafer's seminal book for the combination of belief functions. However, note that "Dempster's rule is not robust to inaccurate estimates of uncertainty values" [28], and thus the normalization procedure should be used with great care.

In the TBM, conditioning by $B \subseteq \Omega$ is equivalent to conjunctive combination with a categorical BBA m_B focused on B, i.e., $m_B(B) = 1$. The result is noted $m[B]$, with $m[B] = m \textcircled{\cap} m_B$. The conditional BBA $m[B]$ quantifies our belief on Ω, in a context where B holds. This operation is called the unnormalized Dempster's rule of conditioning. Its internal mechanism can be better understood using the following equivalent definition of $m[B]$:

$$m[B](A) = \begin{cases} \sum_{B \cap C = A} m(C) & \text{if } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases} \quad (1.7)$$

As can be seen with this other expression of the unnormalized Dempster's rule of conditioning, any mass that is initially given to $C \subseteq \Omega$ is transferred to $C \cap B$ when a given hypothesis B has been ascertained. We may note that this idea of transfer of mass is at the origin of the name Transferable Belief Model. The unnormalized Dempster's rule of conditioning is particularly useful to give interpretations to the plausibility and commonality functions. Indeed, we have $pl(A) = bel[A](A)$, hence $pl(A)$ is the maximum degree of belief that could potentially be assigned to A, if further evidence became available [96]. Similarly, we have $q(A) = m[A](A)$, hence $q(A)$ represents the "share of belief free to potentially support any proposition in the context where [...] A holds true" [34] or, equivalently, $q(A)$ is a "measure of unassigned belief in the context where [...] A holds true" [34]. Eventually, note that the combination by the TBM conjunctive rule $\textcircled{\cap}$ has a simple expression using the unnormalized Dempster's rule of conditioning. Indeed, let m_1 and m_2 be two BBAs. We have

$$m_{1 \textcircled{\cap} 2}(A) = \sum_{B \subseteq \Omega} m_1[B](A)\, m_2(B), \quad \forall A \subseteq \Omega. \quad (1.8)$$

As can easily be seen from (1.8), the conjunctive combination generalizes the conditioning operation: the conditioning by B is obtained when m_2 is a categorical BBA focused on B.

Another important concept related to the TBM conjunctive rule is that of decombination. Let us assume that $m_{1 \textcircled{\cap} 2}$ has been obtained by combining two BBAs m_1 and m_2, and then we learn that m_2 is in fact not supported by evidence and should be "removed" from $m_{1 \textcircled{\cap} 2}$. This operation is called decombination [85] or removal [78]. It is well defined if m_2 is nondogmatic. Let $\textcircled{\not\cap}$ denote this operator. We can write:

$$m_{1 \textcircled{\cap} 2} \textcircled{\not\cap} m_2 = m_1.$$

Let q_1 and q_2 be the commonality functions of two BBAs m_1 and m_2, the decombination is defined as follows:

$$q_{1 \textcircled{\not\cap} 2}(A) = \frac{q_1(A)}{q_2(A)}, \quad \forall A \subseteq \Omega. \quad (1.9)$$

Note that $q_2(A) > 0$ for all A as long as m_2 is nondogmatic. One must also be aware that the pointwise division of two commonality functions is not always a commonality function, hence the resulting function $m_{1 \oslash 2}$ may not be a BBA and its associated function $bel_{1 \oslash 2}$ may not a belief function. In this case, it is called a pseudo belief function [85] or signed belief function [53] (signed belief functions will be discussed at length in Chapter 7). We may further note that the inverse of Dempster's rule was first suggested for binary frames of discernment by Ginsberg in [38], and that it may easily be shown (see Appendix A) that the operator proposed by Ginsberg is just the equivalent to the operator $\oslash$ followed by normalization using (1.1).

1.3.2 Informational comparison of belief functions

The least commitment principle (LCP) of the TBM postulates that, given a set of BBAs compatible with a set of constraints, the most appropriate BBA is the least informative [83]. It is similar to the principle of minimal specificity in possibility theory [105]. Both ordinal and quantitative approaches can be used to make this principle operational; however, we will mainly use ordinal approaches in this thesis.

Several partial orderings, generalizing set inclusion, were proposed in [10, 26, 62, 106] for the informational comparison of belief functions. Their interpretations are discussed from a set-theoretical perspective in [26] and from the point of view of the TBM in [34]. They are defined as follows:

- pl-ordering: $m_1 \sqsubseteq_{pl} m_2$, iff $pl_1(A) \leq pl_2(A)$ for all $A \subseteq \Omega$;
- q-ordering: $m_1 \sqsubseteq_q m_2$, iff $q_1(A) \leq q_2(A)$ for all $A \subseteq \Omega$;
- s-ordering: $m_1 \sqsubseteq_s m_2$, iff there exists a square matrix $\mathbf{S}$ with general term $S(A,B)$, $A, B \subseteq \Omega$ verifying:

$$\sum_{B \subseteq \Omega} S(A,B) = 1, \qquad \forall A \subseteq \Omega,$$
$$S(A,B) > 0 \Rightarrow A \subseteq B, \qquad \forall A, B \subseteq \Omega,$$

  such that

$$m_1(A) = \sum_{B \subseteq \Omega} S(A,B)\, m_2(A), \quad \forall A \subseteq \Omega. \tag{1.10}$$

  The quantity $S(A,B)$ may be seen as the proportion of the mass $m_2(B)$ that is transferred to A. The matrix $\mathbf{S}$ is called a specialization matrix [48], and m_1 is said to be a *specialization* of m_2, or, equivalently, m_2 is said to be a *generalization* of m_1.

A BBA m_1 is said to be x-more committed than m_2, with $x \in \{pl, q, s\}$, if we have $m_1 \sqsubseteq_x m_2$. It was shown in [26] that those definitions are not equivalent: $m_1 \sqsubseteq_s m_2$ implies $m_1 \sqsubseteq_{pl} m_2$ and $m_1 \sqsubseteq_q m_2$, but the converse is not true. Furthermore, the orderings $\sqsubseteq_{pl}$ and $\sqsubseteq_q$ are not comparable. The vacuous BBA m_Ω is the unique greatest element for partial orderings $\sqsubseteq_x$ with $x \in \{s, q, pl\}$, i.e., we have $m \sqsubseteq_x m_\Omega$ for all m. Informally, this latter property means that all beliefs are more informative than ignorance, as they should be. Dubois and Prade show in [26] that when the BBAs are consonant, those three partial orderings come down to the possibilistic ordering of specificity on singletons [105].

We mentioned above that there is also a quantitative approach to the informational comparison of belief functions. As for the ordinal case, there is no consensus on a unique "measure of uncertainty" [50]. Indeed, some measures quantify the amount of nonspecificity of a belief function, whereas other render its conflict [50]. There are also so-called total measures, which are basically combinations of nonspecificity and conflict measures. We provide below an illustrative example of one of these measures of uncertainty, which was proposed by Smets [81] as a measure of the information content of a BBA. The interested reader may find further material on measures of uncertainty in [50].

Smets' information measure will be noted I. It is defined for all nondogmatic BBA m as

$$I(m) = -\sum_{A \subseteq \Omega} \ln(q(A)).$$

From the definition of I, we find that this measure is always nonnegative and that the vacuous BBA contains no information, i.e., $I(m_\Omega) = 0$. When a nondogmatic BBA m contains at least as much information as another BBA m', i.e., when $I(m) \geq I(m')$, we write $m \sqsubseteq_I m'$ [89].

An interesting property of this measure is the following. We have for any two nondogmatic BBAs m_1 and m_2:

$$I(m_1 \bigcirc\!\!\!\!\!\!\cap\, m_2) = I(m_1) + I(m_2). \tag{1.11}$$

This property means that the amount of information of the combination of two distinct BBAs is equal to the sum of the amount of information of those BBAs [104]. We may note that this property can be axiomatically justified [98]. Proposition 1.1 shows that a related property holds for the measure I.

Proposition 1.1. *For any two nondogmatic BBAs m_1 and m_2 we have:*

$$I(m_1 \bigcirc\!\!\!\!\!\!\not\cap\, m_2) = I(m_1) - I(m_2). \tag{1.12}$$

Proof.

$$\begin{aligned} I(m_1 \bigcirc\!\!\!\!\!\!\not\cap\, m_2) &= -\sum_{A \subseteq \Omega} \ln(q_1(A)/q_2(A)) \\ &= -\sum_{A \subseteq \Omega} (\ln(q_1(A)) - \ln(q_2(A))) \\ &= -\sum_{A \subseteq \Omega} \ln(q_1(A)) + \sum_{A \subseteq \Omega} \ln(q_2(A)) \\ &= I(m_1) - I(m_2). \end{aligned}$$

□

The next proposition shows that the partial ordering $\sqsubseteq_q$ implies the order $\sqsubseteq_I$, but that the converse is not true.

Proposition 1.2. *For any two nondogmatic BBAs m_1 and m_2, we have*

$$m_1 \sqsubseteq_q m_2 \Rightarrow m_1 \sqsubseteq_I m_2,$$

but $m_1 \sqsubseteq_I m_2$ does not imply $m_1 \sqsubseteq_q m_2$.

Proof. Let us first prove that we have

$$m_1 \sqsubseteq_q m_2 \Rightarrow m_1 \sqsubseteq_I m_2,$$

for any two nondogmatic BBAs m_1 and m_2.

From the definition of $\sqsubseteq_q$, we have $q_1(A) \leq q_2(A)$ for all $A \subseteq \Omega$, and from the definition of the commonality functions, we have $q_1(A), q_2(A) \in (0, 1]$ for all $A \subseteq \Omega$. It is then clear that $\prod_{A\subseteq\Omega} q_1(A) \in (0, 1]$, $\prod_{A\subseteq\Omega} q_2(A) \in (0, 1]$ and $\prod_{A\subseteq\Omega} q_1(A) \leq \prod_{A\subseteq\Omega} q_2(A)$ holds. Hence, we have $\ln(\prod_{A\subseteq\Omega} q_1(A)) \leq \ln(\prod_{A\subseteq\Omega} q_2(A))$, and thus $I(m_1) \geq I(m_2)$, or, equivalently, $m_1 \sqsubseteq_I m_2$.

We now show using a counterexample that $\sqsubset_I$ does not imply $\sqsubset_q$. Table 1.1 gives two nondogmatic BBAs m_1 and m_2 together with their associated commonality functions. As can easily be seen, the two BBAs are not comparable with respect to the q-ordering. However, we have

$$I(m_1) = -\left(2 \cdot \ln(1) + 4 \cdot \ln(0.5) + 2 \cdot \ln(0.8)\right) \approx 3.2189,$$

and

$$I(m_2) = -\left(2 \cdot \ln(1) + 4 \cdot \ln(0.6) + 2 \cdot \ln(0.7)\right) \approx 2.7567,$$

i.e., $m_1 \sqsubseteq_I m_2$.

Table 1.1: Two nondogmatic BBAs that are not comparable with respect to the q-ordering.

A	m_1	q_1	m_2	q_2
$\emptyset$	0	1	0	1
$\{a\}$	0	0.5	0	0.6
$\{b\}$	0.2	1	0.3	1
$\{a, b\}$	0	0.5	0	0.6
$\{c\}$	0	0.8	0	0.7
$\{a, c\}$	0	0.5	0	0.6
$\{b, c\}$	0.3	0.8	0.1	0.7
Ω	0.5	0.5	0.6	0.6

□

Let us eventually remark that the BBAs m_1 and m_2 of Table 1.1, which are consonant, also allow us to conclude that the $\sqsubset_I$ ordering does not come down to the possibilistic ordering of specificity on singletons, since we have $m_1 \sqsubseteq_I m_2$, $\pi_1(a) <$

$\pi_2(a)$ and $\pi_1(c) > \pi_2(c)$, with π_1 and π_2 the possibility distributions associated, respectively, to m_1 and m_2.

In summary, we thus have, for any two nondogmatic BBAs m_1 and m_2:

$$m_1 \sqsubseteq_s m_2 \Rightarrow \begin{cases} m_1 \sqsubseteq_{pl} m_2 \\ m_1 \sqsubseteq_q m_2 \Rightarrow m_1 \sqsubseteq_I m_2, \end{cases} \tag{1.13}$$

where all implications are strict.

1.3.3 Coarsening, refinement and product spaces

Granularity of the frame of discernment

As remarked by Shafer [77], the degree of granularity of the frame Ω is always, to some extent, a matter of convention, as any element $\omega \in \Omega$ representing a state of nature can always be split into several possibilities. Hence, one should study how a belief function defined on a frame may be expressed in a coarser, or conversely, in a finer frame.

Let Ω and Θ be two finite sets. A mapping $\rho : 2^\Theta \rightarrow 2^\Omega$ is called a refining if it verifies the two following properties:

1. The set $\{\rho(\{\theta\}), \theta \in \Theta\} \subseteq 2^\Omega$ is a partition of Ω.

2. For all $B \subseteq \Theta$, we have:
$$\rho(B) = \bigcup_{\theta \in B} \rho(\{\theta\}).$$

Ω is called a refinement of Θ, and Θ is called a coarsening of Ω. Figure 1.2 shows a coarsening $\Theta = \{\theta_1, \theta_2\}$ of a frame $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, defined by a refining $\rho(\{\theta_1\}) = \{\omega_1, \omega_2\}$, $\rho(\{\theta_2\}) = \{\omega_3, \omega_4\}$.

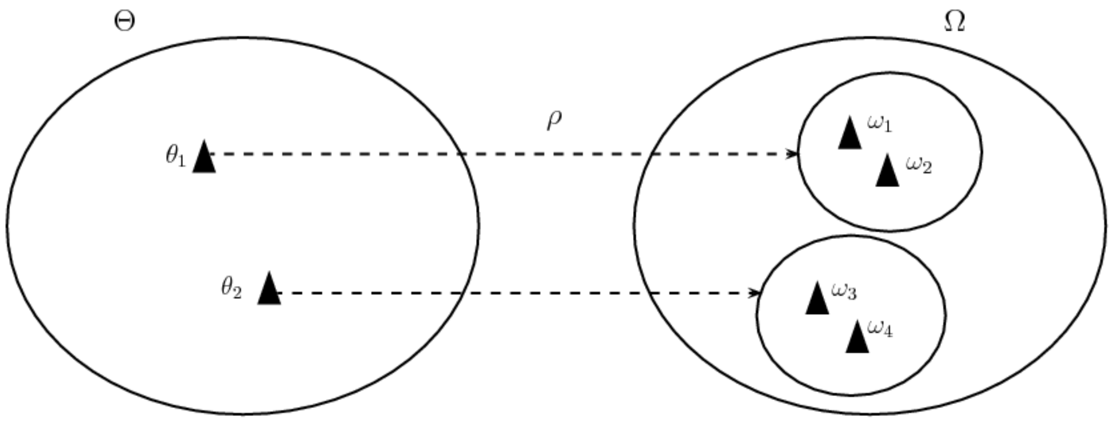

Figure 1.2: Coarsening $\Theta = \{\theta_1, \theta_2\}$ of a frame $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

Formally, defining a coarsening of a frame amounts to defining a partition of that frame. Note that defining the inverse operation, i.e., associating a subset of Θ to each subset of Ω is not easy since a refining $\rho : 2^\Theta \to 2^\Omega$ is not generally onto. This leads to the concepts of inner reduction and outer reduction [77]. We focus only on the outer reduction. It is defined as a mapping $\bar{\theta} : 2^\Omega \to 2^\Theta$ verifying:

$$\bar{\theta}(A) = \{\theta \in \Theta | \rho(\{\theta\}) \cap A \neq \emptyset\},$$

for all $A \subseteq \Omega$.

The mappings ρ and $\bar{\theta}$ can be extended from sets to BBA. A BBA m^Θ on Θ may be transformed into a BBA on a refinement Ω by transferring each mass $m^\Theta(B)$ for $B \subseteq \Theta$ to $A = \rho(B)$. This operation is called the vacuous extension of m^Θ to Ω. It is justified by the LCP [83] and it is noted $m^{\Theta\uparrow\Omega}$. Formally, we have:

$$m^{\Theta\uparrow\Omega}(A) = \begin{cases} m^\Theta(B) & \text{if } A = \rho(B) \text{ for some } B \subseteq \Theta, \\ 0 & \text{otherwise.} \end{cases}$$

A BBA m^Ω on Ω may be transformed into a BBA on a coarsening Θ through an operation called restriction (or outer reduction). It is noted $m^{\Omega\downarrow\Theta}$ and defined as

$$m^{\Omega\downarrow\Theta}(A) = \sum_{\{B \subseteq \Omega | \rho(A) \cap B \neq \emptyset\}} m^\Omega(B),$$

for all $A \subseteq \Theta$.

Operations on Product Spaces

Related to the previous notions of coarsening and refinement is the issue of dealing with BBAs defined on product spaces.

Let $m^{\Omega\times\Theta}$ denote a BBA defined on the Cartesian product $\Omega \times \Theta$ of the frames of two variables $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$. The marginal BBA $m^{\Omega\times\Theta\downarrow\Omega}$ is defined, for all $A \subseteq \Omega$, as

$$m^{\Omega\times\Theta\downarrow\Omega}(A) = \sum_{\{B \subseteq \Omega\times\Theta, (B\downarrow\Omega) = A\}} m^{\Omega\times\Theta}(B), \tag{1.14}$$

where $(B \downarrow \Omega)$ denotes the projection of B onto Ω, defined as

$$(B \downarrow \Omega) = \{\omega \in \Omega | \exists \theta \in \Theta, (\omega, \theta) \in B\}.$$

Marginalization may be seen as going from a frame $\Omega \times \Theta$ to a coarsening Ω.

Conversely, let m^Ω be a BBA defined on Ω. Its vacuous extension on $\Omega \times \Theta$ is defined as:

$$m^{\Omega\uparrow\Omega\times\Theta}(B) = \begin{cases} m^\Omega(A) & \text{if } B = A \times \Theta, \text{ for some } A \subseteq \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{1.15}$$

Note that the vacuous extension on $\Omega \times \Theta$ may be seen as a refining of Ω.

Given two BBAs m_1^Ω and m_2^Θ, their conjunctive combination on $\Omega \times \Theta$ can be obtained by combining their vacuous extensions on $\Omega \times \Theta$ using (1.15). Formally:

$$m_1^\Omega \textcircled{\cap} m_2^\Theta = m_1^{\Omega\uparrow\Omega\times\Theta} \textcircled{\cap} m_2^{\Theta\uparrow\Omega\times\Theta}. \tag{1.16}$$

A similar definition can be given for the combination by Dempster's rule.

Example 1.1 (Illustration of the computations involved by the application of the TBM conjunctive rule, and the marginalization and vacuous extension operations). *Consider two binary frames of discernment* $\Omega = \{\omega_1, \omega_2\}$ *and* $\Theta = \{\theta_1, \theta_2\}$. *Let* m_1^Ω *and* $m_2^{\Omega\times\Theta}$ *be two BBAs defined, respectively, as:*

$$\begin{aligned} m_1^\Omega(\{\omega_1\}) &= 0.6, \\ m_1^\Omega(\Omega) &= 0.4, \end{aligned}$$

and

$$\begin{aligned} m_2^{\Omega\times\Theta}(\{(\omega_1,\theta_2),(\omega_2,\theta_1)\}) &= 0.7, \\ m_2^{\Omega\times\Theta}(\Omega\times\Theta) &= 0.3. \end{aligned}$$

Given m_1^Ω *and* $m_2^{\Omega\times\Theta}$, *one may infer a BBA* m_{12}^Θ *on* Θ *using the following equation:*

$$m_{12}^\Theta = \left(m_1^{\Omega\uparrow\Omega\times\Theta} \text{\textcircled{$\cap$}} m_2^{\Omega\times\Theta}\right)^{\downarrow\Theta}. \tag{1.17}$$

The computation of the right side of (1.17) involves first a vacuous extension of m_1^Ω *to* $\Omega\times\Theta$*:*

$$\begin{aligned} m_1^{\Omega\uparrow\Omega\times\Theta}(\{(\omega_1,\theta_1),(\omega_1,\theta_2)\}) &= 0.6, \\ m_1^{\Omega\uparrow\Omega\times\Theta}(\Omega\times\Theta) &= 0.4. \end{aligned}$$

The BBA $m_1^{\Omega\uparrow\Omega\times\Theta}$ *may then be combined by* $\text{\textcircled{$\cap$}}$ *with* $m_2^{\Omega\times\Theta}$. *The result of this combination is noted* $m_{1\text{\textcircled{$\cap$}}2}^{\Omega\times\Theta}$. *We find:*

$$\begin{aligned} m_{1\text{\textcircled{$\cap$}}2}^{\Omega\times\Theta}(\{(\omega_1,\theta_1),(\omega_1,\theta_2)\}) &= 0.6\cdot 0.3 = 0.18, \\ m_{1\text{\textcircled{$\cap$}}2}^{\Omega\times\Theta}(\{(\omega_1,\theta_2)\}) &= 0.6\cdot 0.7 = 0.42, \\ m_{1\text{\textcircled{$\cap$}}2}^{\Omega\times\Theta}(\{(\omega_1,\theta_2),(\omega_2,\theta_1)\}) &= 0.4\cdot 0.7 = 0.28, \\ m_{1\text{\textcircled{$\cap$}}2}^{\Omega\times\Theta}(\Omega\times\Theta) &= 0.4\cdot 0.3 = 0.12. \end{aligned}$$

Finally, marginalizing $m_{1\text{\textcircled{$\cap$}}2}^{\Omega\times\Theta}$ *on* Θ, *i.e.,* $m_{1\text{\textcircled{$\cap$}}2}^{\Omega\times\Theta\downarrow\Theta}$, *yields* m_{12}^Θ*:*

$$\begin{aligned} m_{12}^\Theta(\{\theta_2\}) &= 0.42, \\ m_{12}^\Theta(\Theta) &= 0.58. \end{aligned}$$

Two other operations that have been defined for BBAs on product spaces are the *conditioning* operation, and its inverse operation called the *ballooning extension.* They are defined as follows. Let $m^{\Omega\times\Theta}$ denote a BBA on $\Omega\times\Theta$, and $m_B^{\Omega\times\Theta}$ the BBA on $\Omega\times\Theta$ with single focal set $\Omega\times B$ with $B\subseteq\Theta$, i.e., $m_B^{\Omega\times\Theta}(\Omega\times B) = 1$. The conditional BBA on Ω given $\boldsymbol{\theta}\in B$ is defined as:

$$m^\Omega[B] = \left(m^{\Omega\times\Theta} \text{\textcircled{$\cap$}} m_B^{\Omega\times\Theta}\right)^{\downarrow\Omega}. \tag{1.18}$$

Now, let $m^\Omega[B]$ denote the conditional BBA on Ω, given $\boldsymbol{\theta}\in B\subseteq\Theta$. The ballooning extension of $m^\Omega[B]$ on $\Omega\times\Theta$ is the least committed BBA, whose conditioning on B yields $m^\Omega[B]$ [83]. It is obtained as:

$$m^\Omega[B]^{\Uparrow\Omega\times\Theta}(C) = \begin{cases} m^\Omega[B](A) & \text{if } C = (A\times B)\cup(\Omega\times(\Theta\backslash B)), \text{ for some } A\subseteq\Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{1.19}$$

1.4 Matrix Notation

The matrix notation can be used to greatly simplify the mathematics of belief function theory. In [91], Smets proposed a review of the application of the matrix calculus to belief functions. This section is devoted to a summary of parts of [91] that are relevant to this thesis.

1.4.1 Belief functions as column vectors

A BBA m (and its associated functions bel, pl, q and b) defined on 2^{Ω} can be seen as a column vector of size $2^{|\Omega|}$. The elements of m can be ordered arbitrarily but the so-called binary order is particularly convenient. The binary order means that the first element of m is related to the empty set, the next to $\{a\}$, the next to $\{b\}$, the next to $\{a, b\}$, etc. Table 1.2 presents the vectors for $\Omega = \{a, b, c\}$. The ith element of the vector $\mathbf{m}$ corresponds to the set with elements indicated by 1 in the binary representation of $i - 1$. For instance, let $\Omega = \{a, b, c, d\}$. The first element ($i = 1$) of the vector $\mathbf{m}$ corresponds to the emptyset since the binary representation of $1 - 1$ is 0000. The twelfth element ($i = 12$) corresponds to $\{a, b, d\}$ since the binary representation of $12 - 1$ is 1011.

Table 1.2: Order of the elements of the vectors $\mathbf{m}$ and $\mathbf{f}$ ($\mathbf{f}$ is used generically to denote $\mathbf{bel}, \mathbf{pl}, \mathbf{q}, \mathbf{b}$) when $\Omega = \{a, b, c\}$.

Position	cba	$\mathbf{m}$	$\mathbf{f}$
1	000	$m(\emptyset)$	$f(\emptyset)$
2	001	$m(\{a\})$	$f(\{a\})$
3	010	$m(\{b\})$	$f(\{b\})$
4	011	$m(\{a, b\})$	$f(\{a, b\})$
5	100	$m(\{c\})$	$f(\{c\})$
6	101	$m(\{a, c\})$	$f(\{a, c\})$
7	110	$m(\{b, c\})$	$f(\{b, c\})$
8	111	$m(\{a, b, c\})$	$f(\{a, b, c\})$

We use the following conventions:

- By default, the length of vectors and matrices are $2^{|\Omega|}$, and vectors are column vectors.
- Matrices and vectors are written in bold type, and their elements in normal type, e.g., a matrix is noted $\mathbf{M}$ and the element on its ith row and jth column is noted $M(i, j)$. Sometimes a matrix will be defined by its general term, in this case we write $\mathbf{M} = [M(i, j)]$. For instance, if $M(i, j)$ is defined by $M(i, j) = 0, \forall i, j$, then $\mathbf{M}$ is a matrix, whose elements are zeros. We may also need to refer to the element on the row A and column B, with $A, B \subseteq \Omega$, of the matrix $\mathbf{M}$. Such an element is noted $M(A, B)$. The notations $M(i, j)$ and $M(A, B)$ are related. For instance, let $\Omega = \{a, b, c\}$ and $A = \{a, b\}$ and $B =$

$\{b, c\}$. The element $M(A, B)$ is, according to the binary notation, the element on the 4th row and the 7th column of the matrix $\mathbf{M}$, i.e., $M(A, B) = M(4, 7)$.

- $\mathbf{0}$ and $\mathbf{1}$ denote the vectors, whose components are, respectively, zeros and ones.
- $\mathbf{1}_A$ denotes the vector, whose components are zeros except the component corresponding to A (with A a subset of Ω) which equals 1.
- $\mathbf{Diag(v)}$ is the diagonal matrix, whose diagonal elements are the elements of vector $\mathbf{v}$.
- To simplify the notation, we write a for $\{a\}$, abc for $\{a, b, c\}$, etc.
- In vectors and matrices, dots replace zeros.
- $\mathbf{I}$ denotes the unitary matrix, i.e., its elements are zeros except those on the main diagonal that are ones.
- $\mathbf{J}$ denotes the square matrix, whose elements are zeros except on the secondary diagonal that are ones. The matrix $\mathbf{J}$ has two major properties when multiplied with a matrix $\mathbf{M}$ by the usual product of matrices: it inverses the order of the rows of $\mathbf{M}$ when placed before it (the first becoming the last, etc), and it inverses the order of the columns of $\mathbf{M}$ when placed behind it. We also have $\mathbf{J} \cdot \mathbf{J} = \mathbf{I}$, $\mathbf{J} = \mathbf{J}^{-1}$, $\overline{\mathbf{m}} = \mathbf{J} \cdot \mathbf{m}$, $\overline{\mathbf{b}} = \mathbf{J} \cdot \mathbf{q}$ and $\overline{\mathbf{q}} = \mathbf{J} \cdot \mathbf{b}$ [91].
- $\mathbf{Kron(A, B)}$ denotes the matrix resulting from the Kronecker product of a $m \times n$ matrix $\mathbf{A}$ with a $p \times q$ matrix $\mathbf{B}$. The matrix $\mathbf{Kron(A, B)}$ is defined by:

$$\mathbf{Kron(A, B)} = \begin{bmatrix} A(1,1)\mathbf{B} & \cdots & A(1,n)\mathbf{B} \\ \vdots & \ddots & \vdots \\ A(m,1)\mathbf{B} & \cdots & A(m,n)\mathbf{B} \end{bmatrix}.$$

  Its size is thus $mp \times nq$.

The transformations between the different representations of a belief function can be represented using the matrix notation. For instance, the classical relation $b(A) = \sum_{C \subseteq A} m(C)$ can be written

$$b(A) \;=\; \sum_{C \subseteq \Omega} B(A, C) m(C),$$

where $B(A, C) = 1$ iff $C \subseteq A$ and 0 otherwise. Letting $\mathbf{B} = [B(A, C)]$, $A, C \subseteq \Omega$, we have $\mathbf{b} = \mathbf{B} \cdot \mathbf{m}$ and $\mathbf{m} = \mathbf{B}^{-1} \cdot \mathbf{b}$ [91]. The matrix $\mathbf{B}$ is given in Table 1.3 when $\Omega = \{a, b, c\}$.

As can be seen from Table 1.3, the matrix $\mathbf{B}$ is built from the following building block:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Table 1.3: Matrix $\mathbf{B}$ when $\Omega = \{a, b, c\}$.

	$\emptyset$	a	b	ab	c	ac	bc	abc
$\emptyset$	1	.	.	.	.	.	.	.
a	1	1	.	.	.	.	.	.
b	1	.	1	.	.	.	.	.
ab	1	1	1	1	.	.	.	.
c	1	.	.	.	1	.	.	.
ac	1	1	.	.	1	1	.	.
bc	1	.	1	.	1		1	.
abc	1	1	1	1	1	1	1	1

This block is what the matrix $\mathbf{B}$ would be if $|\Omega| = 1$. In fact, going from a set Ω with i elements to a set with $i+1$ elements consists in multiplying the building block above by the matrix obtained with i elements using Kronecker multiplication, i.e., we have:

$$\mathbf{B}^{i+1} = \mathbf{Kron}\left(\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}, \quad \mathbf{B}^{i}\right), \mathbf{B}^{1} = 1. \tag{1.20}$$

Let us remark that this is a very simple way to obtain the matrix $\mathbf{B}$.

The matrix that allows the transformation from m to q is noted $\mathbf{Q}$. It can be obtained in a similar manner as the $\mathbf{B}$ matrix is obtained: one merely needs to replace $\mathbf{B}$ by $\mathbf{Q}$ in (1.20), and to change the building block to

$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}.$$

We have $\mathbf{q} = \mathbf{Q} \cdot \mathbf{m}$, $\mathbf{m} = \mathbf{Q}^{-1} \cdot \mathbf{q}$ and $\mathbf{Q} = \mathbf{J} \cdot \mathbf{B} \cdot \mathbf{J}$ [91]. Note that this last relation can easily be proved from the following relation between the building blocks of $\mathbf{B}$ and $\mathbf{Q}$:

$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \mathbf{J} \cdot \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \cdot \mathbf{J}.$$

1.4.2 Transformations of BBA into BBA

In this section, we present how the transformation of a BBA into another BBA, given a piece of evidence, can be expressed using the matrix notation.

Revision of a BBA

Definition 1.1. *A stochastic matrix* $\mathbf{M} = [M(i,j)]$ *is a square matrix with* $M(i,j) \geq 0$ *and* $\sum_i M(i,j) = 1, \forall j$.

Let $\mathcal{M}^{\Omega}$ be the set of BBAs defined on Ω. As shown by [91, Theorem 6.1], the set of matrices that map every element of $\mathcal{M}^{\Omega}$ into an element of $\mathcal{M}^{\Omega}$ is the set of stochastic matrices.

The *revision* of a BBA m_1 by a piece of evidence Ev can be represented by a stochastic matrix $\mathbf{M}(Ev, m_1)$ that transforms m_1 into $m_1[Ev]$:

$$\mathbf{m}_1[Ev] = \mathbf{M}(Ev, m_1) \cdot \mathbf{m}_1 \quad .$$

If the value of the matrix depends only on Ev and not on m_1 (in which case the pieces of evidence that induced m_1 and Ev are said 'distinct' [91]), we can write:

$$\mathbf{m}_1[Ev] = \mathbf{M}(Ev) \cdot \mathbf{m}_1 \quad . \tag{1.21}$$

It may happen that $\mathbf{M}(Ev)$ is a specialization matrix (see Section 1.3.2 for the definition of this type of matrix). In this case, $\mathbf{M}$ is noted $\mathbf{S}$.

The TBM conjunctive rule in the matrix notation

The combination by the TBM conjunctive rule and the TBM disjunctive rule can be expressed using the matrix notation. In the rest of this section, we focus on the combination by the rule $\textcircled{\cap}$.

The conjunctive revision of a BBA m_1 by a distinct piece of evidence inducing a BBA m_2 is achieved by a special kind of specialization matrix, called a Dempsterian specialization matrix [48] and noted $\mathbf{S}_{m_2}$. This matrix is defined as a function of m_2: its general term is $S_{m_2}(A, B) = m_2[B](A)$, $A, B \subseteq \Omega$. We have $\mathbf{m}_2 \textcircled{\cap} \mathbf{m}_1 = \mathbf{S}_{m_2} \cdot \mathbf{m}_1$.

Remember that we have

$$q_{1\textcircled{\cap}2} = q_1 \cdot q_2. \tag{1.22}$$

We now present a proof of this last relation, which will be useful in the next paragraph (the proof below is similar to the one given in [91]).

Proof. Smets [90] shows that the commonalities are the eigenvalues of the Dempsterian specialization matrix $\mathbf{S}_m$, and that the columns of $\mathbf{Q}^{-1}$ are the corresponding *right* eigenvectors or, equivalently, that the rows of $\mathbf{Q}$ are the corresponding *left* eigenvectors[3]. Hence, using the eigen decomposition theorem (see Appendix B), we obtain

$$\mathbf{S}_m = \mathbf{Q}^{-1} \cdot \mathbf{Diag}(\mathbf{q}) \cdot \mathbf{Q}, \tag{1.23}$$

and thus

$$\mathbf{S}_m \cdot \mathbf{Q}^{-1} = \mathbf{Q}^{-1} \cdot \mathbf{Diag}(\mathbf{q}),$$

or, equivalently,

$$\mathbf{Q} \cdot \mathbf{S}_m = \mathbf{Diag}(\mathbf{q}) \cdot \mathbf{Q}. \tag{1.24}$$

We can then show the following. From the eigendecomposition (1.24) of $\mathbf{S}_{m_{1\textcircled{\cap}2}}$, we have

$$\mathbf{Q} \cdot \mathbf{S}_{m_{1\textcircled{\cap}2}} \quad = \quad \mathbf{Diag}(\mathbf{q}_{1\textcircled{\cap}2}) \cdot \mathbf{Q}. \tag{1.25}$$

[3]Appendix B provides the definitions of right and left eigenvectors. Left eigenvectors are an unusual notion in linear algebra. They are used in this thesis in order to be in line in Chapter 6 with the article [87] of Smets on which this chapter will be based.

Multiplying both sides of (1.25) on the right by $\mathbf{1}_\Omega$ ($\mathbf{1}_\Omega$ is the vector corresponding to m_Ω), we obtain

$$\mathbf{Q} \cdot \mathbf{m}_{1 \textcircled{\cap} 2} = \mathbf{Diag}(\mathbf{q}_{1 \textcircled{\cap} 2}) \cdot \mathbf{1}, \qquad (1.26)$$

since m_Ω is a neutral element for $\textcircled{\cap}$ and since $Q(A, \Omega) = 1, \forall A \subseteq \Omega$. From (1.26), we obtain

$$\mathbf{Q} \cdot \mathbf{S}_{m_1} \cdot \mathbf{m}_2 = \mathbf{q}_{1 \textcircled{\cap} 2},$$

which, using the eigendecomposition (1.23) of $\mathbf{S}_{m_1}$, yields

$$\begin{aligned} \mathbf{Q} \cdot \mathbf{Q}^{-1} \cdot \mathbf{Diag}(\mathbf{q}_1) \cdot \mathbf{Q} \cdot \mathbf{m}_2 &= \mathbf{q}_{1 \textcircled{\cap} 2} \\ \mathbf{Diag}(\mathbf{q}_1) \cdot \mathbf{q}_2 &= \mathbf{q}_{1 \textcircled{\cap} 2}. \qquad (1.27) \end{aligned}$$

□

Eventually, we may note that the eigenvalues of $\mathbf{S}_m$ are on its diagonal since $\mathbf{S}_m$ is a triangular matrix. Hence, the commonalities are not only the eigenvalues of $\mathbf{S}_m$ but also its diagonal elements.

$2^{|\Omega|}!$ commonality-like functions and $2^{|\Omega|}!$ Q-like matrices

Equipped with this last proof, we can make the following remark, which does not seem important for the theory of belief functions, but that will nonetheless be central to the reasoning of Section 6.4.2.

Let σ be a permutation of the rows of $\mathbf{Q}$ and of $\mathbf{q}$, and let ${}^\sigma\mathbf{Q}$ and ${}^\sigma\mathbf{q}$ denote the results of this permutation. It is clear that we have ${}^\sigma\mathbf{Q} \cdot \mathbf{S}_m = \mathbf{Diag}({}^\sigma\mathbf{q}) \cdot {}^\sigma\mathbf{Q}$, since ${}^\sigma\mathbf{q}$ and ${}^\sigma\mathbf{Q}$ are just a reordering of the eigenvalues and corresponding left eigenvectors of $\mathbf{S}_m$. Besides, we have

$${}^\sigma\mathbf{q}_{1 \textcircled{\cap} 2} = \mathbf{Diag}({}^\sigma\mathbf{q}_1) \cdot {}^\sigma\mathbf{q}_2. \qquad (1.28)$$

To prove this last relation, simply replace $\mathbf{Q}$ by ${}^\sigma\mathbf{Q}$, $\mathbf{q}_{1 \textcircled{\cap} 2}$ by ${}^\sigma\mathbf{q}_{1 \textcircled{\cap} 2}$, $\mathbf{q}_1$ by ${}^\sigma\mathbf{q}_1$ and $\mathbf{q}_2$ by ${}^\sigma\mathbf{q}_2$ in the preceding proof.

Furthermore, it is also clear that there are $2^{|\Omega|}!$ such permutations σ of the rows of $\mathbf{Q}$ and $\mathbf{q}$ since the vector $\mathbf{q}$ is of length $2^{|\Omega|}$. Hence, there exists $2^{|\Omega|}!$ different functions that can be associated to a BBA m and such that the combination by $\textcircled{\cap}$ can be computed in a similar way as (1.22), i.e., by pointwise product. These functions will be referred to as “commonality-like” functions. Accordingly, there are also $2^{|\Omega|}!$ “**Q**-like” matrices. Example 1.2 illustrates the commonality-like functions.

Example 1.2. *Let $\Omega = \{a, b\}$. Let m_1 and m_2 be two BBAs defined by $m_1(a) = m_1(\Omega) = 0.5$ and $m_2(b) = 0.4, m_2(\Omega) = 0.6$. Their associated commonality functions are $q_1(\emptyset) = q_1(a) = 1, q_1(b) = q_1(\Omega) = 0.5$ and $q_2(\emptyset) = q_2(b) = 1, q_2(a) = q_2(\Omega) = 0.6$. The matrix $\mathbf{Q}$ for a frame of discernment Ω having two elements is given by table 1.4.*

Let σ be a permutation that permutes the first rows of $\mathbf{Q}$, $\mathbf{q}_1$ and $\mathbf{q}_2$ with their last rows, and let ${}^\sigma\mathbf{Q}$, ${}^\sigma\mathbf{q}_1$ and ${}^\sigma\mathbf{q}_2$ denote the results of those permutations. We thus have ${}^\sigma q_1(\Omega) = {}^\sigma q_1(a) = 1$, ${}^\sigma q_1(b) = {}^\sigma q_1(\emptyset) = 0.5$ and ${}^\sigma q_2(\Omega) = {}^\sigma q_2(b) = 1$, ${}^\sigma q_2(a) = {}^\sigma q_2(\emptyset) = 0.6$. The matrix ${}^\sigma\mathbf{Q}$ is given by Table 1.5.

We have $\mathbf{m}_{1 \textcircled{\cap} 2} = \mathbf{Q}^{-1} \cdot \mathbf{Diag}(\mathbf{q}_1) \cdot \mathbf{q}_2$. One can check that we also have $\mathbf{m}_{1 \textcircled{\cap} 2} = {}^\sigma\mathbf{Q}^{-1} \cdot \mathbf{Diag}({}^\sigma\mathbf{q}_1) \cdot {}^\sigma\mathbf{q}_2$.

Table 1.4: Matrix $\mathbf{Q}$ when $\Omega = \{a, b\}$.

	$\emptyset$	a	b	ab
$\emptyset$	1	1	1	1
a	.	1	.	1
b	.	.	1	1
ab	.	.	.	1

Table 1.5: The matrix ${}^{\sigma}\mathbf{Q}$.

	$\emptyset$	a	b	ab
$\emptyset$	.	.	.	1
a	.	1	.	1
b	.	.	1	1
ab	1	1	1	1

1.5 Pignistic Level

In the TBM, it is accepted that decisions are made by maximizing expected utilities (or, conversely, minimizing expected loss) [74]. Let us recall basic material on decision theory, and its associated concept of expected loss. In a decision context, we usually consider a finite set of actions $\mathcal{A} = \{a_1, .., a_K\}$. Typically, the action a_i is associated to the assignment to the singleton ω_i. We define by $\lambda(a_i, \omega_j)$ the cost of choosing a_i when the truth is actually ω_j. Then, the expected loss of an action a_i, relative to a probability measure P, is defined as:

$$R(a_i) = \sum_{\omega_j \in \Omega} \lambda(a_i, \omega_j) P(\omega_j).$$

The Bayesian decision rule consists in choosing the action a_i^* that minimizes the expected loss. Formally,

$$a_i^* = \arg\min_{a_i \in \mathcal{A}} R(a_i).$$

At the pignistic level, the beliefs held by the agent and represented by a belief function must then be transformed to a probability measure. In [93], Smets justifies from a linearity requirement the use of the so-called pignistic transformation which transforms a BBA m into a probability measure noted $BetP_m$ and defined as:

$$BetP_m(\{\omega_k\}) = \sum_{\{A \subseteq \Omega, \omega_k \in A\}} \frac{m(A)}{(1 - m(\emptyset))\,|A|}.$$

This transformation distributes equally each mass $m(A)$ to the singletons of A, for all $A \subseteq \Omega$.

Example 1.3 (Illustration of the pignistic transformation)**.** *Let us transform the BBA $m_{1 \odot 2}^{\Omega \times \Theta}$, computed in Example 1.1, into a probability measure using the pignistic* *transformation. We find:*

$$\begin{aligned}
BetP_{m_{1 \cap 2}^{\Omega\times\Theta}}\left(\{(\omega_1,\theta_1)\}\right) &= \frac{0.18}{2}+\frac{0.12}{4}=0.12,\\
BetP_{m_{1 \cap 2}^{\Omega\times\Theta}}\left(\{(\omega_1,\theta_2)\}\right) &= 0.42+\frac{0.18}{2}+\frac{0.28}{2}+\frac{0.12}{4}=0.68,\\
BetP_{m_{1 \cap 2}^{\Omega\times\Theta}}\left(\{(\omega_2,\theta_1)\}\right) &= \frac{0.28}{2}+\frac{0.12}{4}=0.17,\\
BetP_{m_{1 \cap 2}^{\Omega\times\Theta}}\left(\{(\omega_2,\theta_2)\}\right) &= \frac{0.12}{4}=0.03.
\end{aligned}$$

The pignistic transformation is not the unique method that has been proposed in the literature to transform a BBA m into a probability measure. In particular, Cobb and Shenoy [9] motivate the use of the plausibility transformation, which is noted PlP_m and defined as

$$PlP_m\left(\{\omega_k\}\right) \quad = \quad \kappa^{-1}pl\left(\{\omega_k\}\right),$$

with $\kappa=\sum_{j=1}^{K}pl\left(\{\omega_j\}\right)$. The principal argument of Cobb and Shenoy [9] for the plausibility transformation is that it is invariant with respect to the combination by $\bigcirc\!\!\!\!\cap$ [99], which is not the case of the pignistic transformation. We can remark that this transformation is also invariant with respect to the decombination by $\oslash\!\!\!\!\cap$ [63]. Proposition 1.3 formulates this latter property of the plausibility transformation using the decombination operator in probability theory, noted $\oslash$ and defined in [78] as follows. Let P_1 and P_2 be two probability measures. Assume that $P_2(\omega_k)\neq 0$ for all k, then $P_1\oslash P_2$ is the probability measure defined by:

$$P_1\oslash P_2\left(\{\omega_k\}\right)=\kappa^{-1}P_1\left(\{\omega_k\}\right)/P_2\left(\{\omega_k\}\right)\quad,\forall\omega_k\in\Omega$$

with $\kappa=\sum_{j=1}^{K}P_1\left(\{\omega_j\}\right)/P_2\left(\{\omega_j\}\right)$.

Proposition 1.3 (PlP is invariant with respect to $\oslash\!\!\!\!\cap$)**.** *Let m_1 and m_2 be two nondogmatic BBAs:*

$$PlP_{m_1 \oslash\!\!\!\!\cap m_2}=PlP_{m_1}\oslash PlP_{m_2}\quad .$$

Proof. For all $\omega_k\in\Omega$, let us denote $\alpha_k=pl_1\left(\{\omega_k\}\right)=q_1\left(\{\omega_k\}\right)$, and $\beta_k=pl_2\left(\{\omega_k\}\right)=q_2\left(\{\omega_k\}\right)$. From Equation (1.9) we have:

$$PlP_{m_1 \oslash\!\!\!\!\cap m_2}\left(\{\omega_k\}\right)=\frac{\frac{\alpha_k}{\beta_k}}{\sum_{i=1}^{K}\frac{\alpha_i}{\beta_i}}. \tag{1.29}$$

Besides,

$$PlP_{m_1} \oslash PlP_{m_2}(\{\omega_k\}) = \frac{\frac{\frac{\alpha_k}{\sum_{i=1}^K \alpha_i}}{\frac{\beta_k}{\sum_{i=1}^K \beta_i}}}{\sum_{j=1}^K \frac{\left(\frac{\alpha_j}{\sum_{i=1}^K \alpha_i}\right)}{\left(\frac{\beta_j}{\sum_{i=1}^K \beta_i}\right)}} = \frac{\frac{\alpha_k \cdot \left(\sum_{i=1}^K \beta_i\right)}{\beta_k \cdot \left(\sum_{i=1}^K \alpha_i\right)}}{\sum_{j=1}^K \frac{\alpha_j \cdot \left(\sum_{i=1}^K \beta_i\right)}{\beta_j \cdot \left(\sum_{i=1}^K \alpha_i\right)}} = \frac{\frac{\alpha_k}{\beta_k} \cdot \frac{\left(\sum_{i=1}^K \beta_i\right)}{\left(\sum_{i=1}^K \alpha_i\right)}}{\left(\sum_{j=1}^K \frac{\alpha_j}{\beta_j}\right) \cdot \frac{\left(\sum_{i=1}^K \beta_i\right)}{\left(\sum_{i=1}^K \alpha_i\right)}}. \quad (1.30)$$

(1.29) and (1.30) are equal. □

1.6 Conclusion

In this chapter, the fundamental concepts of the Transferable Belief Model have been exposed. This model distinguishes two cognitive tasks: reasoning and decision-making. Reasoning is handled at the credal level where beliefs of rational agents are represented by belief functions and manipulated using combination rules and the Least Commitment Principle. Decision-making is reserved to the pignistic level where beliefs must be transformed into a probability measure in order to be able to use classical decision theory.

The next chapter is devoted to the presentation of a lesser used notion of the TBM, which will be at the core of some contributions of this thesis.

Conjunctive and Disjunctive Canonical Decompositions

Summary

In this chapter, the conjunctive and disjunctive canonical decompositions of a belief function are introduced. These decompositions allow one to represent a complex belief state as the result of the combination, either by the TBM conjunctive rule or by the TBM disjunctive rule, of elementary and distinct states of belief. Furthermore, they yield two equivalent representations of a belief function, which are called the conjunctive and disjunctive weight functions. As explained in this chapter, these decompositions are interesting because they offer new ways to look at the informational comparison of belief functions, resulting in the definitions of two new partial orderings for the informational comparison of belief functions. Of interest is that these two partial orderings, called the w-ordering and v-ordering, have simple definitions using the conjunctive and disjunctive weight functions, respectively.

This chapter covers also the derivations of the cautious conjunctive rule and the bold disjunctive rule – two combination rules proposed recently by Denœux for the combination of nondistinct belief functions. As shown in this chapter, the least committed conjunctive rule, with respect to the w-ordering, is the cautious conjunctive rule. Conversely, the bold disjunctive rule is the most committed disjunctive rule, with respect to the v-ordering.

Résumé

Les décompositions canoniques conjonctive et disjonctive d'une fonction de croyance sont introduites dans ce chapitre. Ces décompositions permettent de considérer n'importe quelle fonction de croyance comme le résultat de la combinaison, par la règle conjonctive du MCT ou la règle disjonctive du MCT, d'éléments d'évidence simples. De plus, elles permettent d'introduire deux fonctions, appelées fonction de poids conjonctifs et fonction de poids disjonctifs, qui sont deux représentations équivalentes d'une fonction de croyance. Comme expliqué dans ce chapitre, ces décompositions sont intéressantes car elles offrent de nouvelles perspectives pour la comparaison du contenu informationel des fonctions de croyance. En particulier, deux nouveaux ordres partiels, appelés w-ordre et v-ordre, sont définis pour la comparaison de l'information contenue dans des fonctions de croyance. Ces deux ordres

partiels bénéficient notamment de définitions simples utilisant, respectivement, les fonctions de poids conjonctifs et disjonctifs.

Ce chapitre couvre également les dérivations de la règle conjonctive prudente et de la règle disjonctive hardie – deux règles de combinaison proposées récemment par Denœux pour la combinaison de fonctions de croyance non distinctes. Comme montré dans ce chapitre, la règle conjonctive la moins engagée, par rapport au w-ordre, est la règle conjonctive prudente. Inversement, la règle disjonctive hardie est la règle disjonctive la plus engagée, par rapport au v-ordre.

2.1 Introduction

In the preceding chapter, we have seen that there exist some equivalent representations of a BBA m. Here, we will introduce two other ones, called the conjunctive and disjunctive weight functions, which will be important for the rest of this work. Those functions arise from the so-called conjunctive and disjunctive canonical decompositions of a BBA [18, 85]. In essence, these decompositions allow one to represent a complex belief state as the result of the combination, either by $\bigcirc\!\!\!\!\cap$ or by $\bigcirc\!\!\!\!\cup$, of elementary and distinct states of belief [85]. One may note that, to our knowledge, it seems that an interest for those functions has been developed only recently [37, 40, 70, 71, 75], due mainly to the introduction of the *cautious rule* by Denœux [16, 18], whose derivation is recalled in this chapter.

This chapter is organized as follows. Sections 2.2 and 2.3 present, respectively, the conjunctive and disjunctive canonical decompositions of a belief functions. Then, in Section 2.4, two partial orderings for the informational comparison of belief functions are defined based on these decompositions. Finally, Section 2.5 summarizes the relevant parts of [18] related to the cautious rule and its dual, the bold rule, which are two combination operators based on the weight functions.

2.2 Conjunctive Weight Function

A simple BBA (SBBA) m such that $m(A) = 1 - w$ for some $A \neq \Omega$ and $m(\Omega) = w$ can be noted A^w. A categorical BBA can thus be noted A^0 for some $A \neq \Omega$ and the vacuous BBA can be noted A^1 for any $A \subset \Omega$. Let A^{w_1} and A^{w_2} be two SBBAs with the same focal element $A \neq \Omega$. Their combination by $\bigcirc\!\!\!\!\cap$ is the SBBA $A^{w_1 w_2}$.

Shafer [77, Chapter 4] named a BBA separable if it can be written as the combination by Dempster's rule, noted $\oplus$, of SBBAs. For a separable BBA m, one has thus:

$$m = \bigoplus_{\emptyset \neq A \subset \Omega} A^{w(A)},$$

with $w(A) \in [0,1]$ for all $A \subset \Omega$, $A \neq \emptyset$. This representation is unique if m is nondogmatic. In [18], Denœux extends the concept of separability to subnormal BBAs by calling a BBA m u-separable if we have

$$m = \bigcirc\!\!\!\!\!\cap_{A \subset \Omega} A^{w(A)}, \tag{2.1}$$

with $w(A) \in [0,1]$ for all $A \subset \Omega$. The decomposition (2.1) is unique if m is nondogmatic.

In [85], Smets proposed a solution to canonically decompose any nondogmatic BBA. This decomposition uses the concept of a generalized SBBA (GSBBA) which is defined as a function μ from 2^Ω to $\mathbb{R}$ by:

$$\begin{aligned} \mu(A) &= 1 - w, \\ \mu(\Omega) &= w, \\ \mu(B) &= 0 \quad \forall B \in 2^\Omega \setminus \{A, \Omega\}, \end{aligned}$$

for some $A \neq \Omega$ and some $w \in [0, +\infty)$. Extending the SBBA notation, any GSBBA can be noted A^w. When $w \leq 1$, μ is a SBBA. When $w > 1$, μ is no longer a BBA; Smets [85] called such a function an inverse SBBA. An interpretation of an inverse SBBA is proposed in Section 2.2.2.

Smets showed that any nondogmatic BBA can be uniquely represented as the conjunctive combination of GSBBAs:

$$m = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w(A)},$$

with $w(A) \in (0, +\infty)$ for all $A \subset \Omega$. The quantities $w(A)$ for each $A \subset \Omega$ are obtained as follows:

$$w(A) = \prod_{B \supseteq A} q(B)^{(-1)^{|B|-|A|+1}}. \tag{2.2}$$

The function $w : 2^\Omega \setminus \{\Omega\} \to (0, +\infty)$ is called the conjunctive weight function. It is another equivalent representation of a nondogmatic BBA m. Based on the remark that (2.2) can be equivalently written

$$\ln w(A) = -\sum_{B \supseteq A} (-1)^{|B|-|A|} \ln q(B), \quad \forall A \subset \Omega, \tag{2.3}$$

and since (2.3) is similar to (1.2), one may compute the function w from q using any procedure for transforming q to m (such as the Fast Möbius Transform [46] or matrix multiplication [91]).

There exists a convenient analytical formula to compute the weight function associated to a consonant BBA. Let m be a consonant BBA with associated possibility distribution π and associated conjunctive weight function w. Let us note $\pi_k = \pi(\omega_k)$ and let us assume that the elements of $\Omega = \{\omega_1, ..., \omega_K\}$ have been arranged in decreasing order of plausibility, i.e., we have $1 \geq \pi_1 \geq \pi_2 \geq ... \geq \pi_K > \pi_{K+1} = 0$. We have [18, Proposition 2] (note that m is nondogmatic since we have assumed $\pi_K > 0$):

$$w(A) = \begin{cases} \frac{\pi_{k+1}}{\pi_k} & \text{if } A = \{\omega_1, ..., \omega_k\}, 1 \leq k < K, \\ \pi_1 & \text{if } A = \emptyset \\ 1 & \text{otherwise.} \end{cases} \tag{2.4}$$

We may remark that consonant BBAs are u-separable, since they satisfy $w(A) \leq 1$ for all $A \subset \Omega$, as can easily be seen from (2.4) [18].

The methods described in the two preceding parapraghs are convenient ways to obtain the conjunctive weight function w associated to a nondogmatic BBA or to a consonant BBA. Note that the function w is available directly when the BBA m is built from accumulation of SBBAs, which is often the case in practice. It happens for instance in the evidential K-nearest neighbor classification rule [12], which will be summarized in Section 4.6.2.

Let us eventually reproduce a lemma related to the conjunctive weight function, which will be very useful to get results based on this function.

Lemma 2.1 (Lemma 1 of [18]). *Let m be a nondogmatic BBA with conjunctive weight function w, and let w' be a mapping from $2^\Omega \setminus \{\Omega\}$ to $(0, +\infty)$ such that $w'(A) \leq w(A)$ for all $A \subset \Omega$. Then w' is the conjunctive weight function of some BBA m'.*

2.2.1 Rules as pointwise combination of conjunctive weights

The TBM conjunctive rule and its inverse have simple expressions using the conjunctive weight function. Let m_1 and m_2 be two nondogmatic BBAs with conjunctive weight functions w_1 and w_2. We have:

$$\begin{aligned} m_{1 \circledcap 2} &= \left(\bigcircledcap_{A \subset \Omega} A^{w_1(A)}\right) \circledcap \left(\bigcircledcap_{A \subset \Omega} A^{w_2(A)}\right) \\ &= \bigcircledcap_{A \subset \Omega} A^{w_1(A) \cdot w_2(A)}, \end{aligned}$$

and

$$\begin{aligned} m_{1 \oslash 2} &= \left(\bigcircledcap_{A \subset \Omega} A^{w_1(A)}\right) \oslash \left(\bigcircledcap_{A \subset \Omega} A^{w_2(A)}\right) \\ &= \bigcircledcap_{A \subset \Omega} A^{w_1(A)/w_2(A)}. \end{aligned} \quad (2.5)$$

Hence we can write with obvious notations:

$$\begin{aligned} w_{1 \circledcap 2} &= w_1 \cdot w_2, \\ w_{1 \oslash 2} &= w_1 / w_2. \end{aligned}$$

Example 2.1. *Let $\Omega = \{a, b, c\}$ be a frame of discernment, and m_1 and m_2 be the BBAs with the associated commonality functions q_1 and q_2 shown in Table 2.1. Further, let $m_{1 \circledcap 2}$ be the BBA computed using (1.4) with associated commonality function $q_{1 \circledcap 2}$. It may be checked that the conjunctive weight function $w_{1 \circledcap 2}$ associated to $m_{1 \circledcap 2}$ and computed from the commonality function $q_{1 \circledcap 2}$ using (2.2) is indeed equal to the pointwise multiplication of the conjunctive weight functions w_1 and w_2 associated to m_1 and m_2. We may also remark that*

$$\begin{aligned} m_{1 \circledcap 2} &= \{b\}^{w_1(\{b\})} \circledcap \{a, b\}^{w_1(\{a,b\})} \circledcap \{c\}^{w_2(\{c\})} \\ &\quad \circledcap \{a, c\}^{w_2(\{a,c\})} \circledcap \{b, c\}^{w_1(\{b,c\}) \cdot w_2(\{b,c\})} \\ &= \{b\}^{9/5} \circledcap \{a, b\}^{1/3} \circledcap \{c\}^{9/8} \circledcap \{a, c\}^{2/3} \circledcap \{b, c\}^{2/9}. \end{aligned}$$

In other words, $m_{1 \circledcap 2}$ may be represented as the conjunctive combination of three SBBAs $\{a, b\}^{1/3}$ and $\{a, c\}^{2/3}$ and $\{b, c\}^{2/9}$, and two inverse SBBAs $\{b\}^{9/5}$ and $\{c\}^{9/8}$.

2.2.2 Inverse SBBA and latent belief structure

In [85], Smets proposed to interpret an inverse SBBA as representing a state of belief in which one has some reasons *not* to believe in A. This interpretation can be motivated as follows. Suppose that our state of belief is represented by a BBA m. Let us assume that m has been obtained by combining two SBBAs $A^{w'}$ and B^w, and then we learn that B^w is in fact not supported by evidence and should be removed from m; in other words we learn that we should give up our beliefs in B or, equivalently, that we should not believe in B any more. Removing B^w from m is done by decombining B^w from m:

$$m \oslash B^w = m \circledcap B^{1/w}, \quad (2.6)$$

Table 2.1: Combination of two BBAs using the TBM conjunctive rule.

A	m_1	q_1	m_2	q_2	$m_{1 \bigcirc\!\!\!\!\cap 2}$	$q_{1 \bigcirc\!\!\!\!\cap 2}$	$w_{1 \bigcirc\!\!\!\!\cap 2}$	w_1	w_2
$\emptyset$	0	1	0	1	0	1	1	1	1
$\{a\}$	0	0.6	0	0.75	0.1	0.45	1	1	1
$\{b\}$	0	1	0	0.75	0.1	0.75	9/5	9/5	1
$\{a, b\}$	0.4	0.6	0	0.5	0.2	0.3	1/3	1/3	1
$\{c\}$	0	0.6	0	1	0.1	0.6	9/8	1	9/8
$\{a, c\}$	0	0.2	0.25	0.75	0.05	0.15	2/3	1	2/3
$\{b, c\}$	0.4	0.6	0.25	0.75	0.35	0.45	2/9	1/3	2/3
Ω	0.2	0.2	0.5	0.5	0.1	0.1			

which allows us to interpret the inverse SSBA $B^{1/w}$ as representing a state of belief in which one has some reasons not to believe in B. The inverse SBBA $B^{1/w}$ can even be seen as corresponding to a situation where the agent has a "debt of belief" in B [85], since combining $B^{1/w}$ with B^w yields the vacuous BBA, i.e., some evidence has to be accumulated before one can start to believe in B.

Using (2.6), an interesting fact about a nondogmatic BBA m can be unveiled as follows. Let m be a nondogmatic BBA and w its associated conjunctive weight function. For each conjunctive weight $w(A)$, let us define the following quantities:

$$w^c(A) = 1 \wedge w(A), \tag{2.7}$$

and

$$w^d(A) = 1 \wedge \frac{1}{w(A)}, \tag{2.8}$$

where $\wedge$ denotes the minimum operator. It is clear that we have:

$$w(A) = \frac{w^c(A)}{w^d(A)}.$$

Consequently, we can write

$$\begin{aligned} m &= \bigcirc\!\!\!\!\!\cap_{A \subset \Omega} A^{w^c(A)/w^d(A)} \\ &= \left(\bigcirc\!\!\!\!\!\cap_{A \subset \Omega} A^{w^c(A)}\right) \oslash \left(\bigcirc\!\!\!\!\!\cap_{A \subset \Omega} A^{w^d(A)}\right) \\ &= m^c \oslash m^d, \end{aligned} \tag{2.9}$$

with $m^c = \bigcirc\!\!\!\!\!\cap_{A \subset \Omega} A^{w^c(A)}$ and $m^d = \bigcirc\!\!\!\!\!\cap_{A \subset \Omega} A^{w^d(A)}$. Any nondogmatic BBA m can thus be decomposed into two u-separable BBAs m^c and m^d called, respectively, the confidence and diffidence components of m. The pair (m^c, m^d) was referred to as a latent belief structure (LBS) by Smets [85], who interpreted the confidence component as representing good reasons to believe in various propositions $A \subseteq \Omega$, and the diffidence component as representing good reasons not to believe in the same propositions. The proposed interpretation for the diffidence component comes from the idea, developed above, that removal of beliefs corresponds to giving up of beliefs, hence to not believing any more. Note that the BBA m is recovered by removing m^d from m^c and is called the apparent belief structure of the LBS $\left(m^c, m^d\right)$. In Appendix C, we present some exploratory work that take further this concept of LBS. In particular, some notions of belief function theory such as combination rules, informational comparison, and transformation to a probability measure are extended to LBSs.

2.3 Disjunctive Weight Function

Using a similar reasoning as in Section 2.2, Denœux [18] showed that any subnormal BBA m can be uniquely represented as the disjunctive combination of *negative* GSBBAs. A negative GSBBA is the negation of a GSBBA, i.e., a function from 2^Ω to $\mathbb{R}$ assigning a mass $v \geq 0$ to $\emptyset$, a mass $1 - v$ to $A \neq \emptyset$, and a zero mass for all $B \in 2^\Omega \setminus \{A, \emptyset\}$; this function is noted A_v. For any subnormal BBA m, we have:

$$m = \bigcirc\!\!\!\!\cup_{A \neq \emptyset} A_{v(A)},$$

with $v(A) \in (0, +\infty)$ for all $A \neq \emptyset$. The quantities $v(A)$ for each $A \neq \emptyset$ may be obtained from b using a formula similar to (2.3) as follows

$$\ln v(A) = -\sum_{B \subseteq A} (-1)^{|A|-|B|} \ln b(B).$$

The function $v : 2^\Omega \setminus \{\emptyset\} \to (0, +\infty)$ is called the disjunctive weight function. This function is related to the conjunctive weight function $\overline{w}$ associated to the negation $\overline{m}$ of m by the equation:

$$v(A) = \overline{w}(\overline{A}), \quad \forall A \neq \emptyset.$$

The TBM disjunctive rule has a simple expression using the disjunctive weight function. Let m_1 and m_2 be two subnormal BBAs with disjunctive weight functions v_1 and v_2. We have:

$$m_{1 \bigcirc\!\!\!\!\cup 2} = \bigcirc\!\!\!\!\cup_{A \neq \emptyset} A_{v_1(A) \cdot v_2(A)},$$

and, equivalently, $v_{1 \bigcirc\!\!\!\!\cup 2} = v_1 v_2$.

Eventually, we may note that there exists a disjunctive counterpart to the concept of latent belief structure defined in the previous section: it is called a disjunctive LBS [18] and it is based on the inverse of the $\bigcirc\!\!\!\!\cup$ rule, much as the concept of LBS is based on the inverse of the $\bigcirc\!\!\!\!\cap$ rule.

2.4 Informational Comparison Based on the Canonical Decompositions

The idea of decomposing a complex belief state into elementary states of belief can be taken further. Indeed, as will be explained in this section, the canonical decompositions offer a new way to look at the informational comparison of belief functions, which result in the definitions of two new partial orderings based on the two equivalent representations w and v of a BBA m introduced in this chapter.

2.4.1 The w-ordering

In Section 1.3.2, the partial orderings $\sqsubseteq_x$, with $x \in \{s, q, pl, I\}$, were defined. All these partial orderings seem equally well justified and reasonable. Hence, in the absence of any definite argument to eliminate any of them, the choice of a particular partial ordering for a given problem may be delicate. Nonetheless, we may identify situations that make this choice simpler, due to the equivalence between some partial orderings; for instance, we have seen in Section 1.3.2 that the orderings $\sqsubseteq_x$, with $x \in \{s, q, pl\}$, become equivalent in the case of consonant BBAs. One may wonder whether there exist situations where all the partial orderings $\sqsubseteq_x$, with $x \in \{s, q, pl, I\}$, become equivalent. This is interesting since the question of the appropriate ordering to be used among these four becomes then irrelevant and, in that sense, the informational comparison becomes "unquestionable". In the remainder of this section, such situations are described and used to propose a definition for the informational comparison of generalized SBBAs. This definition is then used to explain the behavior of the so-called w-ordering introduced in [18].

Informational comparison of SBBAs

Let us first consider the informational comparison of two SBBAs A^w and $A^{w'}$, i.e., two SBBAs focused on the same subset. In this context, the following proposition holds.

Proposition 2.1. *We have, for all $A \subset \Omega$ and all $w, w' \in (0, 1]$:*

$$w \leq w' \Leftrightarrow A^w \sqsubseteq_x A^{w'}, \quad \forall x \in \{s, pl, q, I\}\,. \tag{2.10}$$

Proof. Let q and q' denote the commonality functions associated, respectively, to A^w and $A^{w'}$. We have

$$q(B) = \begin{cases} 1 & \text{if } B \subseteq A, \\ w & \text{otherwise,} \end{cases} \tag{2.11}$$

$$q'(B) = \begin{cases} 1 & \text{if } B \subseteq A, \\ w' & \text{otherwise.} \end{cases} \tag{2.12}$$

Let us show that $w \leq w' \Rightarrow A^w \sqsubseteq_x A^{w'}$ holds for all $x \in \{s, pl, q, I\}$. From (2.11) and (2.12), we have $q(A) \leq q'(A), \forall A \subseteq \Omega$ if $w \leq w'$. Hence, $w \leq w' \Rightarrow A^w \sqsubseteq_q A^{w'}$. From this latter implication and (1.13), we further obtain $w \leq w' \Rightarrow A^w \sqsubseteq_I A^{w'}$. Since A^w and $A^{w'}$ are consonant BBAs, we also have $w \leq w' \Rightarrow A^w \sqsubseteq_x A^{w'}$, with $x \in \{s, pl\}$, due to the equivalence between the partial orderings $\sqsubseteq_s$, $\sqsubseteq_{pl}$ and $\sqsubseteq_q$ in case of consonant BBAs.

Let us now show that $A^w \sqsubseteq_x A^{w'} \Rightarrow w \leq w'$ holds for all $x \in \{s, pl, q, I\}$. It may easily be seen from (2.11) and (2.12) that $A^w \sqsubseteq_q A^{w'}$, i.e., $q(A) \leq q'(A)$ for all $A \subseteq \Omega$, implies $w \leq w'$. Since A^w and $A^{w'}$ are consonant BBAs, $A^w \sqsubseteq_s A^{w'}$ and

$A^w \sqsubseteq_{pl} A^{w'}$ imply $A^w \sqsubseteq_q A^{w'}$ and thus imply $w \leq w'$. Eventually, we have

$$\begin{aligned} A^w &\sqsubseteq_I A^{w'} \\ \log(\prod_{A\subseteq\Omega} q(A)) &\leq \log(\prod_{A\subseteq\Omega} q'(A)) \\ w^n &\leq w'^n \end{aligned} \tag{2.13}$$

with n the number of sets $B \subseteq \Omega$ such that $B \not\subseteq A$. From (2.13), one obtains $w \leq w'$, which completes the proof. □

The consequences of Proposition 2.1 are twofold. First, this proposition shows that all the partial orderings become equivalent when comparing SBBAs focused on the same subset. Hence, in this particular setting, one does not need to specify a particular partial ordering and we may simply say that a SBBA is more committed (instead of x-more committed) or, equivalently, more "informed" [34] than another SBBA. Second, the equivalence in (2.10) means that, if we consider two SBBAs focused on the same subset $A \subset \Omega$, it is equivalent to say that A^w is more informed than $A^{w'}$ or that $w \leq w'$.

Informational comparison of inverse SBBAs

Let us now discuss the case of inverse SBBAs. For that, we need the following proposition.

Proposition 2.2. *We have, for all $A \subset \Omega$ and all $w, w' \in [1, +\infty)$:*

$$w \leq w' \Leftrightarrow A^{1/w'} \circledcap A^w \sqsubseteq_x A^{1/w'} \circledcap A^{w'}, \quad \forall x \in \{s, pl, q, I\}. \tag{2.14}$$

Proof. Equation (2.14) can be written

$$w \leq w' \Leftrightarrow A^{w/w'} \sqsubseteq_x A^1 = m_\Omega, \quad \forall x \in \{s, pl, q, I\}, \tag{2.15}$$

or, equivalently,

$$x \leq 1 \Leftrightarrow A^x \sqsubseteq_x m_\Omega, \quad \forall x \in \{s, pl, q, I\}. \tag{2.16}$$

with $x = w/w'$. One can easily show that (2.16) holds using Proposition 2.1. □

This proposition allows us to motivate a definition of the informational comparison of inverse SBBAs as follows.

Suppose the agent is in a belief state represented by the SBBA $A^{1/w'}$, with $w' \in [1, +\infty)$, and that she receives a new piece of information represented by the inverse SBBA A^w, with $w \in [1, +\infty)$. Suppose further that after adding A^w to $A^{1/w'}$, the agent is in a state of belief that is at least as x-committed, with $x \in \{s, q, pl, I\}$, as if she had added $A^{w'}$. This means that the inverse SBBA A^w must clearly represent at least as much information as the inverse SBBA $A^{w'}$. Proposition 2.2 then basically shows that it is equivalent to say an inverse SBBA A^w is at least as informed as an inverse BBA $A^{w'}$ or that $w \leq w'$.

Informational comparison of SBBAs with inverse SBBAs

Proposition 2.3 unveils another situation where the partial orderings $\sqsubseteq_x$, with $x \in \{s, q, pl, I\}$, become equivalent.

Proposition 2.3. *We have, for all $A \subset \Omega$ and all $w \in (0, 1], w' \in [1, +\infty)$:*

$$A^{1/w'} \circledcap A^{w} \sqsubseteq_x A^{1/w'} \circledcap A^{w'}, \quad \forall x \in \{s, pl, q, I\}. \tag{2.17}$$

Proof. Equation (2.17) can be written

$$A^{w/w'} \sqsubseteq_x A^1 = m_\Omega, \quad \forall x \in \{s, pl, q, I\}. \tag{2.18}$$

As $w/w' \leq 1$, $A^{w/w'}$ is a SBBA. Since the vacuous BBA m_Ω is the unique greatest for partial orderings $\sqsubseteq_x$, with $x \in \{s, q, pl, I\}$, Equation (2.18) holds. □

From this latter proposition and using a similar reasoning to the one developed in the previous paragraph, one gets to the conclusion that a SBBA A^w must be at least as committed as an inverse SBBA $A^{w'}$.

Informational comparison based on the conjunctive canonical decomposition

Based on the three preceding paragraphs, the informational comparison of two simple items of evidence supporting the same subset A of the frame of discernment and represented by two generalized SBBA A^w and $A^{w'}$, with $w, w' \in (0, +\infty)$, may be reasonably defined as follows: A^w is said to be at least as informed as $A^{w'}$ iff $w \leq w'$.

Equipped with this definition of the information content of simple items of evidence, it becomes possible, using the conjunctive canonical decomposition, to informationally compare BBAs according to the information content of their underlying simple items of evidence. In particular, consider the situation where we have two nondogmatic BBAs m_1 and m_2 such that each simple item of evidence underlying m_1 is at least as committed as the corresponding item of evidence underlying m_2. In this situation, it is clear that m_1 must be at least as committed as m_2, hence the following definition.

Definition 2.1 (w-ordering [18]). *Given two nondogmatic BBAs m_1 and m_2, m_1 is said to be w-more committed than m_2 iff $w_1(A) \leq w_2(A)$, for all $A \subset \Omega$.*

When m_1 is w-more committed than m_2, we write $m_1 \sqsubseteq_w m_2$. We present below properties of this new partial ordering.

In [18], Denœux proves that the w-ordering is stronger than the s-ordering. Hence, we have, for any two nondogmatic BBAs m_1 and m_2:

$$m_1 \sqsubseteq_w m_2 \Rightarrow m_1 \sqsubseteq_s m_2 \Rightarrow \begin{cases} m_1 \sqsubseteq_{pl} m_2 \\ m_1 \sqsubseteq_q m_2 \Rightarrow m_1 \sqsubseteq_I m_2, \end{cases} \tag{2.19}$$

where all implications are strict.

The relation $\sqsubseteq_w$ may be seen as generalizing set inclusion [18, Proposition 3], much as x-orderings with $x \in \{s, q, pl\}$ do. Furthermore, letting $\mathcal{M}_{nd}$ be the set of nondogmatic BBAs, $(\mathcal{M}_{nd}, \circledcap, \sqsubseteq_w)$ is a partially ordered commutative monoid, i.e., for all m_1, m_2 and m_3, $m_1 \sqsubseteq_w m_2$ implies $m_1 \circledcap m_3 \sqsubseteq_w m_2 \circledcap m_3$.

In contrast to the partial orderings $\sqsubseteq_x$, with $x \in \{s, q, pl, I\}$, the vacuous BBA m_Ω (with conjunctive weight function $w_\Omega(A) = 1$, for all $A \subset \Omega$) is only a maximal element of the $\sqsubseteq_w$ relation, i.e., we have $m_\Omega \sqsubseteq_w m \Rightarrow m = m_\Omega$, for all nondogmatic BBA m. All u-separable BBAs are w-less committed than m_Ω, whereas non u-separable BBAs are not comparable with m_Ω according to the w-ordering. As can be seen from (2.19), this ordering is also stronger than all the other orderings; a behavior that one might expect as this ordering, by looking into the entrails of a belief function, seems much more demanding than the other ones. The w-ordering is still stronger than the classical ones in the case of consonant BBAs, i.e., the possibilistic ordering does not imply the w-ordering in the case of consonant BBAs, a startling fact at first sight. However, the lack of recovery of the possibilistic ordering should not hurt our intuition, because one should remember that the w-ordering is meaningful only for comparing BBAs in terms of underlying pieces of information. Keeping that in mind, we may remark that the w-ordering has actually an interesting behavior when comparing consonant BBAs. Indeed, let m_1 and m_2 be two consonant and nondogmatic BBAs such that $m_2 \sqsubseteq_{pl} m_1$. Then, in order to have $m_2 \sqsubseteq_w m_1$ in this particular setting of consonant BBAs, we must also have $A^{w_2(A)} \sqsubseteq_{pl} A^{w_1(A)}$ for all $A \subset \Omega$, i.e., in order to have $m_2 \sqsubseteq_w m_1$, the BBA m_2 must not only be more pl-specific than m_1 but its underlying SBBAs must also be more pl-specific than those of m_1.

2.4.2 The v-ordering

Using a symmetrical reasoning to the one of Section 2.4.1, another partial ordering, which is the disjunctive counterpart of $\sqsubseteq_w$, can be defined. This ordering, called the v-ordering, is based on the disjunctive weight function. It is defined as follows [18]: given two subnormal BBAs m_1 and m_2, $m_1 \sqsubseteq_v m_2$ iff $v_1(A) \geq v_2(A)$ for all $A \neq \emptyset$. Let $\mathcal{M}_s$ be the set of subnormal BBAs. We can note that $(\mathcal{M}_s, \circledcup, \sqsubseteq_v)$ is a partially ordered commutative monoid, with $m_\emptyset$ as neutral element. Furthermore, we have, for any two subnormal BBAs m_1 and m_2 (cf. Section 4.2 of [18]):

$$m_1 \sqsubseteq_v m_2 \Rightarrow m_1 \sqsubseteq_s m_2 \Rightarrow \begin{cases} m_1 \sqsubseteq_{pl} m_2 \\ m_1 \sqsubseteq_q m_2 \Rightarrow m_1 \sqsubseteq_I m_2, \end{cases} \tag{2.20}$$

where all implications are strict.

In the following section, we will review how the $\sqsubseteq_w$ and $\sqsubseteq_v$ orderings can be used to derive two rules of combination.

2.5 Two Idempotent Rules Based on the Weight Functions

We have seen that the TBM conjunctive and TBM disjunctive rules are based on pointwise combination of conjunctive and disjunctive weights, respectively, using the product. Recently, Denœux [18] proposed two other rules, called the cautious and bold rules of combination, which are based on pointwise combination of conjunctive and disjunctive weights, respectively, using the minimum. This section summarizes necessary material on those two rules.

2.5.1 The cautious rule of combination

The TBM conjunctive rule is justified only when it is safe to assume that the items of evidence combined are distinct or, in other words, that the information sources are independent. When this assumption does not hold, an alternative consists in adopting a cautious, or conservative, attitude to the merging of belief functions by applying the LCP [18, 21, 32]. Let us now recall the building blocks of the cautious merging of belief functions.

As remarked in [26], it is possible to think of $\sqsubseteq_x$ as generalizing set inclusion. This reasoning can be used to see conjunctive combination rules as generalizing set intersection. Let us consider the following situation. Suppose two sources of information that are assumed to tell the truth. One states that ω is in $A \subseteq \Omega$, whereas the other states that it is in $B \subseteq \Omega$. It is then certain that ω is in C such that $C \subseteq A$ and $C \subseteq B$. The largest subset C satisfying those constraints is $A \cap B$. Now, suppose that the two sources of information provide the BBAs m_1 and m_2. Upon receiving those two pieces of information, the agent's state of belief should be represented by a BBA m_{12} more informative than m_1 and m_2. Let $\mathcal{S}_x(m)$ be the set of BBAs m' such that $m' \sqsubseteq_x m$, with $x \in \{v, w, s, q, pl, I\}$. Hence $m_{12} \in \mathcal{S}_x(m_1)$ and $m_{12} \in \mathcal{S}_x(m_2)$, or equivalently $m_{12} \in \mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$. According to the LCP, the x-least committed BBA should be chosen in $\mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$. This defines a conjunctive combination rule if the x-least committed BBA exists and is unique. Choosing the w-ordering yields an interesting solution [18, Proposition 4] which Denœux uses to define the so-called cautious rule of combination.

Definition 2.2 (Definition 1 of [18])**.** *Let m_1 and m_2 be two nondogmatic BBAs, and let $m_{1\owedge 2} = m_1 \owedge m_2$ denote the result of their combination by the cautious rule. The BBA $m_{1\owedge 2}$ has the following conjunctive weight function:*

$$w_{1\owedge 2}(A) \;=\; w_1(A) \wedge w_2(A), \quad \forall A \subset \Omega.$$

We thus have:

$$m_{1\owedge 2} \;=\; \textcircled{\cap}_{A \subset \Omega} A^{w_1(A) \wedge w_2(A)}.$$

The cautious rule is commutative, associative, idempotent and monotonic with respect to $\sqsubseteq_w$. This last property means that if a BBA m_1 is less informative than a BBA m_2 according to the $\sqsubseteq_w$ ordering, then this order is unchanged after combination by $\owedge$ with a third BBA.

As we have seen, any non dogmatic BBA m can be decomposed into two u-separable BBAs m^c and m^d, called the confidence and diffidence components of m. Let m_1 and m_2 be two nondogmatic BBAs. Proposition 6 of [18] gives the expression of the confidence and diffidence components of $m_{1 \owedge 2}$:

$$\begin{aligned} m^c_{1 \owedge 2} &= \textcircled{\cap}_{A \subset \Omega} A^{w^c_1(A) \wedge w^c_2(A)}, \\ m^d_{1 \owedge 2} &= \textcircled{\cap}_{A \subset \Omega} A^{w^d_1(A) \vee w^d_2(A)}, \end{aligned}$$

where $\vee$ denotes the maximum. Using this decomposition it is clear that we have [18, Proposition 7]:

$$m \owedge m_\Omega = m^c,$$

which means that the vacuous BBA is not a neutral element for the cautious rule. For any nondogmatic BBA m, we have $m \owedge m_\Omega = m$ iff m is u-separable [18, Proposition 8]. Furthermore, the cautious conjunctive rule has no neutral element, since the only BBA m_0 such that $m \owedge m_0 = m$ for any u-separable BBA m is the vacuous BBA, and this property is not satisfied for non u-separable BBAs [18]. Hence, $(\mathcal{M}_{nd}, \owedge, \sqsubseteq_w)$ is not a monoid but a partially ordered commutative semigroup.

Another property verified by the cautious rule is the following:

$$m_1 \textcircled{\cap} (m_2 \owedge m_3) \quad = \quad (m_1 \textcircled{\cap} m_2) \owedge (m_1 \textcircled{\cap} m_3),$$

for all nondogmatic BBAs m_1, m_2 and m_3. This last property is called distributivity of $\textcircled{\cap}$ with respect to $\owedge$. Interpretations of all of these properties are proposed in [18]. Of interest for Chapter 4 of this report is the fact that these properties are due to similar properties of the minimum on $(0, +\infty]$, much as the properties of the TBM conjunctive rule may be seen as consequences of the properties of the product on $(0, +\infty]$.

To conclude this presentation of the cautious rule, we may remark that selecting the $\sqsubseteq_w$ ordering in its definition imposes a severe restriction on the search space for choosing the combined BBA [18], which has a consequence on its "cautiousness". Indeed, let $\mathcal{X}_{12}$ denote the set $\mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$, with $x \in \{w, s, pl, q, I\}$. Then, the cautious rule chooses the least committed element in the set $\mathcal{W}_{12}$, and in this respect it may be termed "cautious". However, due to (2.19), we have

$$\mathcal{W}_{12} \subseteq \mathcal{S}_{12} \subseteq \mathcal{PL}_{12} \tag{2.21}$$

and

$$\mathcal{W}_{12} \subseteq \mathcal{S}_{12} \subseteq \mathcal{Q}_{12} \subseteq \mathcal{I}_{12}. \tag{2.22}$$

As can be seen with (2.21) and (2.22), $\mathcal{W}_{12}$ is the smallest possible set to search for the combined BBA. As a consequence, one may find outside of $\mathcal{W}_{12}$ a BBA that is x-less committed, with $x \in \{w, s, pl, q, I\}$, than $m_1 \owedge m_2$. In particular, when either m_1 or m_2 is not an u-separable BBA, then $m_{1 \textcircled{\cap} 2}$ is not in $\mathcal{W}_{12}$, and it is possible to choose two BBAs m_1 and m_2 such that $m_{1 \owedge 2} \sqsubseteq_x m_{1 \textcircled{\cap} 2}$ with $x \in \{w, s, q, pl, I\}$ (cf [18, Example 3]), which shows that the cautious rule is not really more "cautious" than the TBM conjunctive rule [18]. As remarked in [18], "these two rules actually belong to two different families of rules with distinct algebraic properties, and as such they cannot easily be compared". More information on those two different families are provided in Chapter 4.

2.5.2 The bold rule of combination

We have seen that the cautious rule of combination extends set intersection, and that it supposes the sources of information to tell the truth. Let us now consider another situation. Suppose we get two sources of information and that it is known that at least one of the two sources tells the truth, but we do not know which one. One of them states that ω is in $A \subseteq \Omega$, whereas the other one states that it is in $B \subseteq \Omega$. The smallest set containing both A and B is $A \cup B$. This reasoning is used in [18] to derive a disjunctive merging of belief functions based on the LCP, which can be summarized as follows. Suppose we get two sources of information that provide two BBAs m_1 and m_2, and that at least one of the sources tells the truth but it is not known which one. Then, the BBA m_{12} resulting from the merging of m_1 and m_2 should be the x-most committed BBA amongst the BBAs which are x-less committed than m_1 and m_2, with $x \in \{v, w, s, pl, q, I\}$ [18]. Denœux showed that using the v-ordering yields an interesting solution [18, Proposition 13], from which he defined the so-called bold rule of combination.

Definition 2.3 (Definition 2 of [18])**.** *Let m_1 and m_2 be two subnormal BBAs, and let $m_{1 \circledvee 2} = m_1 \circledvee m_2$ denote the result of their combination by the bold rule. The disjunctive weight function of the BBA $m_{1 \circledvee 2}$ is:*

$$v_{1 \circledvee 2}(A) \quad = \quad v_1(A) \wedge v_2(A), \quad \forall A \neq \emptyset.$$

We thus have:

$$m_{1 \circledvee 2} \quad = \quad \bigcirc\!\!\!\!\cup_{A \neq \emptyset} A_{v_1(A) \wedge v_2(A)}.$$

The bold rule has similar properties as the cautious rule since they are both based on the minimum; in particular $(\mathcal{M}_s, \circledvee, \sqsubseteq_v)$ is a partially ordered commutative semigroup with no neutral element. Furthermore, the cautious and bold rules are related by De Morgan's laws [18]:

$$\overline{m_1 \circledvee m_2} \quad = \quad \overline{m_1} \circledwedge \overline{m_2}, \tag{2.23}$$

$$\overline{m_1 \circledwedge m_2} \quad = \quad \overline{m_1} \circledvee \overline{m_2}. \tag{2.24}$$

2.6 Conclusion

In this chapter, two equivalent representations of a belief function have been introduced. They are called the conjunctive and disjunctive weight functions and stem, respectively, from the conjunctive and disjunctive canonical decompositions of a belief function. Those decompositions allow one to see a complex belief state as originating from the combination and decombination of simple BBAs, which represent the simplest form of evidence.

Building on the semantics of those decompositions, two other partial orderings for belief functions were proposed and their properties investigated. Those two orderings were then used to derive two idempotent combination rules for belief functions, called the cautious rule and the bold rule.

Those last two rules have a simple expression based on the conjunctive and disjunctive weight functions, much as the TBM conjunctive and TBM disjunctive rules do. It thus appears interesting to study combinations rules based on these rarely exploited functions. A major difference between those two pairs of rules is that the cautious and bold rules do not have a neutral element. This may be seen as a drawback of these rules, as having a neutral element for a conjunctive or disjunctive operator is generally a required property. The next chapter will shed some light on combination rules based on weight functions and that have a neutral element.

Part II

Weight-Based Combination Rules

A New Justification of the TBM Conjunctive Rule

Summary

The TBM conjunctive rule and the cautious conjunctive rule can both be expressed using the conjunctive weight function. It seems thus interesting to study rules based on this function. The fundamental difference between the TBM conjunctive rule and the cautious conjunctive rule is that the vacuous belief funtion is a neutral element for the former, whereas it is not for the latter. However, this property may be regarded as desirable for a conjunctive operator, as the vacuous belief function represents total ignorance. In this chapter, the rules based on the conjunctive weight function and that admit the vacuous belief function as neutral element, are characterized. In particular, it is shown that, among those rules, the TBM conjunctive rule is the least committed one. This can be seen as a new formal justification of the TBM conjunctive rule as a rule that respects a central principle of the TBM. A counterpart to this result is also obtained for the TBM disjunctive rule. This new justification of the TBM conjunctive rule is put into perspective by comparing it to the previously proposed justifications in the literature.

Résumé

La règle conjonctive du MCT et la règle conjonctive prudente peuvent toutes les deux être exprimées avec la fonction de poids conjonctifs. Il semble donc intéressant d'étudier les règles de combinaison basées sur cette fonction. La différence fondamentale entre la règle conjonctive du MCT et la règle conjonctive prudente est que la fonction de croyance vide est un élément neutre pour la première, alors que cela n'est pas le cas pour la seconde. Cependant, cette propriété peut être considérée comme souhaitable pour un opérateur conjonctif, car la fonction de croyance vide représente l'ignorance totale. Dans ce chapitre, les règles de combinaison basées sur la fonction de poids conjonctifs et qui ont pour élément neutre la fonction de croyance vide, sont caractérisées. En particulier, nous montrons que, parmi ces règles, la règle conjonctive du MCT est la moins engagée. Ceci peut être vu comme une nouvelle justification formelle de cette règle comme respectant un principe central du MCT. Une justification similaire est également proposée dans ce chapitre pour la règle disjonctive du MCT. De plus, cette nouvelle justification de la règle conjonctive

du MCT est mise en perspective avec les autres justifications trouvées dans la littérature.

3.1 Introduction

The most often used combination rule in the TBM is the TBM conjunctive rule. This is due to its intuitive appeal and to its numerous justifications [25, 42, 47, 48, 82][1]. It seems indeed reasonable to favor a principled rule over "ad hoc" ones [95], hence the necessity for such justifications. In this chapter, yet another justification is proposed, following a completely different approach to the existing ones.

This chapter is organized as follows. The previous justifications are reviewed in Section 3.2. A new justification is proposed in Section 3.3. In Section 3.4, a counterpart to this justification is provided for the TBM disjunctive rule. Eventually, our justification of the TBM conjunctive rule is compared to the existing ones in Section 3.5.

The work presented in this chapter was published in [64].

3.2 Previous Justifications

This section is devoted to a summary of the various axiomatic derivations of the TBM conjunctive rule that have been proposed in the literature.

Formally, the problem of axiomatically justifying the TBM conjunctive rule may be formulated as follows. Let m_1 and m_2 be two BBAs defined on a frame of discernment Ω and provided, respectively, by the sources of information S_1 and S_2. Let $m_{1\otimes 2} = m_1 \otimes m_2$ denote the result of their combination, where $\otimes$ symbolizes the combination operator. One must show that $m_{1\otimes 2} = m_{1\bigcirc\!\!\!\!\cap\, 2}$ holds when the combination satisfies a given set of axioms.

3.2.1 Dubois and Prade's justification

In [25], Dubois and Prade study conjunctive rules of the form

$$(m_1 \otimes m_2)(A) = \sum_{B\cap C=A} m_1(B)\Box m_2(C), \quad \forall A \subseteq \Omega, \tag{3.1}$$

where $\Box$ is a binary operator on $[0,1]$. They show that $m_{1\otimes 2}$ is a BBA, i.e., $\sum_{A\subseteq\Omega} m_{1\otimes 2}(A) = 1$, if and only if the binary operator $\Box$ is the product. They conclude that there is only one possible conjunctive rule in the theory of belief functions, provided that the combination is asked to satisfy the algebraic property given by Equation (3.1).

[1]Although those justifications use loosely the expression "Dempster's rule", one should note that they actually prove the unicity of the unnormalized version of Dempster's rule. Furthermore, they are not based on probability concepts, hence they can readily be used to justify the TBM conjunctive rule as already remarked by Smets [97, 95].

3.2.2 Smets' justification and related justifications

Smets' justification

In [82], Smets shows that we have $m_1 \otimes m_2 = m_1 \textcircled{\cap} m_2$ if $\otimes$ satisfies the following eight axioms[2]:

S1: $(bel_1 \otimes bel_2)(A)$ is a function of m_1, m_2 and A only.

S2: $\otimes$ is commutative.

S3: $\otimes$ is associative.

S4: If $m_2(B) = 1$, $\otimes$ amounts to the conditioning operation (1.7), i.e., $m_1 \otimes m_2 = m_1[B]$.

S5: $\otimes$ is invariant with respect to a permutation of the elements of Ω (property called internal symmetry in [82]).

S6: For all $A \neq \Omega$, $(m_1 \otimes m_2)(A)$ does not depend on $m_1(B)$, for all $B \subset \bar{A}$ (autofunctionality property [82]).

S7: There are at least three elements in Ω.

S8: Let m_A be a categorical BBA such that $m_A(A) = 1$ and let m_2^ε be a simple BBA defined by $m_2^\varepsilon(A) = 1-\varepsilon, m_2^\varepsilon(\Omega) = \varepsilon$. $\otimes$ satisfies the following continuity property, for any BBA m_1:

$$\forall B \subseteq \Omega, \lim_{\varepsilon \to 0}(m_1 \otimes m_2^\varepsilon)(B) = (m_1 \otimes m_A)(B).$$

Axiom S1 is essentially a formulation of the idea of distinctness [82]. Axioms S2 and S3 mean that the result of the combination is independent of the order in which the pieces of evidence are processed. Axiom S4 dictates the behavior of the combination when a given hypothesis B has been ascertained: any mass given to $C \subseteq \Omega$ is transferred to $C \cap B$. Axiom S5 means that the result of the combination must not be modified by a permutation of the elements of Ω. Axiom S6 expresses the idea that the mass given to A after combination is independent of the masses given by m_1 and m_2 to the subsets $B \subseteq \Omega$, which are included in $\bar{A}$. Eventually, axioms S7 and S8 were added by Smets for technical reasons.

Klawonn and Schwecke's justification

The six axioms found in [47] are for the most part quite similar to the ones of Smets [82], as already remarked by Bloch [6]. The notable differences are that there are no counterparts to S7 and S8 in [47] and that Klawonn and Schwecke [47] use relations between frames of discernment whereas Smets [82] only works in a single frame of discernment. Eventually, we may note that Klawonn and Schwecke argue that their proof is simpler than the one of Smets.

[2] The formulation of Smets' axioms used here is the one of [6].

Hájek's justification

In [42], Hájek revisits Smets' justification [82]. Hájek's major contribution is to show that some of Smets' axioms are redundant and that one can derive the TBM conjunctive rule merely from axioms S1, S2, S4 and S5 (S5 is actually slightly modified in [42]) or from axioms S1, S2, S3, S4, S6 and S7.

3.2.3 Klawonn and Smets' justification

The justification of Klawonn and Smets [48] may be summarized as follows[3].

Let m_1 and m_2 be two BBAs. Let $\otimes$ be an operator such that:

$$m_1 \otimes m_2 = \mathbf{S}_1 \cdot m_2,$$

and

$$m_1 \otimes m_2 = \mathbf{S}_2 \cdot m_1,$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are specialization matrices. If it is required that the following properties be satisfied

- Distinctness: $\mathbf{S}_1$ does not depend on m_2 and $\mathbf{S}_2$ does not depend on m_1,
- Generalization of conditioning: $m_1 \otimes m_2 = m[B]$, when m_2 is a categorical BBA focused on some subset $B \subseteq \Omega$,
- Commutativity: $m_1 \otimes m_2 = m_2 \otimes m_1$,

then $m_1 \otimes m_2 = m_1 \textcircled{\cap} m_2$.

This justification means that the TBM conjunctive rule follows from the notions of commutativity, distinctness, specialization and conditioning (conditioning can itself be derived from the specialization concept and the LCP [48]). It may be compared to Hájek's justification. Indeed, the commutativity, distinctness, and conditioning axioms are shared by these two justifications. They thus differ on only one point: one postulates that the combination must result in a specialization, whereas the other requires axiom S5 to be satisfied. Arguably, the specialization requirement seems more defendable.

3.3 New Justification of the TBM Conjunctive Rule

In Part I of this thesis, it was explained that two important rules, the cautious rule and the TBM conjunctive rule, share a remarkable property: they are based on pointwise combination of conjunctive weights using a binary operator on $(0, +\infty]$ (respectively, the minimum and the product). It is thus interesting to study rules based on pointwise combination of conjunctive weights using a binary operator on $(0, +\infty]$ (such rules will be hereafter referred to as w-based).

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[3]The formulation of Klawonn and Smets' justification used here is the one of [19].

The fundamental difference between the TBM conjunctive rule and the cautious rule is that the vacuous BBA is a neutral element for the former, whereas it is not for the latter. However, this property may be regarded as desirable in the context of conjunctive mergings, as the vacuous BBA represents total ignorance. In the remainder of this section, we characterize the w-based rules that admit the vacuous BBA as neutral element. A new justification of the TBM conjunctive rule will immediately follow from this characterization.

It is clear that a w-based rule based on a binary operator on $(0, +\infty]$, has the vacuous BBA as neutral element if and only if 1 is a neutral element of the binary operator. We may remark that the product on $(0, +\infty]$ satisfies this property, whereas the minimum on $(0, +\infty]$ does not, hence the difference between the TBM conjunctive rule and the cautious rule.

As will be shown by Proposition 3.1 and Theorem 3.1 below, a binary operator must satisfy another property in order to obtain a w-based rule that has the vacuous BBA as neutral element. We may already note that this property is essential since it is the one that will lead to the justification of the TBM conjunctive rule.

Proposition 3.1. *Let $\circ$ be a binary operator on $(0, +\infty]$ such that $1 \circ x = x \circ 1 = x$ for all $x \in (0, +\infty)$ and $x \circ y \leq xy$ for all $x, y \in (0, +\infty)$. Then, for any conjunctive weight functions w_1 and w_2, the function $w_{1\circ 2}$ defined by:*

$$w_{1\circ 2}(A) \quad = \quad w_1(A) \circ w_2(A), \quad \forall A \subset \Omega,$$

is a conjunctive weight function associated to some nondogmatic BBA $m_{1\circ 2}$.

Proof. We have

$$w_{1\circ 2}(A) \quad \leq \quad w_{1 \bigcirc\!\!\!\!\cap\, 2}(A), \quad \forall A \subset \Omega.$$

From Lemma 2.1, $w_{1\circ 2}$ is a conjunctive weight function since $w_{1 \bigcirc\!\!\!\!\cap\, 2}$ is a conjunctive weight function. □

Proposition 3.1 has shown that if a binary operator is below the product and has 1 as neutral element, then it can be used to define a w-based rule that has the vacuous BBA as neutral element. One may wonder if the constraint of being below the product can be relaxed. The answer is negative, as shown by the following theorem.

Theorem 3.1. *Let $\circ$ be a binary operator on $(0, +\infty]$ such that*

- *$1 \circ x = x \circ 1 = x$ for all $x \in (0, +\infty)$ and*
- *$x \circ y > xy$ for some $x, y \in (0, +\infty)$.*

Then, there exist two nondogmatic BBAs m_1 and m_2 on a frame Ω such that the function obtained by pointwise combination using $\circ$ of the conjunctive weight functions associated to m_1 and m_2 is not a conjunctive weight function.

Proof. See Appendix E.1. □

We may illustrate this theorem with an example.

Example 3.1. *Let us combine the conjunctive weight functions* w_1 *and* w_2 *of Example 2.1 using a binary operator* $\circ$ *having* 1 *as neutral element and such that* $\frac{1}{3} \circ \frac{2}{3} > \frac{1}{3} \cdot \frac{2}{3}$, *e.g.,* $\frac{1}{3} \circ \frac{2}{3} = \frac{1}{3} \cdot \frac{2}{3} + 0.01$. *We denote the function resulting from this combination by* $w_1 \circ w_2 = w_{1\circ 2}$. *Then, the function* $w_{1\circ 2}$ *is equal to* $w_{1 \textcircled{\cap} 2}$ *for all* $A \in 2^\Omega \setminus \{\Omega, \{b,c\}\}$ *and we have* $w_{1\circ 2}(\{b,c\}) = w_{1 \textcircled{\cap} 2}(\{b,c\}) + 0.01$. *Setting* $m_{1\circ 2} = \textcircled{\cap}_{A \subset \Omega} A^{w_{1\circ 2}(A)}$, *we find that* $m_{1\circ 2}(\emptyset) = -0.0035 < 0$, *hence* $m_{1\circ 2}$ *is not a BBA.*

The immediate corollary of this theorem constitutes the central result of this chapter.

Corollary 3.1. *The TBM conjunctive rule* $\textcircled{\cap}$ *is the* x*-least committed rule, with* $x \in \{w, s, pl, q, I\}$, *among the* w*-based rules that have the vacuous BBA* m_Ω *as neutral element.*

Proof. From Theorem 3.1 and Proposition 3.1, it is clear that any w-based rule that has the vacuous BBA as neutral element is based on a binary operator $\circ$ on $(0, +\infty]$ with 1 as neutral element and such that $x \circ y \leq xy$ for all $x, y \in (0, +\infty)$. For all nondogmatic BBAs m_1 and m_2, we thus have

$$w_1(A) \circ w_2(A) \quad \leq \quad w_{1 \textcircled{\cap} 2}(A), \quad \forall A \subset \Omega.$$

Consequently, the rule based on $\circ$ is at least as w-committed as the rule $\textcircled{\cap}$. The corollary follows then directly from (2.19). □

According to this corollary, the TBM conjunctive rule thus respects a central principle of the TBM: the least commitment principle, under the two requirements of being based on pointwise combination of conjunctive weights and having the vacuous BBA as neutral element. Corollary 1 further implies that it is the only rule satisfying these properties. We thus have provided a new formal justification of the TBM conjunctive rule.

In Section 3.5, this justification will be compared to the other ones proposed in the literature. Before that, we will first state a corresponding result for the TBM disjunctive rule, which is a consequence of the duality between these two rules.

3.4 The Disjunctive Case

The TBM disjunctive rule of combination can be justified using a 'disjunctive version' of Dubois and Prade's justification of the TBM conjunctive rule recalled in Section 3.2.1: if it required that the combination be disjunctive then the mass $m(B, C)$ is assigned to the union $B \cup C$, and the TBM disjunctive rule is recovered. There exists also a disjunctive version of Klawonn and Smets' justification: it is based on the concepts of generalization and deconditioning, which are duals of the specialization and conditioning concepts. In this section, we provide a disjunctive version of the justification of the TBM conjunctive rule proposed in the preceding section.

The bold and TBM disjunctive rules are based on pointwise combination of disjunctive weights using a binary operator on $(0,+\infty]$ (respectively, the minimum and the product). The difference between these two rules is that the or-vacuous BBA $m_\emptyset$ is a neutral element for the TBM disjunctive rule, whereas it is not for the bold rule. This property is important in the context of a disjunctive merging. Indeed, it is a direct consequence of generalizing set union, as is the use of the principle of maximal commitment (instead of the LCP). The study of rules based on pointwise combination of *disjunctive* weights (v-based rules for short) and which admit the or-vacuous BBA as neutral element leads to the following conclusion.

Corollary 3.2. *The TBM disjunctive rule* $\ocircle\!\!\!\cup$ *is the* x*-most committed rule, with* $x \in \{v, s, pl, q, I\}$ *among the* v*-based rules that have the or-vacuous BBA* $m_\emptyset$ *as neutral element.*

Proof. Let $\circ$ be a binary operator on $(0,+\infty]$ having 1 as neutral element. Let v_1 and v_2 be the disjunctive weight functions associated to two subnormal BBAs m_1 and m_2. Let $\overline{w_1}$ and $\overline{w_2}$ be the conjunctive weight functions associated to $\overline{m_1}$ and $\overline{m_2}$. We have:

$$\begin{aligned}\overline{\ocircle\!\!\!\cup_{A\neq\emptyset} A_{v_1(A)\circ v_2(A)}} &= \ocircle\!\!\!\cap_{A\neq\emptyset} \overline{A_{v_1(A)\circ v_2(A)}}\\ &= \ocircle\!\!\!\cap_{A\neq\emptyset} \bar{A}^{\overline{w_1}(\bar{A})\circ\overline{w_2}(\bar{A})}\\ &= \ocircle\!\!\!\cap_{A\subset\Omega} A^{\overline{w_1}(A)\circ\overline{w_2}(A)} \qquad (3.2)\end{aligned}$$

From Theorem 3.1 and Lemma 2.1, (3.2) is guaranteed to be a BBA iff $\circ$ is such that $x \circ y \leq xy$. For any operator $\circ$ on $(0,+\infty]$ having 1 as neutral element and such that $x \circ y \leq xy$ for all $x, y \in (0,+\infty)$, we have

$$v_1(A) \circ v_2(A) \leq v_{1\ocircle\!\!\!\cup 2}(A), \quad \forall A \neq \emptyset.$$

Consequently, the disjunctive rule based on $\circ$ is at most as v-committed as the rule $\ocircle\!\!\!\cup$. The corollary follows then directly from (2.20). □

This corollary shows that the TBM disjunctive rule respects the principle of maximal commitment, which is the one to be followed in the context of disjunctive merging. It may thus be seen as a new justification for the TBM disjunctive rule.

3.5 Discussion

The justification of the TBM conjunctive rule proposed in this chapter as well as the ones proposed in [25, 42, 47, 48, 82] completely fit the TBM since they are obtained without introducing any underlying probability concepts. However, it is interesting to remark that our approach is completely different from the other ones. Indeed, the main properties required of the combination operator in the justifications presented in [42, 47, 48, 82] are: commutativity, distinctness and generalization of the conditioning operation. The justification of Dubois and Prade [25] is based on the requirement that the combination should satisfy a particular algebraic property.

In comparison, our justification is based on two other requirements: the combination should be w-based and it should have the vacuous BBA as neutral element. Whereas the latter requirement is intuitively appealing, the former may seem more difficult to interpret. However, some justification may be found in the meaning of the canonical decomposition of a belief function, which breaks down a belief function into elementary pieces of evidence pertaining to single propositions. It may be argued that the combination of two belief functions should be performed by considering in turn each proposition and combining the two elementary pieces of evidence pertaining to it, which leads to the w-based requirement. As a further motivation for introducing this requirement, we may notice that w-based combinations offer a rarely considered, yet promising outlook on the combination of belief functions, as demonstrated by the recent introduction of the cautious rule.

In summary, all those justifications seem reasonable and have their merits, even though the requirements on which they are based can always be subject to discussion. Globally, however, there seems to be a convergence of arguments in favor of the TBM conjunctive rule, even if other rules may be valuable in some situations as explained in the introduction of this thesis. As a matter of fact, the next chapter will reveal the existence of infinite families of combination rules in which the TBM conjunctive rule and the cautious rule are particular members.

Four Infinite Families of Combination Rules

Summary

This chapter continues the study of rules based on weight functions. It is shown that the cautious conjunctive rule and the TBM conjunctive rule are particular members of two different families of conjunctive rules. More precisely, the cautious conjunctive rule is the least committed element of an infinite family of conjunctive rules based on extended triangular norms, whereas the TBM conjunctive rule is the least committed element of an infinite family of conjunctive rules based on extended uninorms. Similar results are obtained for the TBM disjunctive rule and the bold disjunctive rule. The introduction of those infinite families of conjunctive and disjunctive rules sheds some new light on the fundamentally different behaviors of the cautious conjunctive rule and the TBM conjunctive rule, as well as the bold disjunctive rule and the TBM disjunctive rule. It also shows that the TBM is not poorer than possibility theory in terms of conjunctive and disjunctive operators. Some experiments with one of those families of rules are also conducted in this chapter. They show that this family of rules may improve the performances in some classification applications.

Résumé

Ce chapitre continue l'étude des règles basées sur les fonctions de poids. Nous montrons que la règle conjonctive prudente et la règle conjonctive du MCT sont deux membres particuliers de deux familles différentes de règles. Plus précisément, la règle conjonctive prudente est l'élément le moins engagé d'une famille infinie de règles conjonctives basée sur des normes triangulaires étendues, alors que la règle conjonctive du MCT est l'élément le moins engagé d'une famille infinie de règles conjonctives basée sur des uninormes étendues. Des résultats similaires sont obtenus pour la règle disjonctive hardie et la règle disjonctive du MCT. La mise à jour de ces familles infinies de règles conjonctives et disjonctives apporte un nouvel éclairage sur les comportements fondamentalement différents de la règle conjonctive prudente et la règle conjonctive du MCT, ainsi que la règle disjonctive hardie et la règle disjonctive du MCT. Cela montre aussi que le MCT dispose d'autant d'opérateurs conjonctifs et disjonctifs que la théorie des possibilités. Quelques expériences avec une de ces familles de règles sont également conduites dans ce chapitre. Elles montrent que

cette famille de règle peut être utile afin d'améliorer les performances dans des applications de classification.

4.1 Introduction

As discussed in the introduction of this thesis, having only one rule is not the ideal situation to cope with real-world problems. It could thus be useful to have other rules of combination. Such rules should at least satisfy a few basic properties such as commutativity and associativity. This chapter shows that the cautious and TBM conjunctive rules can be seen as particular members of two distinct families of combination rules, which are based on binary operators having similar properties as triangular norms [49] and uninorms [108], respectively. It thus provides an answer to the need for more flexibility in terms of combination rules, and sheds some new light on the fundamentally different behaviors of the cautious and TBM conjunctive rules. It also allows us to put the result of the preceding chapter in a broader perspective.

This chapter is organized as follows. In Section 4.2, the definitions of triangular norms (t-norms for short) and uninorms, which are usually defined as binary operators on the unit interval, are extended to the interval $(0, +\infty]$. A family of rules based on conjunctive weights and t-norms on $(0, +\infty]$ is then defined in Section 4.3. A family of rules based on conjunctive weights and uninorms on $(0, +\infty]$ having 1 as neutral element is introduced in Section 4.4. Section 4.5 presents counterparts to those families for rules based on disjunctive weights. Eventually, Section 4.6 demonstrates the usefulness of one of those families of rules in two classification applications.

The work presented in this chapter was published in [65].

4.2 T-Norms and Uninorms on $(0, +\infty]$

4.2.1 Extended definitions

The key to the family of rules that will be introduced in this chapter is to remark that the cautious rule and the bold rule are based on the minimum of weights, whereas the TBM conjunctive rule and the TBM disjunctive rule are based on the product of weights. Furthermore, the minimum and the product on $(0, +\infty]$ are binary operators that essentially differ by the position of their neutral element. Indeed, on the one hand, the minimum on $(0, +\infty]$ is commutative, associative, and monotonic. In addition, the upper bound of $(0, +\infty]$ serves as neutral element for the minimum. The minimum on $(0, +\infty]$ has thus similar properties as t-norms [49], which are defined as commutative, associative, monotonic operators on $[0, 1]$ that admit the upper bound of $[0, 1]$, i.e., 1, as neutral element. On the other hand, the product on $(0, +\infty]$ is commutative, associative, monotonic and has one as neutral element. It has thus similar properties as uninorms [108], which are defined as commutative, associative, monotonic operators on $[0, 1]$ that admit a number $e \in [0, 1]$ as neutral element. This comparison between the minimum and the product leads us to extend the definitions of t-norms and uninorms on $(0, +\infty]$ as follows.

Definition 4.1. *A t-norm on* $(0, +\infty]$ *is a binary operator on* $(0, +\infty]$*, which is commutative, associative, monotonic, and which admits* $+\infty$ *as neutral element.*

Definition 4.2. *A uninorm on* $(0,+\infty]$ *is a binary operator on* $(0,+\infty]$*, which is commutative, associative, monotonic, and which admits some positive real number* $e \in (0,+\infty]$ *as neutral element.*

4.2.2 Construction of t-norms and 1-uninorms on $(0,+\infty]$

This section provides means of obtaining other t-norms on $(0,+\infty]$ and uninorms on $(0,+\infty]$ having one as neutral element (1-uninorms for short) than, respectively, the minimum and the product. Propositions 4.1 and 4.2 below give two construction mechanisms to obtain such t-norms and 1-uninorms out of t-norms on $[0,1]$. Note that a t-norm $\top$ on $[0,1]$ verifying $x \top y > 0$ whenever $x > 0$ and $y > 0$ will be said *positive.*

Proposition 4.1. *Let* $\top$ *be a positive t-norm on* $[0,1]$*, and let* $\top'$ *be a t-norm on* $[0,1]$*. Then the operator* $\mathcal{T}_{(\top,\top')}$ *defined by*

$$x \,\mathcal{T}_{(\top,\top')}\, y \;=\; \begin{cases} x \top y & \text{if } x \vee y \leq 1, \\ \left(1 - \left(\left(1 - \frac{1}{x}\right) \top' \left(1 - \frac{1}{y}\right)\right)\right)^{-1} & \text{if } x \wedge y > 1, \\ x \wedge y & \text{otherwise,} \end{cases} \tag{4.1}$$

for all $x, y \in (0,+\infty]$ *is a t-norm on* $(0,+\infty]$.

Proof. The notation is simplified in this proof: the operator $\mathcal{T}_{(\top,\top')}$ defined by (4.1) is simply noted $\mathcal{T}$. To prove this proposition, we also need to introduce some usual conventions related to the use of the extended real line (see [49, pp xviii]): we have $1/+\infty = 0$ and $1/0 = +\infty$.

Let us first remark that the operator defined by (4.1) can be equivalently written

$$x \,\mathcal{T}\, y \;=\; \begin{cases} x \top y & \text{if } x \vee y \leq 1, \\ \left(\frac{1}{x} \perp' \frac{1}{y}\right)^{-1} & \text{if } x \wedge y > 1, \\ x \wedge y & \text{otherwise,} \end{cases} \tag{4.2}$$

where $\perp'$ is the *dual t-conorm* of $\top'$ [49]. We recall that a t-conorm is a commutative, associative, monotonic operator on $[0,1]$, whose neutral element is the lower bound of $[0,1]$. Furthermore, the dual t-conorm $\perp$ of a t-norm $\top$ is defined by:

$$x \perp y = N\left(N\left(x\right) \top N\left(y\right)\right), \quad x, y \in [0,1],$$

with N a mapping from $[0,1]$ to $[0,1]$ defined by $N(x) = 1 - x$, for all $x \in [0,1]$. The commutativity, associativity, and monotonicity of the operator defined by (4.2) can be proved using Lemma 2 of [18]. Let us show that $+\infty$ is a neutral element for this operator: for $x \leq 1$, $+\infty \,\mathcal{T}\, x = +\infty \wedge x = x$, and for $x > 1$, $+\infty \,\mathcal{T}\, x = \left(0 \perp' \frac{1}{x}\right)^{-1} = x$. Eventually, we must show that $x \,\mathcal{T}\, y > 0$ holds for all $x, y \in (0,+\infty]$:

- Suppose $x \,\mathcal{T}\, y = x \top y$, hence $x \vee y \leq 1$. The positivity of $\top$ ensures that $x \,\mathcal{T}\, y > 0$.

- Suppose $x \mathcal{T} y = \left(\frac{1}{x} \bot' \frac{1}{y}\right)^{-1}$, hence $x \wedge y > 1$. Clearly, $\left(\frac{1}{x} \bot' \frac{1}{y}\right)^{-1} > 1$ for all x, y such that $x \wedge y > 1$.
- Suppose $x \mathcal{T} y = x \wedge y$. Then, $x \mathcal{T} y > 0$ since $x \wedge y > 0$ for all $x > 0$ and $y > 0$.

□

Proposition 4.2. *Let $\top$ be a positive t-norm on $[0,1]$, and let $\top'$ be a t-norm on $[0,1]$. Then, the operator $\mathcal{U}_{(\top,\top')}$ defined by*

$$x \, \mathcal{U}_{(\top,\top')} \, y = \begin{cases} x \top y & \text{if } x \vee y \leq 1, \\ ((1/x) \top' (1/y))^{-1} & \text{if } x \wedge y \geq 1, \\ x \wedge y & \text{otherwise,} \end{cases} \tag{4.3}$$

for all $x, y \in (0, +\infty]$ is a uninorm on $(0, +\infty]$ having 1 as neutral element.

Proof. See Appendix E.2. □

The construction mechanisms provided by Propositions 4.1 and 4.2 are illustrated, respectively, by Examples 4.1 and 4.2.

Example 4.1. *The t-norm $\mathcal{T}$ on $(0, +\infty]$ defined by*

$$x \mathcal{T} y = \begin{cases} \left(\frac{1}{x} + \frac{1}{y} - \frac{1}{x} \cdot \frac{1}{y}\right)^{-1} & \text{if } x \wedge y > 1, \\ x \wedge y & \text{otherwise,} \end{cases}$$

for all $x, y \in (0, +\infty]$, is obtained by setting $\top = \wedge$ and $\top' = \cdot$ (product), with $\top$ and $\top'$ the t-norms involved in Proposition 4.1.

Example 4.2. *The 1-uninorm $\mathcal{U}$ defined by*

$$x \, \mathcal{U} \, y = \begin{cases} ((1/x) \wedge (1/y))^{-1} & \text{if } x \wedge y \geq 1, \\ x \wedge y & \text{otherwise,} \end{cases}$$

for all $x, y \in (0, +\infty]$, is obtained by setting $\top = \wedge$ and $\top' = \wedge$, with $\top$ and $\top'$ the t-norms involved in Proposition 4.2.

4.3 Conjunctive T-Rules

As previously mentioned, the minimum is a t-norm on $(0, +\infty]$. The cautious rule thus belongs to a family of rules based on pointwise combination of conjunctive weights using t-norms on $(0, +\infty]$. In order to characterize this family, we need to remark that the minimum is the largest t-norm on $(0, +\infty]$, much as it is the largest t-norm on $[0, 1]$.

Lemma 4.1. *The minimum is the largest t-norm on $(0, +\infty]$.*

Proof. Any t-norm $\mathcal{T}$ on $(0,+\infty]$ has by definition $+\infty$ as neutral element and is monotonic, hence we have $x \mathcal{T} y \leq x \mathcal{T} +\infty = x$ and $x \mathcal{T} y \leq +\infty \mathcal{T} y = y$, so $x \mathcal{T} y \leq x \wedge y$, for all $x, y \in (0,+\infty]$.

□

We may then show the following.

Proposition 4.3. *Let $\mathcal{T}$ be a t-norm on $(0,+\infty]$. Then, for any conjunctive weight functions w_1 and w_2, the function $w_{1 \circledT_w 2}$ defined by:*

$$w_{1 \circledT_w 2}(A) = w_1(A) \, \mathcal{T} \, w_2(A), \quad \forall A \subset \Omega,$$

is a conjunctive weight function associated to some nondogmatic BBA $m_{1 \circledT_w 2}$.

Proof. From Lemma 4.1, we have

$$w_{1 \circledT_w 2}(A) \quad \leq \quad w_{1 \owedge 2}(A), \quad \forall A \subset \Omega.$$

From Lemma 2.1, $w_{1 \circledT_w 2}$ is a conjunctive weight function since $w_{1 \owedge 2}$ is a conjunctive weight function.

□

Proposition 4.3 allows us to define combination rules for belief functions which can be formally defined as follows.

Definition 4.3 (T-norm-based conjunctive combination rule)**.** *Let $\mathcal{T}$ be a t-norm on $(0,+\infty]$. Let m_1 and m_2 be two nondogmatic BBAs. Their combination using the t-norm-based conjunctive combination rule, or conjunctive t-rule for short, is noted $m_{1 \circledT_w 2} = m_1 \circledT_w m_2$. It is defined as a BBA with the following conjunctive weight function:*

$$w_{1 \circledT_w 2}(A) \quad = \quad w_1(A) \, \mathcal{T} \, w_2(A), \quad \forall A \subset \Omega.$$

We thus have:

$$m_{1 \circledT_w 2} \quad = \quad \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w_1(A) \, \mathcal{T} \, w_2(A)}.$$

Proposition 4.4. *Any conjunctive t-rule $\circledT_w$ has the following properties:*

- *Commutativity: for all m_1 and m_2, $m_1 \circledT_w m_2 = m_2 \circledT_w m_1$;*
- *Associativity: for all m_1, m_2 and m_3,*

$$m_1 \circledT_w (m_2 \circledT_w m_3) \quad = \quad (m_1 \circledT_w m_2) \circledT_w m_3;$$

- *Monotonicity with respect to $\sqsubseteq_w$: for all m_1, m_2 and m_3, we have $m_1 \sqsubseteq_w m_2 \Rightarrow m_1 \circledT_w m_3 \sqsubseteq_w m_2 \circledT_w m_3$;*

Proof. These properties follow directly from corresponding properties of the t-norm $\mathcal{T}$.

□

From this proposition it is clear that for any conjunctive t-rule $\textcircled{\scriptsize T}_w$, the algebraic structure $(\mathcal{M}_{nd}, \textcircled{\scriptsize T}_w, \sqsubseteq_w)$ is a commutative, partially ordered semigroup.

Finally, the following proposition situates the cautious rule in the family of conjunctive t-rules.

Proposition 4.5. *Among all conjunctive t-rules, the cautious rule is the x-least committed, with $x \in \{w, s, pl, q, I\}$:*

$$m_1 \textcircled{\scriptsize T}_w m_2 \sqsubseteq_x m_1 \textcircled{\wedge} m_2, \quad \forall m_1, m_2.$$

Proof. Since the minimum is the largest t-norm on $(0, +\infty]$, we have $m_1 \textcircled{\scriptsize T}_w m_2 \sqsubseteq_w m_1 \textcircled{\wedge} m_2$ for all nondogmatic BBAs m_1 and m_2. The result follows then directly from (2.19).

□

Given a positive t-norm $\top$ on $[0,1]$ and a t-norm $\top'$ on $[0,1]$, Proposition 4.1 allows us to obtain a conjunctive t-rule based on the t-norm $\mathcal{T}_{(\top,\top')}$ on $(0,+\infty]$ defined by (4.1). This conjunctive t-rule is noted $\textcircled{\scriptsize T}_w^{\top,\top'}$. Explicitly, we have

$$m_1 \textcircled{\scriptsize T}_w^{\top,\top'} m_2 = \textcircled{\cap}_{A \subset \Omega} A^{w_1(A) \; \mathcal{T}_{(\top,\top')} w_2(A)}. \tag{4.4}$$

This conjunctive t-rule will be useful in Section 4.6.

4.4 Conjunctive U-Rules

We have seen that the TBM conjunctive rule is based on the product and that the product is a 1-uninorm. Hence, the TBM conjunctive rule belongs to a family of rules characterized by pointwise combination of conjunctive weights using 1-uninorms[1].

Definition 4.4 (Uninorm-based conjunctive combination rule)**.** *Let $\mathcal{U}$ be a 1-uninorm, such that $x \; \mathcal{U} \; y \leq xy$ for all $x, y \in (0, +\infty)$. Let m_1 and m_2 be two nondogmatic BBAs. Their combination using the uninorm-based conjunctive combination rule, or conjunctive u-rule for short, is noted $m_{1 \textcircled{\scriptsize U}_w 2} = m_1 \textcircled{\scriptsize U}_w m_2$. It is defined as a BBA with the following conjunctive weight function:*

$$w_{1 \textcircled{\scriptsize U}_w 2}(A) = w_1(A) \, \mathcal{U} \, w_2(A), \quad \forall A \subset \Omega.$$

We thus have:

$$m_{1 \textcircled{\scriptsize U}_w 2} = \textcircled{\cap}_{A \subset \Omega} A^{w_1(A) \, \mathcal{U} \, w_2(A)}.$$

Proposition 4.6. *Any conjunctive u-rule $\textcircled{\scriptsize U}_w$ is commutative, associative, monotonic with respect to $\sqsubseteq_w$, and such that: $m \textcircled{\scriptsize U}_w m_\Omega = m$, for all m.*

[1] The idea of using the setting of uninorms to generalize the product of weights was first mentioned in [21].

Proof. These properties follow directly from corresponding properties of the uninorm $\mathcal{U}$.

□

From this proposition it is clear that for any conjunctive u-rule $ⓤ_w$, the algebraic structure $(\mathcal{M}_{nd}, ⓤ_w, \sqsubseteq_w)$ is a commutative, partially ordered monoid, with the vacuous BBA as neutral element.

The next proposition shows that the TBM conjunctive rule has a special position in the family of the conjunctive u-rules.

Proposition 4.7. *Among all conjunctive u-rules, the TBM conjunctive rule is the x-least committed, with $x \in \{w, s, pl, q, I\}$:*

$$m_1 ⓤ_w m_2 \ \sqsubseteq_x \ m_1 \cap\!\!\!\!\bigcirc m_2, \quad \forall m_1, m_2.$$

Proof. From the definition of the conjunctive u-rules, we have $m_1 ⓤ_w m_2 \sqsubseteq_w m_1 \cap\!\!\!\!\bigcirc m_2$ for all nondogmatic BBAs m_1 and m_2. The result follows then directly from (2.19).

□

Finally, we complete the characterization of the conjunctive u-rules by the following remark.

Remark 4.1. *Conjunctive u-rules are not idempotent.*

Proof. This follows from the fact that idempotence and having the vacuous BBA as neutral element are incompatible properties for w-based rules. Indeed, from Proposition 3.1 and Theorem 3.1, a w-based rule that as the vacuous BBA as neutral element is based on a binary operator $\circ$ satisfying $x \circ y \leq xy$, for all $x, y > 0$. Let $z \in (0, 1)$, we have $z \circ z \leq z^2 < z$, hence $\circ$ is not idempotent. □

In order to obtain other conjunctive u-rules than the TBM conjunctive rule, one needs 1-uninorms $\mathcal{U}$ such that $x \ \mathcal{U} \ y \leq xy$ for all $x, y \in (0, +\infty]$. Proposition 4.8 below gives a construction mechanism to obtain such 1-uninorms out of t-norms on $[0, 1]$.

Proposition 4.8. *Let $\top$ be a positive t-norm on $[0, 1]$ verifying $x \top y \leq xy$ for all $x, y \in [0, 1]$, and let $\top'$ be a t-norm on $[0, 1]$ verifying $x \top' y \geq xy$ for all $x, y \in [0, 1]$. Then the operator $\mathcal{U}_{(\top,\top')}$ defined by*

$$x \, \mathcal{U}_{(\top,\top')} \, y \;=\; \begin{cases} x \top y & \text{if } x \vee y \leq 1, \\ ((1/x) \top' (1/y))^{-1} & \text{if } x \wedge y \geq 1, \\ x \wedge y & \text{otherwise,} \end{cases} \tag{4.5}$$

for all $x, y \in (0, +\infty]$ is a uninorm on $(0, +\infty]$ having 1 as neutral element and verifying $x \, \mathcal{U}_{(\top,\top')} \, y \leq xy$ for all $x, y \in (0, +\infty]$.

Proof. The operator $\mathcal{U}_{(\top,\top')}$ defined by (4.5) is simply noted $\mathcal{U}$ in this proof. Using the proof of Proposition 4.2, it may be shown that the operator $\mathcal{U}$ is a uninorm on $(0, +\infty]$ having 1 as neutral element. Hence, we merely have to show that $x \ \mathcal{U} \ y \leq xy$ holds for all $x, y \in (0, +\infty]$, i.e., $\mathcal{U}$ is below the product.

- Suppose $x \,\mathcal{U}\, y = x \top y$. Then $x \,\mathcal{U}\, y \leq xy$ by the definition of $\top$.
- Suppose $x \,\mathcal{U}\, y = ((1/x) \top' (1/y))^{-1}$. By the definition of $\top'$, we have

$$\begin{aligned} (1/x) \top' (1/y) &\geq (1/x)(1/y) \\ ((1/x) \top' (1/y))^{-1} &\leq ((1/x)(1/y))^{-1} = xy. \end{aligned}$$

- Suppose $x \,\mathcal{U}\, y = x \wedge y$. Hence, we have $x \vee y > 1$ and $x \wedge y < 1$, and thus $x \wedge y \leq xy$ clearly holds.

□

The construction mechanism provided by Proposition 4.8 is illustrated by the following example.

Example 4.3. *The* 1*-uninorm* $\mathcal{U}$ *defined by*

$$x \,\mathcal{U}\, y = \begin{cases} x \cdot y & \text{if } x \vee y \leq 1, \\ ((1/x) \wedge (1/y))^{-1} & \text{if } x \wedge y \geq 1, \\ x \wedge y & \text{otherwise}, \end{cases}$$

for all $x, y \in (0, +\infty]$*, is obtained by setting* $\top = \cdot$ *(product) and* $\top' = \wedge$*, with* $\top$ *and* $\top'$ *the t-norms involved in Proposition 4.8.*

Given a positive t-norm $\top$ on $[0,1]$ verifying $x \top y \leq xy$ for all $x, y \in [0,1]$, and a t-norm $\top'$ on $[0,1]$ verifying $x \top' y \geq xy$ for all $x, y \in [0,1]$, Proposition 4.8 allows us to obtain a conjunctive u-rule based on the 1-uninorm $\mathcal{U}_{(\top,\top')}$ defined by (4.5). This conjunctive u-rule is noted $\circledcirc_w^{\top,\top'}$. Explicitly, we have

$$m_1 \circledcirc_w^{\top,\top'} m_2 = \bigcircledcap_{A \subset \Omega} A^{w_1(A) \,\mathcal{U}_{(\top,\top')} w_2(A)}. \tag{4.6}$$

4.5 Disjunctive T-Rules and U-Rules

For the sake of completeness, this section presents results corresponding to the previous ones for v-based rules. Obvious results are stated succinctly, whereas De Morgan relations between conjunctive and disjunctive rules are more detailed.

4.5.1 Disjunctive t-rules

The bold rule is based on the minimum. Hence, it belongs to a family of rules based on pointwise combination of disjunctive weights using t-norms on $(0, +\infty]$. The counterpart of Proposition 4.3 for disjunctive weights allows us to define a belief function combination rule $\circledT_v$, called a disjunctive t-rule, as

$$m_{1 \circledT_v 2} = \bigcircledcup_{A \neq \emptyset} A_{v_1(A) \,\mathcal{T}\, v_2(A)},$$

where $\mathcal{T}$ is a t-norm on $(0, +\infty]$, and m_1 and m_2 are two subnormal BBAs.

Any disjunctive t-rule $\circledT_v$ is commutative, associative and monotonic with respect to $\sqsubseteq_v$. Hence $(\mathcal{M}_s, \circledT_v, \sqsubseteq_v)$ is a commutative, partially ordered semigroup.

Furthermore, it may easily be shown, using similar arguments as developed in Section 4.3, that the bold rule is the x-most committed disjunctive t-rule, with $x \in \{v, s, pl, q, I\}$.

Finally, the following proposition shows that the $\circledT_w$ and $\circledT_v$ operations are dual to each other with respect to complementation, i.e., they are linked by De Morgan laws analogous to (2.23) and (2.24).

Proposition 4.9. *Let $\circledT_w$ and $\circledT_v$ be respectively, conjunctive and disjunctive t-rules based on a t-norm $\mathcal{T}$ on $(0, +\infty]$. We have:*

$$\overline{m_1 \circledT_v m_2} = \overline{m_1} \circledT_w \overline{m_2},$$

for all subnormal BBAs m_1 and m_2, and

$$\overline{m_1 \circledT_w m_2} = \overline{m_1} \circledT_v \overline{m_2}, \qquad (4.7)$$

for all nondogmatic BBAs m_1 and m_2.

Proof. Let m_1 and m_2 be two subnormal BBAs. We have

$$\begin{aligned}
\overline{m_1 \circledT_v m_2} &= \overline{\textcircled{\cup}_{A \neq \emptyset} A_{v_1(A) \,\mathcal{T}\, v_2(A)}} \\
&= \textcircled{\cap}_{A \neq \emptyset} \overline{A_{v_1(A) \,\mathcal{T}\, v_2(A)}} \\
&= \textcircled{\cap}_{A \neq \emptyset} \bar{A}^{\overline{w_1}(\bar{A}) \,\mathcal{T}\, \overline{w_2}(\bar{A})} \\
&= \textcircled{\cap}_{A \subset \Omega} A^{\overline{w_1}(A) \,\mathcal{T}\, \overline{w_2}(A)} \\
&= \overline{m_1} \circledT_w \overline{m_2}.
\end{aligned}$$

The proof of (4.7) is similar.

□

4.5.2 Disjunctive u-rules

The TBM disjunctive rule is based on the product of disjunctive weights. Hence, it belongs to a family of rules defined by pointwise combination of disjunctive weights using 1-uninorms. From Corollary 3.2, the condition that those uninorms must respect is known. We may thus define a belief function combination rule $\circledU_v$, called a disjunctive u-rule, as

$$m_{1 \circledU_v 2} = \textcircled{\cup}_{A \neq \emptyset} A_{v_1(A) \,\mathcal{U}\, v_2(A)},$$

where $\mathcal{U}$ is a 1-uninorm, such that $x \,\mathcal{U}\, y \leq xy$ for all $x, y \in (0, +\infty)$, and where m_1 and m_2 are two subnormal BBAs.

Any disjunctive u-rule $\circledU_v$ is commutative, associative, monotonic with respect to $\sqsubseteq_v$ and has the BBA $m_\emptyset$ as neutral element. Hence $(\mathcal{M}_s, \circledU_v, \sqsubseteq_v)$ is a commutative, partially ordered monoid. Furthermore, the TBM disjunctive rule is the x-most committed disjunctive u-rule, with $x \in \{v, s, pl, q, I\}$.

Finally, the following proposition shows that the $\circledU_w$ and $\circledU_v$ operations are dual to each other with respect to complementation.

Proposition 4.10. *Let* $\textcircled{u}_w$ *and* $\textcircled{u}_v$ *be respectively, conjunctive and disjunctive u-rules based on a* 1*-uninorm* $\mathcal{U}$*. We have:*

$$\overline{m_1 \textcircled{u}_v m_2} = \overline{m_1} \textcircled{u}_w \overline{m_2}, \tag{4.8}$$

for all subnormal BBAs m_1 *and* m_2*, and*

$$\overline{m_1 \textcircled{u}_w m_2} = \overline{m_1} \textcircled{u}_v \overline{m_2}, \tag{4.9}$$

for all nondogmatic BBAs m_1 *and* m_2*.*

Proof. The proof of (4.8) is direct using the proof of Proposition 3.2. The proof of (4.9) is similar.

□

4.6 Application to Classification Problems

Besides their nice mathematical properties, one may wonder if the families of rules defined in this chapter might be useful in some applications. In this section, we present two classification applications showing that the conjunctive t-rules are more efficient on some data sets than the TBM conjunctive rule and the cautious rule. We may already remark that, in both of these applications, the following idea will be used. The existence of parameterized families of t-norms on $[0, 1]$ [49] makes it possible, using Proposition 4.1, to obtain parameterized families of t-norms on $(0, \infty]$, which in turn lead to parameterized families of conjunctive t-rules. Considering a classification system based on a parameterized family of conjunctive t-rules, one may learn the conjunctive t-rule in this family that will give the best classification results on new data, by selecting the rule optimizing the performances of the classification system on a set of data whose actual class is known.

4.6.1 Classifier fusion

Overall scheme

Consider a classification problem[2] with Q classes and D continuous features $X_1, ..., X_D$. The set of classes is denoted by $\Omega = \{\omega_1, ...\omega_Q\}$. Assume that the available information is a training set $\mathcal{L}$ of N objects $\mathbf{x}^n = (x_1^n, ..., x_D^n)$, $n = 1, ..., N$, whose class labels are singletons and are represented by vectors $\mathbf{u}^n = (u_1^n, ..., u_Q^n)$, $n = 1, ..., N$, of binary indicator variables u_q^n: we have $u_q^n = 1$ if object $\mathbf{x}^n$ belongs to class ω_q, and $u_q^n = 0$ otherwise. Suppose that a classifier is trained using $\mathcal{L}$ on each feature. We thus have D classifiers, which we denote by $C_1, ..., C_D$. Eventually, suppose that the values $x_1^s, ..., x_D^s$ of features $X_1, ..., X_D$ have been observed for a new object $\mathbf{x}^s = (x_1^s, ..., x_D^s)$, whose class is unknown, and that classifiers $C_1, ..., C_D$ are able to provide partial information on the class of object $\mathbf{x}^s$ in the form of D u-separable BBAs $m_1, ..., m_D$. Given the information provided by the classifiers, we wish to classify $\mathbf{x}^s$.

[2]The formulation of this problem is inspired from [18, Section 6.1].

In [70], this problem of classifier fusion was considered. In order to classify $\mathbf{x}^s$, Quost et al. [70] proposed to express first our belief $m[\mathbf{x}^s]$ on Ω concerning the class of object $\mathbf{x}^s$ and then, assuming $\{0,1\}$ costs[3], to assign $\mathbf{x}^s$ to the class of maximal pignistic probability (draws are resolved by randomly choosing a class among the classes with equal pignistic probability). In [70], various ways to obtain the BBA $m[\mathbf{x}^s]$ were proposed. These different ways are summarized below.

Provided that the classifiers can be assumed to be distinct, $m[\mathbf{x}^s]$ may be obtained by combining the BBAs $m_1, ..., m_D$ using the TBM conjunctive rule. In this case, we have

$$\begin{aligned} m[\mathbf{x}^s] &= \textcircled{\cap}_{d=1}^{D} m_d \\ &= \textcircled{\cap}_{A\subset\Omega} A^{\prod_{d=1}^{D} w_d(A)}. \end{aligned} \tag{4.10}$$

If the classifiers cannot be assumed distinct, we may use the cautious rule to combine the classifier outputs:

$$\begin{aligned} m[\mathbf{x}^s] &= \textcircled{\wedge}_{d=1}^{D} m_d \\ &= \textcircled{\cap}_{A\subset\Omega} A^{\bigwedge_{d=1}^{D} w_d(A)}. \end{aligned} \tag{4.11}$$

Since the classifiers C_d are assumed to produce u-separable BBAs, we may remark that the weights $w_d(A)$ are all smaller or equal to 1. Hence, the weights $w_d(A)$ are combined in (4.10) using the product t-norm on $[0,1]$ and in (4.11) using the minimum t-norm on $[0,1]$. Using a conjunctive t-rule $\textcircled{\scriptsize T}_w^{\top,\top'}$ such as introduced in Section 4.3 (Equation (4.4)) amounts to combining the $w_d(A)$ using the t-norm $\top$ on $[0,1]$, i.e., we have

$$\begin{aligned} m[\mathbf{x}^s] &= m_1 \textcircled{\scriptsize T}_w^{\top,\top} ... \textcircled{\scriptsize T}_w^{\top,\top} m_D \\ &= \textcircled{\cap}_{A\subset\Omega} A^{\top_{d=1}^{D} w_d(A)}. \end{aligned} \tag{4.12}$$

Since the result of the combination by $\textcircled{\scriptsize T}_w^{\top,\top'}$ does not depend on $\top'$ in this setting, $\textcircled{\scriptsize T}_w^{\top,\top'}$ will be simply noted $\textcircled{\scriptsize T}_w^{\top}$ in the sequel. If $\top$ is chosen in the Frank family of parameterized t-norms defined, for all $x, y \in [0,1]$, by

$$x \top_\lambda y = \begin{cases} x \wedge y & \text{if } \lambda = 0, \\ xy & \text{if } \lambda = 1, \\ \log_\lambda\left(1 + \frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}\right) & \text{otherwise,} \end{cases}$$

where λ is a positive parameter, then the TBM conjunctive rule is recovered for $\lambda = 1$, whereas the cautious rule is recovered for $\lambda = 0$. Choosing λ between 0 and 1 results in a combination rule $\textcircled{\scriptsize T}_w^{\top_\lambda}$ somewhere between these two rules.

In [70], Quost et al. proposed a method to learn the value $\hat{\lambda}$ of the parameter λ that will give the best classification results on new data. Their method consists in using a validation set of M objects and finding the value $\hat{\lambda}$ of the parameter λ that minimizes the error criterion $Err(\lambda)$ defined by:

$$Err(\lambda) = \sum_{m=1}^{M} \sum_{q=1}^{Q} \left(BetP^m(\{\omega_q\}) - u_q^m\right)^2, \tag{4.13}$$

[3]See Section 1.5 for basic material on decision theory.

where u_q^m is the $0-1$ class membership indicator variable for object $\mathbf{x}^m$ in class ω_q, and where $BetP^m$ is the pignistic probability measure computed from the BBA $m[\mathbf{x}^m]$, which is in turn given by

$$m[\mathbf{x}^m] \quad = \quad m_1[\mathbf{x}^m] \circledT_w^{\top_\lambda} ... \circledT_w^{\top_\lambda} m_D[\mathbf{x}^m],$$

where $m_d[\mathbf{x}^m]$ is the BBA, provided by classifier C_d, on the class of object $\mathbf{x}^m$.

Numerical experiments

Quost et al. [70] considered five synthetic data sets and five real-world data sets to experiment with this classifier fusion scheme. The general procedure may be summarized as follows. Each data set was divided into two sets: a training set and a test set. The D classifiers C_d were learnt on each variable using the training set. For each object of the test set, the outputs of the D classifiers were combined using the TBM conjunctive rule, the cautious rule and a learnt conjunctive t-rule $\circledT_w^{\top_{\hat\lambda}}$ somewhere between these two rules. The value $\hat\lambda$ involved in the conjunctive t-rule $\circledT_w^{\top_{\hat\lambda}}$ was learnt by taking the average of several values of the error defined by (4.13) computed on several validation sets[4]. These validation sets were generated apart in the case of the synthetic data sets or obtained from the training sets via 5×2 cross-validation in the case of real-world data sets.

Over the ten data sets considered, a conjunctive t-rule different from the TBM conjunctive rule was always learnt, and on four data sets the cautious rule was learnt. The test error rates of the TBM conjunctive rule were lower than the test error rates of the learnt conjunctive t-rules on only one synthetic data set and one real-world data set. However, on these two data sets, the test error rates of the TBM conjunctive rule and those of the learnt conjunctive t-rules were not judged significantly different by a McNemar test [22] at level 5%. On five out of the remaining eight data sets, where the test error rates of the learnt conjunctive t-rules were strictly lower than the test error rates of the TBM conjunctive rule, the test error rates of the learnt conjunctive t-rules were judged significantly different from the test error rates of the TBM conjunctive rule by a McNemar test at level 5%. In summary, Quost et al. [70] showed that the conjunctive t-rules and the cautious rule can lead, in some instances of this classifier fusion application, to better classification performances than the TBM conjunctive rule.

As a concluding remark to this first classification application, we may note that the classifier fusion scheme studied in [70] was further refined in [71], where classifiers were clustered according to some measure of similarity. Interestingly, the work presented in [71] confirmed that the TBM conjunctive rule can be outperformed by the cautious rule. In addition, it was shown in [71] that using multiple conjunctive t-rules to combine the classifiers C_d can lead to better classification results than using merely the TBM conjunctive rule or the cautious rule or a single conjunctive t-rule.

[4]Note that the search space for $\hat\lambda$ was restrained in [70] to the interval $[0,1]$. This is the reason why the learnt conjunctive t-rule $\circledT_w^{\top_{\hat\lambda}}$ was an intermediate between the TBM conjunctive rule and the cautious rule.

4.6.2 Evidential K-nearest neighbor classification rule

Overall scheme

In [12], Denœux proposed an evidential version of the K-nearest neighbor classification rule [35]. This method works as follows.

Similarly to the previous section, consider a classification problem with set of classes $\Omega = \{\omega_1, ..., \omega_Q\}$ and continuous features $X_1, ..., X_D$. Assume that the available information is a training set $\mathcal{L}$ of N objects $\mathbf{x}^n$, $n = 1, ..., N$, whose class labels are indicated by vectors $\mathbf{u}^n$, $n = 1, ..., N$, of binary indicator variables. Eventually, suppose that the values $x_1^s, ..., x_D^s$ of features $X_1, ..., X_D$ have been observed for a new object $\mathbf{x}^s = (x_1^s, ..., x_D^s)$, whose class is unknown. Given the training set information, we wish to classify $\mathbf{x}^s$. Using the algorithm proposed by Denœux [12], we may obtain an expression of our belief on the class of $\mathbf{x}^s$ in the form of a BBA $m[\mathbf{x}^s]$. As was the case in the classifier fusion application, this BBA can then be transformed into a pignistic probability measure in order to classify $\mathbf{x}^s$. We review below how $m[\mathbf{x}^s]$ is built from the training set information using Denœux' scheme.

Suppose that object $\mathbf{x}^n$, $n \in \{1, ..., N\}$, of the training set is close (according to some relevant distance measure δ) in feature space to $\mathbf{x}^s$. Denœux [12] proposed to regard the knowledge $u_q^n = 1$ as a piece of evidence that increases, to some degree, our belief that $\mathbf{x}^s$ also belongs to class ω_q. In the TBM, such a belief may be modeled by a BBA $m[\mathbf{x}^s|\mathbf{x}^n]$ focused on ω_q and Ω, due to the fact that $u_q^n = 1$ only points to the hypothesis ω_q. Furthermore, the mass assigned to ω_q can reasonably be assumed to be a decreasing function of $\delta(\mathbf{x}^s, \mathbf{x}^n)$, the distance between $\mathbf{x}^s$ and $\mathbf{x}^n$. We then arrive at the following expression for $m[\mathbf{x}^s|\mathbf{x}^n]$:

$$m[\mathbf{x}^s|\mathbf{x}^n](A) = \begin{cases} \beta\varphi_q[\delta(\mathbf{x}^s, \mathbf{x}^n)] & \text{if } A = \{\omega_q\}, \\ 1 - \beta\varphi_q[\delta(\mathbf{x}^s, \mathbf{x}^n)] & \text{if } A = \Omega, \\ 0 & \text{otherwise,} \end{cases} \tag{4.14}$$

where β is a parameter such that $0 < \beta < 1$ and φ_q is a decreasing function verifying $\varphi_q(0) = 1$ and $\lim_{\delta\to\infty} \varphi_q(\delta) = 0$ [12]. Note that the index q of φ_q indicates that the influence of $\delta(\mathbf{x}^s, \mathbf{x}^n)$ may depend on the class of $\mathbf{x}^n$. When δ denotes the euclidean distance, a rational choice for φ_q was shown in [13] to be:

$$\varphi_q(\delta) = e^{-\gamma_q \delta^2}, \tag{4.15}$$

where γ_q is a positive parameter associated to class ω_q.

As $m[\mathbf{x}^s|\mathbf{x}^n]$ is a simple BBA, we may obtain the following simple expression for $m[\mathbf{x}^s|\mathbf{x}^n]$, which will be useful in the remainder of this section:

$$m[\mathbf{x}^s|\mathbf{x}^n] = \left\{\omega_{(n)}\right\}^{d^{s,n}},$$

where $\omega_{(n)}$ denotes the class of object $\mathbf{x}^n$ and with $d^{s,n} = 1 - \beta\varphi_{(n)}[\delta(\mathbf{x}^s, \mathbf{x}^n)]$, where the index (n) of $\varphi_{(n)}$ is given by the class of object $\mathbf{x}^n$, e.g., if the class of object $\mathbf{x}^n$ is ω_q, then $\varphi_{(n)} = \varphi_q$.

As a result of considering the N training objects, we obtain N BBAs. Regarding these training objects as distinct items of evidence, we may use the TBM conjunctive rule to combine these N BBAs in order to produce a global BBA $m[\mathbf{x}^s|\mathcal{L}]$

representing our belief on the class of $\mathbf{x}^s$:

$$\begin{aligned} m[\mathbf{x}^s|\mathcal{L}] &= \textcircled{\cap}_{n=1}^{N} m[\mathbf{x}|\mathbf{x}^n] \\ &= \textcircled{\cap}_{q=1}^{Q} \left(\textcircled{\cap}_{\{n,u_q^n=1\}} \{\omega_q\}^{d^{s,n}} \right) \\ &= \textcircled{\cap}_{q=1}^{Q} \{\omega_q\}^{\prod_{\{n,u_q^n=1\}} d^{s,n}} . \end{aligned}$$

To improve the computational efficiency, we may modify this latter equation so as to consider only the K nearest neighbors of $\mathbf{x}^s$ in the feature space. We then have:

$$m[\mathbf{x}^s|K] = \textcircled{\cap}_{q=1}^{Q} \{\omega_q\}^{\prod_{\{n \in S_K(\mathbf{x}^s), u_q^n=1\}} d^{s,n}} ,$$

where $S_K(\mathbf{x}^s)$ denotes the set of K nearest neighbors of $\mathbf{x}^s$.

When the training objects cannot be considered distinct, we may use the cautious rule instead of the TBM conjunctive rule to pool the pieces of evidence. We then have:

$$\begin{aligned} m[\mathbf{x}^s|\mathcal{L}] &= \textcircled{\wedge}_{n=1}^{N} m[\mathbf{x}|\mathbf{x}^n] \\ &= \textcircled{\wedge}_{q=1}^{Q} \left(\textcircled{\wedge}_{\{n,u_q^n=1\}} \{\omega_q\}^{d^{s,n}} \right) \\ &= \textcircled{\wedge}_{q=1}^{Q} \{\omega_q\}^{\bigwedge_{\{n,u_q^n=1\}} d^{s,n}} . \\ &= \textcircled{\cap}_{q=1}^{Q} \{\omega_q\}^{\bigwedge_{\{n,u_q^n=1\}} d^{s,n}} . \end{aligned} \tag{4.16}$$

The last line comes from $A^x \textcircled{\wedge} B^y = A^x \textcircled{\cap} B^y$ if $A \neq B$. From (4.16), we may remark that pooling the pieces of evidence using the cautious rule actually amounts to considering only the nearest neighbor of $\mathbf{x}^s$ in each class.

As was done in the previous section, we may consider using the conjunctive t-rule $\textcircled{\scriptsize T}_w^{\top,\top'}$ introduced in Section 4.3 instead of the conjunctive rule and the cautious rule. Since the $d^{s,n}$ are smaller than 1, using $\textcircled{\scriptsize T}_w^{\top,\top'}$ amounts to combining the $d^{s,n}$ using the t-norm $\top$ on $[0,1]$, i.e., we have

$$m[\mathbf{x}^s|K] = \textcircled{\cap}_{q=1}^{Q} \{\omega_q\}^{\top_{\{n \in S_K(\mathbf{x}^s), u_q^n=1\}} d^{s,n}} . \tag{4.17}$$

If $\top$ is chosen in the Dubois and Prade family of parameterized t-norms defined, for all $x, y \in [0,1]$, by

$$x \top_\theta y = \frac{xy}{\max(x, y, \theta)},$$

where $\theta \in [0,1]$, then the TBM conjunctive rule is recovered for $\theta = 1$, whereas the cautious rule is recovered for $\theta = 0$.

Numerical experiments

A comparison between Denœux' original scheme based on the TBM conjunctive rule and Denœux' extended scheme as defined by (4.17) (with $\top$ of (4.17) chosen in the Dubois and Prade family of parameterized t-norms), was performed using real-world classification problems. Before presenting the results of some of these experiments, practical issues related to the implementation of these schemes need to be addressed. Indeed, the parameters K, β, θ and γ_q, $q = 1, ..., Q$ have to be fixed in order to allow the use of the schemes.

A method for optimizing the parameters β and γ_q has been described in [110]. However, the results that will be presented in this section are based on the following simple heuristic used in [12]: $\beta = 0.95$ and γ_q is equal to the inverse of the mean distance between training patterns belonging to class ω_q. We also choose $K = 30$ in order to reduce computations, yet still considering relatively many neighbors. Eventually, the parameter θ is left free for the moment, since the purpose of the two following experiments is to compare the original evidential K-nearest neighbor classification rule with its extended version.

Experiment 1

In this first experiment, we considered three real-world data sets[5]: Cleveland heart disease, mammographic mass, and vehicle silhouettes. The first two data sets are two-class problems, and the last data set is a four-class problem. For each of these data sets, we computed the leave-one-out (LOO) cross-validation error rate of the extended scheme for different values of θ: $\theta = 0, 0.1, ..., 1$. These LOO error rates are given in Figures 4.1 to 4.3. Let us stress that, in these figures, the figure shown for $\theta = 0$ is the LOO error rate of the cautious rule, and the figure shown for $\theta = 1$ is the LOO error rate of the TBM conjunctive rule.

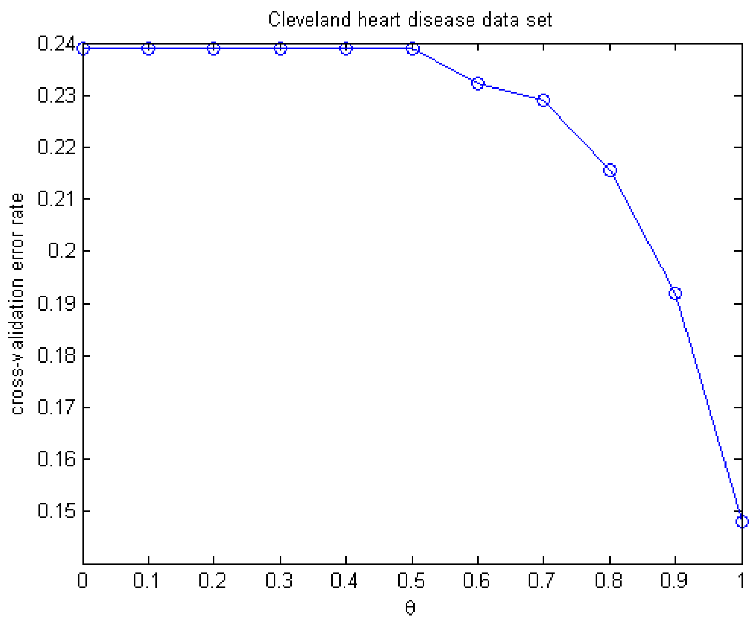

Figure 4.1: Cleveland heart disease data set. Best performance obtained for $\theta = 1$.

[5]These data sets were obtained from the UCI Machine Learning repository at http://archive.ics.uci.edu/ml. Patterns with missing feature values in the used data sets were discarded.

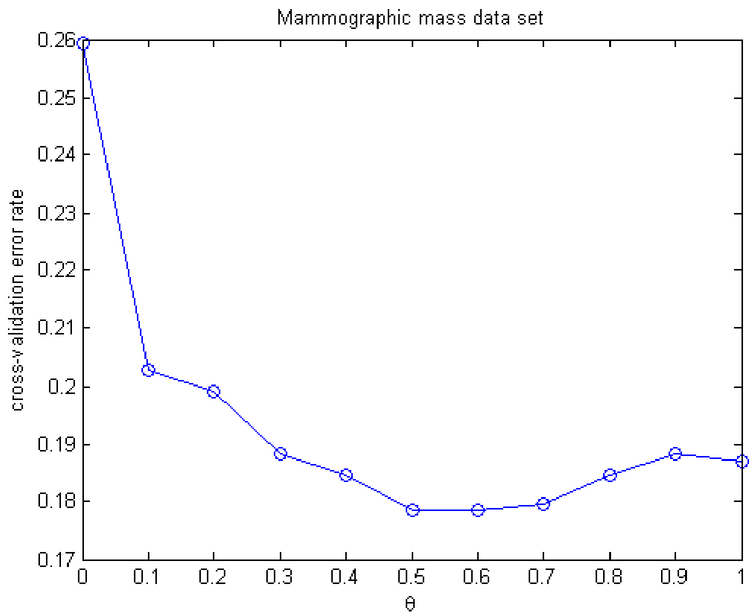

Figure 4.2: Mammographic mass data set. Best performance obtained for $\theta = 0.5$.

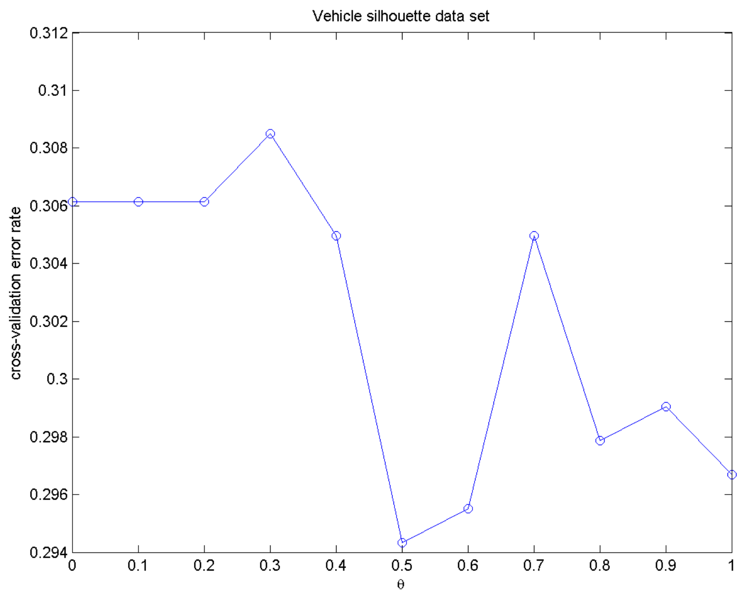

Figure 4.3: Vehicle silhouettes data set. Best performance obtained for $\theta = 0.5$.

As can be seen in Figures 4.2 and 4.3, a conjunctive t-rule may have a lower LOO error rate than the TBM conjunctive rule, which shows that the family of conjunctive t-rules is useful in this classification application. Note that on the Cleveland heart disease data set, the lowest LOO error rate is obtained for the TBM conjunctive rule, i.e., the conjunctive t-rules between the cautious rule and the TBM conjunctive rule do not lead to an improvement for this particular data set. Let us already remark that, in Chapter 6, we will show that on this latter data set, it is possible to obtain with another family of rules than the conjunctive t-rules, a lower LOO error rate than the one obtained with the TBM conjunctive rule.

Experiment 2

This experiment is similar to the one conducted by Quost et al. to test their classifier fusion scheme and reported in the previous section: we first split several real-world data sets into a training set and a test set, and then we computed the test error rates of the evidential K-nearest neighbor classification schemes based (1) on the TBM conjunctive rule, (2) the cautious rule and (3) a conjunctive t-rule based on a t-norm $\top_{\hat{\theta}}$ learnt using the training set. For each of these real-world data sets, the t-norm $\top_{\hat{\theta}}$ learnt was chosen as the one minimizing the LOO error rate on the training set (the parameter space was restricted to $\theta = 0, 0.1, ..., 1$).

We used the four following data sets of the UCI Machine Learning repository: ionosphere, liver disorders, wine, and segment. The first two data sets are two-class problems, the third data set is a three-class problem and the last data set is a seven-class problem. These four data sets were split into a training set and a test set with the following proportions: 2/3 of the data went into the training set and the remaining 1/3 of the data were used as test data.

Error rates obtained with the TBM conjunctive rule, the cautious rule, and the learnt conjunctive t-rule corresponding to the optimal parameter value $\hat{\theta}$ are provided in Table 4.1, together with 95% confidence intervals. The best result is underlined. Whenever a result obtained with the cautious rule or a learnt conjunctive t-rule is significantly different by a McNemar test at level 5%, from the result obtained with the TBM conjunctive rule, it is printed in bold.

Table 4.1: Error rates of the TBM conjunctive rule, the cautious rule, and the learnt conjunctive t-rule, together with 95% confidence intervals.

Data	TBM conjunctive rule	Cautious rule	Learnt conjunctive t-rule
Ionosphere	0.1466 ± 0.0644	0.1121 ± 0.0574	$\underline{\mathbf{0.0948}} \pm 0.0533$ ($\hat{\theta} = 0.8$)
Liver disorders	$\underline{0.3130} \pm 0.0848$	0.3391 ± 0.0865	$\underline{0.3130} \pm 0.0848$ ($\hat{\theta} = 1$)
Wine	$\underline{0.0333} \pm 0.0454$	$\underline{0.0333} \pm 0.0454$	$\underline{0.0333} \pm 0.0454$ ($\hat{\theta} = 1$)
Segment	0.0870 ± 0.0199	$\underline{\mathbf{0.0519}} \pm 0.0157$	$\underline{\mathbf{0.0519}} \pm 0.0157$ ($\hat{\theta} = 0$)

The first striking remark that can be made from Table 4.1, is that the classification results obtained with the learnt conjunctive t-rules are always at least as good as the classification results obtained with the TBM conjunctive rule and the cautious rule. When we look into the details, we see that the conjunctive t-rule learnt on the liver disorders and wine data sets was the TBM conjunctive rule. Hence, the conjunctive t-rules between the cautious rule and the TBM conjunctive rule do not bring any improvement for these two data sets. However, on the other two data sets (ionosphere and segment), the cautious rule and an intermediate rule were learnt, yielding better classification results than the TBM conjunctive rule. Furthermore, the results obtained with the cautious rule in the case of the segment data set or with the intermediate rule in the case of the ionosphere data set, are significantly different from the ones obtained with the TBM conjunctive rule. In summary, we have yet another experimental evidence that the conjunctive t-rules are useful in a classification application. Eventually, let us already remark that on the very data sets where the TBM conjunctive rule is not outperformed by the conjunctive t-rules, it is possible to find some rules, belonging to another family of rules than the conjunctive t-rules, that do outperform the TBM conjunctive rule; this result will be presented in Chapter 6.

4.6.3 Limitations of the experiments: discussion

The numerical experiments reported in the previous two sections have shown that the conjunctive t-rules may improve the performances in two classification applications. However, these experiments suffer from at least three limitations. The first important remark is that the BBAs involved in these applications were all u-separable. Hence, these experiments have actually shown the usefulness of conjunctive t-rules, only for classification schemes based on combination of u-separable BBAs. This may be seen as a limitation on the good results reported, since classification schemes based on combination of nonseparable BBAs were not studied. However, it may be argued that this limitation is not so important since most BBAs encountered in practice are u-separable. This is for instance the case with the method proposed in [57], where the confusion matrix of a classifier is used to transform the classifications made by this classifier into consonant BBAs, or with the supervised classification version of the evidential K-nearest neighbor classification rule[6] [12] or with the TBM model-based classifier [20]. This is also the case when one transforms the probabilistic output of a classifier into a consonant BBA, by choosing the q-least committed BBA among the set of so-called isopignistic BBAs[7] [34], as done in [70].

To our knowledge, the only situations where one may encounter nonseparable BBAs are (1) the expert opinions elicitation method proposed in [5], (2) the BBA construction mechanism proposed in [17] and based on the observation of a realization of an independent, identically distributed random sample from the same distribution, and (3) the application of the ballooning extension as done in [72] where a BBA m defined on a frame Ω is built from a BBA m' defined on a frame Ω' with $\Omega' \subseteq \Omega$. Admittedly, we have not yet been able to design a realistic classification experiment, which would use standard pattern recognition data sets and that would

[6]The evidential K-nearest neighbor classification rule is in its supervised classification version if the classes of the learning objects are known with certainty and precision, as is the case in Section 4.6.2.

[7]BBAs that induce the same pignistic probability measure are called isopignistic.

require combining nonseparable BBAs. The only classification experiment, involving the combination of nonseparable BBAs, that we could think of and that is left for future research is the following. Consider a classification problem with Q classes and D continuous features $X_1, ..., X_D$. Assume that the available information is a training set $\mathcal{L}$ of N objects, whose class labels have been determined from expert opinions using the method proposed in [5]. Hence, the class labels of the training objects are in the form of BBAs, which may possibly be nonseparable. Eventually, suppose that the values $x_1^s, ..., x_D^s$ of features $X_1, ..., X_D$ have been observed for a new object $\mathbf{x}^s$, whose class is unknown. Given the training set information, we wish to classify $\mathbf{x}^s$. Using a more general version of the evidential K-nearest neighbor classification rule than the version used in this chapter (see [12, Section 3.3] for this general version), it is possible to obtain a BBA $m[\mathbf{x}^s]$ expressing our beliefs on the class of $\mathbf{x}^s$, and thus $\mathbf{x}^s$ may then be classified using, e.g., the pignistic transformation. The computation of $m[\mathbf{x}^s]$ using this general version of the evidential K-nearest neighbor classification rule, involves combining with the TBM conjunctive rule, the possibly nonseparable BBAs associated to the training objects. The TBM conjunctive rule could then be replaced by the new conjunctive rules defined in this chapter, in order to investigate whether these new rules lead to better performances than the TBM conjunctive rule, when nonseparable BBAs are involved.

The second limitation of the experiments presented above is that they do not provide an answer to the following question: are the conjunctive u-rules also useful for classification applications? The numerical experiments that we have conducted, use indeed conjunctive t-rules only. The reason behind this choice is that the conjunctive t-rules $\circledT_w^{\top,\top'}$ defined by (4.4) are more relevant for classification tasks involving u-separable BBAs, than the conjunctive u-rules $\circledT_w^{\top,\top'}$ defined by (4.6). This claim may be motivated as follows.

Proposition 4.11. *Let $\top$ be a positive t-norm on $[0,1]$ verifying $x\top y \leq xy$ for all $x, y \in [0,1]$, and let $\top'$ be a t-norm on $[0,1]$ verifying $x\top' y \geq xy$ for all $x, y \in [0,1]$. Let $\circledT_w^{\top,\top'}$ be a conjunctive t-rule based on the t-norm $\mathcal{T}_{(\top,\top')}$ defined by (4.1), and let $\circledU_w^{\top,\top'}$ be a conjunctive u-rule based on the 1-uninorm $\mathcal{U}_{(\top,\top')}$ defined by (4.5). Let m_1 and m_2 be two u-separable BBAs. We have:*

$$m_1 \circledT_w^{\top,\top'} m_2 = m_1 \circledU_w^{\top,\top'} m_2. \tag{4.18}$$

Proof. Since m_1 and m_2 are u-separable, we have

$$\begin{aligned} m_1 \circledT_w^{\top,\top'} m_2 &= \bigcirc\!\!\!\!\cap_{A\subset\Omega} A^{w_1(A)\top w_2(A)} \\ &= m_1 \circledU_w^{\top,\top'} m_2. \end{aligned}$$

□

This proposition means that a conjunctive t-rule based on a t-norm $\mathcal{T}(\top,\top')$ coincides, when combining u-separable BBAs, with a conjunctive u-rule based on a 1-uninorm $\mathcal{U}(\top,\top')$, if $\top$ is a positive t-norm on $[0,1]$ such that $x\top y \leq xy$, for all $x, y \in [0,1]$. Now, let us remark that the t-norm $\top$ involved in the definition of $\mathcal{T}(\top,\top')$ may be any positive t-norm on $[0,1]$, whereas the t-norm $\top$ involved in the definition of $\mathcal{U}(\top, \top')$ must be below the product. This implies that the conjunctive t-rules $\circledT_w^{\top,\top'}$ encompass, when only u-separable BBAs are considered, the conjunctive u-rules $\circledU_w^{\top,\top'}$ and thus they represent a larger spectrum of rules. In addition, we may remark that a desirable property for the combination rule involved in the experiments presented above is to have the vacuous BBA as neutral element. Even though the conjunctive t-rules $\circledT_w^{\top,\top'}$ do not possess this property in general, they do have the vacuous BBA as neutral element if they are used with u-separable BBAs only. Hence, the conjunctive t-rules should be preferred over conjunctive u-rules in classification applications that require u-separable BBAs to be combined. This is the reason why conjunctive t-rules rather than conjunctive u-rules were used in our experiments.

Note that in the case of classification schemes involving nonseparable BBAs combinations, it may argued that the conjunctive u-rules represent, theoretically, a better suited family of rules than the conjunctive t-rules. This is due to the fact that the former rules have the vacuous BBA as neutral element, which is not the case of the latter rules. This property may be regarded as essential in, e.g., the experiment involving expert opinions sketched above: if the expert cannot determine the class of a training object, which is modeled by a vacuous BBA, then this training object must not have an impact on our beliefs regarding the class of the object $\mathbf{x}^s$. In the general version of the evidential K-nearest neighbor classification rule, this latter property will be verified only if the combination rule used has the vacuous BBA as neutral element.

The third limitation of the experiments is that only the conjunctive t-rules, which are intermediate between the TBM conjunctive rule and the cautious rule, were considered. Besides looking at the combination of nonseparable BBAs, future experiments will also focus on the combination of u-separable BBAs using conjunctive t-rules based on t-norms that are below the product, i.e., conjunctive t-rules that are not intermediate between the TBM conjunctive rule and the cautious rule.

4.7 Conclusion

This chapter has brought forward that the TBM conjunctive rule $\textcircled{\cap}$ and the more recent cautious rule $\textcircled{\wedge}$ have fundamental different algebraic properties: the former is based on a uninorm on $(0, +\infty]$ and has a neutral element while the latter is based on a t-norm on $(0, +\infty]$ and has no neutral element. Similar properties hold for the disjunctive duals of these two rules, namely the TBM disjunctive rule $\textcircled{\cup}$ and the bold rule $\textcircled{\vee}$.

In addition, it was revealed that to each of those four basic rules corresponds one infinite family of combination rules. Indeed, there exist two t-norm-based families that are based, respectively, on the conjunctive and disjunctive weight functions. There exist also two uninorm-based families that are based, respectively, on the conjunctive and disjunctive weight functions. Those families of rules yield different algebraic structures: partially ordered commutative semigroups and partially ordered commutative monoids. It was also shown that t-norm-based conjunctive and disjunctive rules, as well as uninorm-based conjunctive and disjunctive rules, are related by De Morgan laws. The existence of such families of rules suggests that the TBM is not poorer than Possibility theory [27] in terms of fusion operators, as already noted in [18].

Of particular interest is that the four basic rules occupy a special position in each of their respective family: the $\textcircled{\cap}$ and $\textcircled{\wedge}$ rules are the least committed elements, whereas the $\textcircled{\cup}$ and $\textcircled{\vee}$ rules are the most committed elements. This is summarized in Figure 4.4. Eventually, numerical experiments have shown that the t-norm-based conjunctive rules may improve the performances of classification applications.

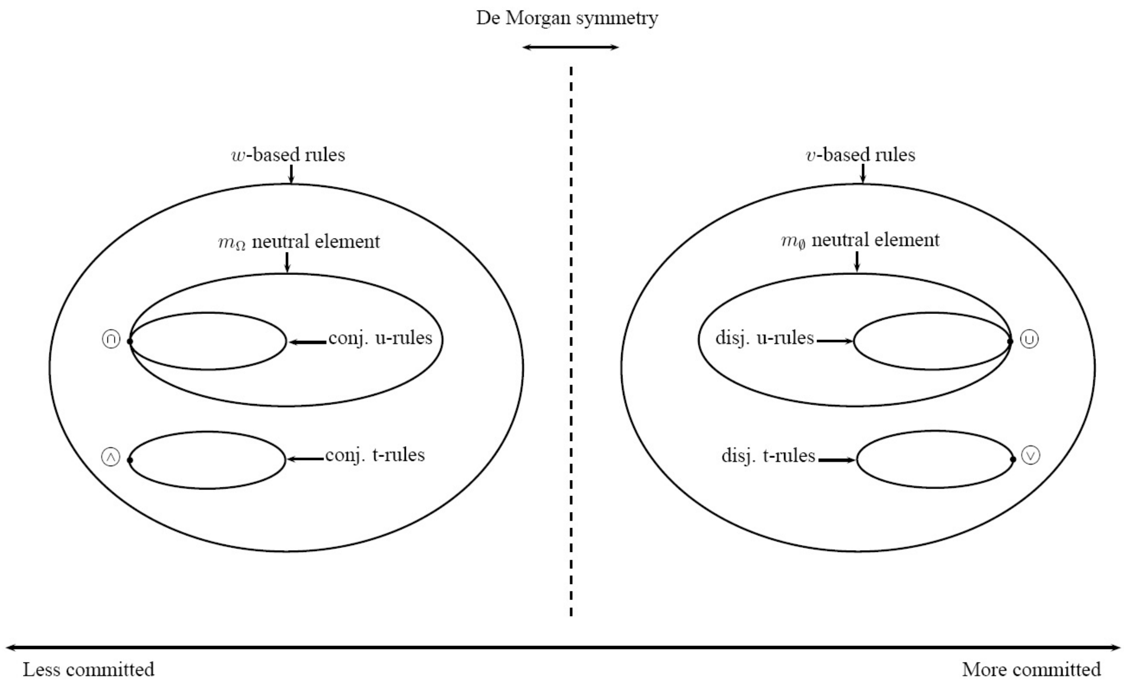

Figure 4.4: The four families of combination rules studied in this chapter, and the singular positions of the four basic rules $\textcircled{\cap}$, $\textcircled{\wedge}$, $\textcircled{\cup}$ and $\textcircled{\vee}$.

We have seen in Chapter 1 that belief functions may be defined on product spaces. It was also explained that it is possible to marginalize a belief function to a narrower domain and that one may combine belief functions defined on different frames using the TBM conjunctive rule. A problem with this latter kind of combination is that complexity grows with the size of the frames of the BBAs combined. However, if one is only interested in a marginal of the combination of some BBAs, then one can benefit from a nice property of the TBM conjunctive rule: the sought marginal can be obtained without explicitly computing the combination of the BBAs. This chapter has shown that the TBM conjunctive rule share many algebraic properties with the conjunctive u-rules. The next chapter will investigate whether the property of the TBM conjunctive rule related to operations on product spaces is also satisfied by the conjunctive u-rules.

Another Singular Property of the TBM Conjunctive Rule

Summary

The conjunctive rules based on extended uninorms share many properties with the TBM conjunctive rule. This latter rule fits the valuation algebra framework, which is a useful formalism to reduce computations when dealing with belief functions defined on product spaces. In this chapter, it is shown that the conjunctive rules based on extended uninorms and different from the TBM conjunctive rule do not satisfy an axiom of the valuation algebra framework. This result means that these rules will be difficult to use in problems involving a large number of variables. It also brings a new argument in favor of the TBM conjunctive rule. Finally, it is also shown that the cautious conjunctive rule does not satisfy an axiom of the valuation algebra framework, which means that the cautious conjunctive rule will also be difficult to use in high dimensional problems.

Résumé

Les règles conjonctives basées sur des uninormes étendues partagent de nombreuses propriétés avec la règle conjonctive du MCT. Cette dernière règle peut bénéficier d'un formalisme appelé algèbre de valuation afin de réduire les calculs lors de la combinaison de fonctions de croyance définies sur des espaces produits. Dans ce chapitre, nous montrons que les règles conjonctives basées sur des uninormes étendues, différentes de la règle conjonctive du MCT, ne satisfont pas un axiome de ce formalisme. Ce résultat implique que ces règles seront difficiles à utiliser dans des problèmes impliquant un grand nombre de variables. Il fournit aussi un nouvel argument en faveur de la règle conjonctive du MCT. Enfin, nous montrons également que la règle conjonctive prudente ne vérifie pas l'un des axiomes des algèbres de valuation, ce qui signifie que cette règle sera aussi difficile à utiliser pour des problèmes de grande dimension.

5.1 Introduction

The preceding chapter has introduced a family of rules based on the conjunctive weight function and uninorms. All the rules in this family share some properties: commutativity, associativity, monotonicity, and the same neutral element (the vacuous BBA). It was also shown that the TBM conjunctive rule is a particular member of this family.

A valuation algebra [51] is an abstract, yet useful, framework for many different AI formalisms. In particular, it can be used to manage efficiently information represented by belief functions defined on product spaces, if the belief functions are combined using the TBM conjunctive rule. It is shown in this chapter that, despite the numerous properties shared by the TBM conjunctive rule and the conjunctive u-rules, the TBM conjunctive rule is the only rule that satisfies an axiom of the valuation algebra framework. This property further singles out the TBM conjunctive rule in this family of rules and also exhibits a weakness of the conjunctive u-rules. Additionally, we discuss in this chapter the problem of combining belief functions defined on product spaces and induced by items of evidence, which cannot be assumed distinct.

This chapter is organized as follows. Basic notions on valuation algebras are first recalled in Section 5.2. Our contribution on the inadequacy of the conjunctive u-rules to the valuation algebra framework is then given in Section 5.3. Eventually, the problem of combining belief functions induced by items of evidence, which cannot be assumed distinct, is discussed in Section 5.4.

The work presented here was published in [66].

5.2 Valuation Algebras

Many formalisms dealing with information, such as the TBM, share an underlying algebraic structure with the essential algebraic operations of combination and marginalization [51]. Combination corresponds to aggregation of knowledge and marginalization corresponds to focusing of knowledge to a narrower domain. A recurrent task in these formalisms is that of *inference*, that is, aggregation of all available pieces of information, in order to focus the result afterwards on the actual questions of interest [68].

A problem faced by the inference process is that in many cases, computing the combination of the available information is computationally intractable. Provided that combination and marginalization satisfy some axioms, one can however benefit from *local computation* algorithms, such as the Shenoy-Shafer algorithm [79]. In essence, these algorithms make it possible to compute marginals of a combination of valuations without explicitly computing the combination (see, e.g., [68] for a clear presentation of the most important local computation algorithms).

The algebraic structures with the operations of combination and marginalization satisfying these axioms are called valuation algebras [51]. In this section, these axioms are stated. It is also shown how the TBM fits into the framework of valuation algebras.

5.2.1 Basic definitions

In the valuation algebra framework, it is considered that reasoning is concerned with a finite set of variables. Each variable is associated with a finite set of possible values called its frame; a variable is noted using an upper-case letter, e.g. X, and the frame of the variable is noted Ω_X. Sets of variables are noted using a lower-case letter, e.g. s. Let s be a non empty set of variables. We note Ω_s the Cartesian product of the frames Ω_X of the variables $X \in s$, and we call configurations the elements of Ω_s. Knowledge about the possible values of a set s of variables is represented by a valuation. Valuations are noted using lower-case greek letters such as φ and ψ. If φ is valuation for s, then we call s the domain of φ and we write $d(\varphi) = s$. Given a set s of variables, we may consider that there is a set Φ_s of valuations. We note r the set of all variables, and $\Phi = \cup_{s \subseteq r} \Phi_s$ the set of all valuations. Eventually, we use D to denote the power set of r.

In the TBM, valuations are BBAs. Indeed, let m be a BBA with domain s. Then, the BBA m represents some evidence regarding the actual value in Ω_s. A BBA thus fits the notion of valuation.

Two operations are defined for valuations in the valuation algebra framework. The combination of valuations is a binary operation $\otimes : \Phi \times \Phi \rightarrow \Phi$, which is assumed to be commutative and associative, hence Φ is a commutative semigroup under combination. The marginalization of a valuation is a binary operation $\downarrow : \Phi \times D \rightarrow \Phi$. For any valuation φ and domain $s \subseteq d(\varphi)$, a valuation $\varphi^{\downarrow s}$ is associated. $\varphi^{\downarrow s}$ is called the marginal of φ for s. Marginalization corresponds to focusing of the knowledge represented by φ for $d(\varphi)$ to the smaller domain s.

In the literature concerned with the belief functions instantiation of the valuation algebra framework, combination is assimilated to the combination by the TBM conjunctive rule (or by Dempster's rule), as defined by (1.16), and marginalization is given by the marginalization operation defined by (1.14).

5.2.2 The problem of inference

Equipped with the definitions of the preceding section, the *problem of inference* [51] (or *projection problem* [68]) can be formally stated as follows. Suppose a knowledge base consisting of a finite set of valuations $\varphi_1, ..., \varphi_m$, and let $\varphi_1 \otimes ... \otimes \varphi_m$ represent the combined knowledge, which we call the joint valuation. The problem of inference is to marginalize the joint valuation to a domain s of interest: $(\varphi_1 \otimes ... \otimes \varphi_m)^{\downarrow s}$. Example 5.1 below illustrates the problem of inference.

Example 5.1 (Chest Clinic)**.** *The following paragraph is a quote from [55], which formulates the chest clinic problem.*

> *"Shortness of breath (dyspnoea) may be due to tuberculosis, lung cancer or bronchitis, or none of them, or more than one of them. A recent visit to Asia increases the chances of tuberculosis, while smoking is known to be a risk factor for both lung cancer and bronchitis. The results of a single chest X-ray do not discriminate between lung cancer and tuberculosis, as neither does the presence or absence of dyspnoea" [55].*

In the context of the valuation algebra framework, the chest clinic problem may be modeled as follows [51]. There are eight binary variables: A denotes "visit to Asia?", S denotes "Smoker?", T denotes "has Tuberculosis?", L denotes "has Lung cancer?", B denotes "has Bronchitis?", E denotes "has Either bronchitis or tuberculosis?", X denotes "has positive X-ray?", and D denotes "has Dysponea?". The knowledge prior to any observation is represented by eight valuations: α for $\{A\}$, σ for $\{S\}$, τ for $\{A, T\}$, λ for $\{S, L\}$, β for $\{S, B\}$, ε for $\{T, L, E\}$, ξ for $\{E, X\}$, and δ for $\{E, B, D\}$.

Suppose a patient is observed and that this patient visited Asia recently and is suffering from dysponea. These two observations may be represented by two valuations o_A for $\{A\}$ and o_D for $\{D\}$. Suppose we are interested by the answer to the following question: what is our belief that the patient is suffering from tuberculosis? Answering this question consists in marginalizing the joint valuation to the domain T of interest.

A qualitative description of a knowledge base can be provided using a so-called *valuation network*, which is a graphical display of a set of valuations, where variables are represented by circular nodes and valuations are represented by square nodes. Figure 5.1 shows the valuation network corresponding to Example 5.1.

The straightforward way to perform inference is to compute first the joint valuation and then marginalize to the domain of interest. However, if a valuation is a belief function, its size increases exponentially in the number of variables in its domain and also in the sizes of the frames of these variables. Hence, the computation of the joint valuation, whose domain is the set r of all variables, may be intractable, even if all the valuations φ_i in the knowledge base are defined on small domains. Consider for instance the chest clinic example and suppose the valuations are BBAs. Computing the joint valuation requires finding $2^{(2^8)}$ values, an infeasible task.

However, if combination and marginalization satisfy some axioms, then the marginal $(\varphi_1 \otimes ... \otimes \varphi_m)^{\downarrow s}$ can be computed without explicitly computing the joint valuation. These axioms are the following [68]:

1. *Commutative Semigroup*: Φ is associative and commutative under $\otimes$.

2. *Domain of combination*: for $\varphi, \psi \in \Phi$,

$$d(\varphi \otimes \psi) = d(\varphi) \cup d(\psi).$$

3. *Marginalization*: for $\varphi \in \Phi$, $s \in D$ and $s \subseteq d(\varphi)$.

$$d(\varphi^{\downarrow s}) \quad = \quad s,$$

4. *Transitivity of marginalization*: for $\varphi \in \Phi$ and $t \subseteq s \subseteq d(\varphi)$,

$$(\varphi^{\downarrow s})^{\downarrow t} \quad = \quad \varphi^{\downarrow t}.$$

5. *Distributivity of marginalization over combination*: for $\varphi, \psi \in \Phi$ with $d(\varphi) = s$, $d(\psi) = t$, and $z \in D$ such that $t \subseteq z \subseteq s \cup t$,

$$(\varphi \otimes \psi)^{\downarrow z} = \varphi^{\downarrow z \cap s} \otimes \psi. \tag{5.1}$$

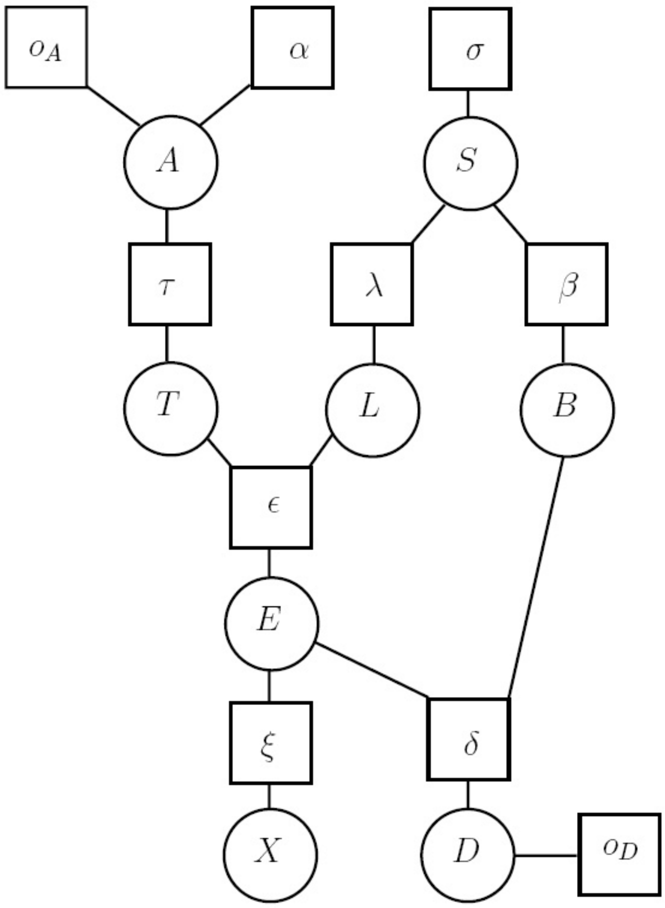

Figure 5.1: The valuation network corresponding to Example 5.1.

6. *Domain*: for $\varphi \in \Phi$ with $d(\varphi) = s$,

$$\varphi^{\downarrow s} = \varphi.$$

A set Φ of valuations with set D of domains, combination $\otimes$, and marginalization $\downarrow$, which satisfies these six axioms is called a valuation algebra. It is denoted by $(\Phi, D, \otimes, \downarrow)$.

The fifth axiom is the crucial one for local computation. Indeed, it basically means that if we have two valuations on two different domains and we want to perform inference onto the domain of one of the valuations, we can first marginalize the other valuation to the intersection of the two domains and then combine, instead of first combining the two valuations and then marginalizing [51].

Let $\mathcal{M}$ be the set of BBAs, and let $\downarrow$ be the projection operation as defined by (1.14). It was originally shown in [79] that the structures $(\mathcal{M}, D, ⓒ, \downarrow)$ and $(\mathcal{M}, D, \oplus, \downarrow)$ satisfy all the axioms of valuation algebras. Let us note that the axioms in [79] are not exactly the ones given in this chapter; the relations between the axiom system of [79] and the one used in this chapter is explained in [68, Section 2.1]. The proof that $(\mathcal{M}, D, ⓒ, \downarrow)$ and $(\mathcal{M}, D, \oplus, \downarrow)$ satisfy the axioms used in this chapter is given by the proof of [68, Theorem 3.3]. Informally, we may see that Axioms 3, 4 and 6 follow from the definition of the marginalization operation. Furthermore, the TBM conjunctive rule and Dempster's rule are associative and commutative, and thus Axiom 1 is satisfied. Axiom 2 is verified by the definition of the combination by the TBM conjunctive rule on product spaces and the definition of the combination by Dempster's rule on product spaces. The proof for Axiom 5 is, however, more complex (see, e.g., [68, p.145]).

5.3 Conjunctive U-Rules and Valuation Algebras

We have seen that the TBM conjunctive rule belongs to the family of the conjunctive u-rules, hence the rule ⓒ has common properties with the conjunctive u-rules. It is thus interesting to know whether more properties are shared by those rules. In order to investigate properties of the conjunctive u-rules in the context of the valuation algebra framework, the combination on product spaces by a conjunctive u-rule needs to be defined. A natural way to define this combination is given by Definition 5.1; this definition is based on the vacuous extension of BBAs, as is the case for the TBM conjunctive rule (see Equation (1.16) of Section 1.3.3).

Definition 5.1. *Let $m_1^{\Omega_s}$ and $m_2^{\Omega_t}$ be two BBAs defined, respectively, on the frames Ω_s and Ω_t. Let $ⓤ_w$ be a conjunctive u-rule based on a 1-uninorm $\mathcal{U}$. The combination of $m_1^{\Omega_s}$ and $m_2^{\Omega_t}$ by $ⓤ_w$ on $\Omega_s \times \Omega_t$ is defined as:*

$$m_1^{\Omega_s} ⓤ_w m_2^{\Omega_t} = m_1^{\Omega_s \uparrow \Omega_s \times \Omega_t} ⓤ_w m_2^{\Omega_t \uparrow \Omega_s \times \Omega_t}. \tag{5.2}$$

Equipped with this definition, we can study whether the algebraic structure $(\mathcal{M}_{nd}, D, ⓤ_w, \downarrow)$ is a valuation algebra for at least one conjunctive u-rule based on a 1-uninorm $\mathcal{U}$ different from the product.

As for the TBM conjunctive rule, determining whether this structure is a valuation algebra depends mainly on the fifth axiom. Indeed, we can first remark that Axioms 3, 4 and 6 do not need to be studied since they are independent from the combination rule used. Furthermore, Axiom 2 is satisfied by the definition given by (5.2) of the combination by a conjunctive u-rule on product spaces. Axioms 1 is a direct consequence of $(\mathcal{M}_{nd}, \textcircled{u}_w)$ being a commutative monoid.

Proposition 5.1. *Let $\textcircled{u}_w$ be a conjunctive u-rule. If the binary operator $\mathcal{U}$ underlying the conjunctive u-rule $\textcircled{u}_w$ is different from the product, i.e., $\exists x, y \in (0, +\infty)$ such that $x \,\mathcal{U}\, y \neq xy$, then the algebraic structure $(\mathcal{M}_{nd}, D, \textcircled{u}_w, \downarrow)$ does not satisfy Axiom 5.*

Proof. See Appendix E.3. □

Proposition 5.1 tells us that Axiom 5 is not satisfied by conjunctive u-rules different from the TBM conjunctive rule, i.e., among conjunctive u-rules, marginalization is distributive only over the combination by the rule $\textcircled{\cap}$.

The practical consequence of this proposition is that the conjunctive u-rules will be difficult to use in problems involving many variables, since we cannot benefit from the valuation algebra framework and thus we must always work with the joint valuation. However, note that the conjunctive u-rules remain applicable for many problems, such as the classification applications presented in the preceding chapter.

5.4 The Cautious Rule and Valuation Algebras

As we have seen, we may resort to the cautious rule when the items of evidence to be combined cannot be assumed distinct. It thus seems interesting to know whether the set of nondogmatic BBAs equipped with the combination by the cautious rule, fit the axioms of the valuation algebra framework. If the combination on product spaces by the cautious rule is defined in a similar manner as (5.2) and (1.16), i.e., if it is based on the vacuous extension, then it is relatively easy to check whether $(\mathcal{M}_{nd}, D, \owedge, \downarrow)$ satisfies the axioms of the valuation algebra framework: as for the conjunctive u-rules, we merely need to check the fifth axiom.

Proposition 5.2. *Marginalization is not distributive over the combination by the cautious rule.*

Proof. Suppose m_1 and m_2 are the BBAs of Case 1 of the proof of Proposition 5.1 (see Appendix E.3), for some $x, y \in (0,1)$ fixed. We have $(m_1 \owedge m_2)^{\downarrow t} = (m_1 \textcircled{\cap} m_2)^{\downarrow t}$ and $m_1^{\downarrow t} \owedge m_2 \neq m_1^{\downarrow t} \textcircled{\cap} m_2$. Hence, we have

$$(m_1 \owedge m_2)^{\downarrow t} \neq m_1^{\downarrow t} \owedge m_2,$$

and thus

$$(m_1 \owedge m_2)^{\downarrow z} \neq m_1^{\downarrow z \cap s} \owedge m_2,$$

according to the choice of t, s and z in the proof of Proposition 5.1. □

The practical consequence of this proposition is similar to the consequence of Proposition 5.1 for the conjunctive u-rules: it will be difficult to use the cautious rule in problems involving many variables. Let us stress that, despite this drawback, the cautious rule remains nonetheless relevant for many other problems, such as classification applications, as demonstrated in Chapter 4.

5.5 Conclusion

The TBM conjunctive rule belongs to an infinite family of rules based on generalized uninorms. All the rules in this family share some basic properties. An interesting fact related to this family is that the TBM conjunctive rule is its least committed element. In this chapter, we have shown that the TBM conjunctive rule is also the only rule in this family for which marginalization is distributive over the combination. On the one hand, this second singular property of the TBM conjunctive rule strengthens the fact that this rule has a special position in this family and may be seen as yet another argument in favor of this rule. On the other hand, it may be seen as a restriction to the breadth of problems that can be tackled by the conjunctive u-rules. A similar conclusion holds for the cautious rule.

Interestingly, in the next part of this thesis, we will see the TBM conjunctive rule surfacing once again in another family of combination rules, called α-junctions.

Part III

α-Junctions

Interpretation and Computation of α-Junctions

Summary

The TBM conjunctive rule, the TBM disjunctive rule, the exclusive disjunctive rule and its negation are particular cases of an infinite family of combination rules called α-junctions. Until now, the α-junctions suffered from two main limitations. First, they did not have an interpretation in the general case. Second, it was difficult to compute a combination by an α-junction. In this chapter, it is first shown that these operators correspond to a particular form of knowledge on the truthfulness of the sources of information. Then, it is shown that there exist several simple means to compute a combination by an α-junction. These means are mere generalizations of expressions that allow one to compute the combination by the TBM conjunctive rule. These new results on the interpretation and the computation of the α-junctions make this family of rules potentially interesting for applications. At the end of this chapter, the usefulness of these rules in a classification application is also investigated.

Résumé

La règle conjonctive du MCT, la règle disjonctive du MCT, la règle disjonctive exclusive et sa négation sont des cas particuliers d'une famille infinie de règles de combinaison appelées α-jonctions. Ces règles souffraient jusqu'ici de deux problèmes majeurs. Tout d'abord, aucune interprétation des α-jonctions n'était connue dans le cas général. De plus, il était difficile de calculer la combinaison par une α-jonction. Dans ce chapitre, nous montrons que ces règles correspondent à une connaissance particulière quant à la véracité des sources d'information. Ensuite, nous donnons plusieurs nouvelles formules simples permettant de calculer la combinaison par une α-jonction. Ces formules généralisent des expressions permettant de calculer la combinaison par la règle conjonctive du MCT. Ces nouveaux résultats sur l'interprétation et le calcul des α-jonctions rendent cette famille de règles intéressante d'un point de vue applicatif. À la fin de ce chapitre, nous étudions aussi l'intérêt de ces règles dans une application de classification.

6.1 Introduction

In [87], Smets introduced a family of combination rules, which he called α-junctions. This family basically represents the set of associative, commutative and linear operators for belief functions with a neutral element. It includes as particular cases the TBM conjunctive rule, the TBM disjunctive rule, as well as the exclusive disjunctive rule [26] and its negation [87] (the definitions of these two latter rules will be given in this chapter). We recall that the use of the TBM conjunctive rule is appropriate when one can assume that all the sources tell the truth. On the other hand, the TBM disjunctive rule should be used when it is known that at least one of the sources tells the truth, but it is not known which one. The uses of the exclusive disjunctive rule and its negation are also conditioned by knowledge on the truthfulness of the sources of information: the former fits with the case where exactly one of the sources is known to tell the truth, but it is not known which one, whereas the latter corresponds to a situation where either all or none of the sources are known to tell the truth [87]. The behavior of an α-junction, or α-junctive rule, is determined by a parameter, noted α, and the chosen neutral BBA for the combination, noted m_{vac}, which can be either m_Ω or $m_\emptyset$[1]; the rule $\bigcirc\!\!\!\!\cap$, the rule $\bigcirc\!\!\!\!\cup$, the exclusive disjunctive rule and its negation are recovered for particular values of α and m_{vac}. For other values of the parameter α, the α-junctive rules did not have an interpretation.

To our knowledge, this infinite family of rules has never been exploited in the literature. A possible explanation is that, until now, these rules suffered from two main limitations. First, those operators did not have an interpretation in the general case. Second, it was difficult to compute a combination by an α-junction using the methods proposed in [87], as already remarked in [91].

In our search for alternatives to the TBM conjunctive rule, we carefully reexamine in this chapter this theoretical contribution of Smets: some new light on the meaning of the α-junctions is shed and their mathematics are simplified so that their computation be easier. More precisely, it is first shown that these operators correspond to a particular form of knowledge, determined by the parameter α, about the truthfulness of the sources. The α-junctions becomes thus suitable as flexible combination rules that allows one to take into account some particular knowledge about the sources. Several efficient and simple ways of computing a combination by an α-junction are then presented, making the practical use of the α-jonctions in applications possible. These new means are based on generalizations of mechanisms that can be used to compute combinations by the TBM conjunctive and TBM disjunctive rules. In particular, the conditioning operation is generalized in the context of the α-junctions. It is also shown that there exists a simple way to obtain two other equivalent representations of a belief function, that do not have a name in Smets' paper [87] but that we will call the *α-commonality* and *α-implicability* functions for reasons that will become apparent. This is important since the combination by an α-junctive rule can easily be computed once one has these functions. This contribution will be based on a generalization of the matrices **Q** and **B** (see Chapter 1), that can be used to obtain, in a simple manner, the commonality and implicability functions

[1] Recall that m_Ω and $m_\emptyset$ are used, respectively, in place of $m(\Omega) = 1$ and $m(\emptyset) = 1$.

associated to a belief function. We will also propose a light technical modification to a part of Smets' presentation of the α-junctions so that the so-called α-implicability function is a complete generalization of the implicability function, which is not the case with Smets' presentation as will be shown in this chapter.

This chapter is organized as follows. In Section 6.2, basic notions on α-junctions are given. The interpretation of this family of rules is discussed in Section 6.3. Simple means to perform a combination by an α-junctive rule are proposed in Section 6.4. Eventually, the usefulness of these rules in a classification application is investigated in Section 6.5.

6.2 α-Junctions: Basic Notions

In [87], Smets studies the set of possible linear combination rules. Smets calls this set the α-junctions because, as we will see, they cover the conjunction, the disjunction and the exclusive disjunction. We report in this section the summary of [87] given in [91].

Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two BBAs on Ω. Suppose we want to build a BBA $\mathbf{m}_{12}$ such that $\mathbf{m}_{12} = f(\mathbf{m}_1, \mathbf{m}_2)$, i.e., $\mathbf{m}_{12}$ depends only on $\mathbf{m}_1$ and $\mathbf{m}_2$. Smets [87] determines the operators that map $\mathcal{M}^\Omega \times \mathcal{M}^\Omega$ to $\mathcal{M}^\Omega$ and that satisfy the following requirements (the origins of those requirements are summarized in [91, p.25]).

- Linearity: $f(\mathbf{m}, p\mathbf{m}_1 + q\mathbf{m}_2) = pf(\mathbf{m}, \mathbf{m}_1) + qf(\mathbf{m}, \mathbf{m}_2)$, $p \in [0,1]$, $q = 1 - p$.
- Commutativity: $f(\mathbf{m}_1, \mathbf{m}_2) = f(\mathbf{m}_2, \mathbf{m}_1)$.
- Associativity: $f(f(\mathbf{m}_1, \mathbf{m}_2), \mathbf{m}_3) = f(\mathbf{m}_1, f(\mathbf{m}_2, \mathbf{m}_3))$.
- Neutral element: existence of a belief function $\mathbf{m}_{vac}$ such that $f(\mathbf{m}, \mathbf{m}_{vac}) = \mathbf{m}$ for any $\mathbf{m}$.
- Anonymity: relabeling the elements of Ω does not affect the results.
- Context preservation: if $pl_1(X) = 0$ and $pl_2(X) = 0$ for some $X \subseteq \Omega$, then $pl_{12}(X) = 0$.

It is shown in [87] that the solutions are stochastic matrices. We have:

$$\mathbf{m}_{12} = \mathbf{K}_{m_1} \cdot \mathbf{m}_2,$$

where

$$\mathbf{K}_{m_1} = \sum_{X \subseteq \Omega} m_1(X) \cdot \mathbf{K}_X. \tag{6.1}$$

Smets [87] proves that the $2^{|\Omega|} \times 2^{|\Omega|}$ matrices $\mathbf{K}_X$ depend only on $\mathbf{m}_{vac}$ and one parameter $\alpha \in [0,1]$. Furthermore, he shows that there are only two solutions for $\mathbf{m}_{vac}$: either $\mathbf{m}_{vac} = \mathbf{m}_\Omega = \mathbf{1}_\Omega$ or $\mathbf{m}_{vac} = \mathbf{m}_\emptyset = \mathbf{1}_\emptyset$. Hence, there are only two sets of solutions, which are presented now.

6.2.1 Case $\mathbf{m}_{vac} = \mathbf{m}_\Omega$

The definition of the matrices $\mathbf{K}_X$ that satisfy the above requirements when $\mathbf{m}_{vac} = \mathbf{m}_\Omega$ is the following.

$$\begin{aligned} \mathbf{K}_\Omega &= \mathbf{I}, \\ \mathbf{K}_X &= \prod_{x \notin X} \mathbf{K}_{\{\overline{x}\}}, \quad \forall X \subset \Omega, \end{aligned}$$

where

$$\mathbf{K}_{\{\overline{x}\}} = \left[k_{\overline{x}}(A,B)\right], \quad \forall x \in \Omega, \tag{6.2}$$

with

$$k_{\overline{x}}(A,B) = \begin{cases} 1 & \text{if } x \notin A, \quad B = A \cup \{x\}, \\ \alpha & \text{if } x \notin B, \quad B = A, \\ 1-\alpha & \text{if } x \notin B, \quad A = B \cup \{x\}, \\ 0 & \text{otherwise}, \end{cases} \tag{6.3}$$

where $\alpha \in [0,1]$ and is constant for all $\mathbf{K}_X$.

When $\mathbf{m}_{vac} = \mathbf{m}_\Omega$, the matrix $\mathbf{K}_m$ computed using (6.1) is noted $\mathbf{K}_m^{\cap,\alpha}$. The index $\cap$ is used because when $\alpha = 1$, $\mathbf{K}_m^{\cap,1}$ becomes the Dempsterian specialization matrix (Section 1.4.2) and we have $\mathbf{K}_{m_1}^{\cap,1} \cdot \mathbf{m}_2 = \mathbf{m}_{1 \textcircled{\cap} 2}$ [91, p. 26]. Furthermore, when $\mathbf{m}_{vac} = \mathbf{m}_\Omega$, an α-junction is referred to as an α-conjunction since $\mathbf{m}_\Omega$ is the neutral element of the conjunction [87, p. 144]; we denote an α-conjunctive rule by $\textcircled{\cap}^\alpha$. Note that we will give in the next chapter another motivation for using the term α-conjunction when $\mathbf{m}_{vac} = \mathbf{m}_\Omega$.

The case $\alpha = 0$ corresponds to the combination rule noted $\textcircled{=}$ and defined by

$$m_{1 \textcircled{=} 2}(A) = \sum_{A = (B \cap C) \cup (\overline{B} \cap \overline{C})} m_1(B)\, m_2(C), \quad \forall A \subseteq \Omega. \tag{6.4}$$

This rule corresponds to the situation where it is known that either both or none of the sources of information tell the truth [87].

Example 6.1 illustrates the various matrices $\mathbf{K}_X$ when $\Omega = \{a, b\}$ (this example is a more detailed version of Example 12.1 of [91]).

Example 6.1. *Let* $\overline{\alpha} = 1 - \alpha$. *We have*

$$\begin{aligned}
\mathbf{m}_{12} &= (m_1(\emptyset)\cdot\mathbf{K}_\emptyset)\cdot\mathbf{m}_2 + (m_1(a)\cdot\mathbf{K}_a)\cdot\mathbf{m}_2 + (m_1(b)\cdot\mathbf{K}_b)\cdot\mathbf{m}_2 \\
&\quad + (m_1(\Omega)\cdot\mathbf{K}_\Omega)\cdot\mathbf{m}_2 \\
&= \left(m_1(\emptyset)\cdot\mathbf{K}_{\{\overline{a}\}}\cdot\mathbf{K}_{\{\overline{b}\}}\right)\cdot\mathbf{m}_2 + \left(m_1(a)\cdot\mathbf{K}_{\{\overline{b}\}}\right)\cdot\mathbf{m}_2 + \left(m_1(b)\cdot\mathbf{K}_{\{\overline{a}\}}\right)\cdot\mathbf{m}_2 \\
&\quad + (m_1(\Omega)\cdot\mathbf{I})\cdot\mathbf{m}_2 \\
&= \left(m_1(\emptyset)\cdot\begin{bmatrix}\alpha & 1 & . & . \\ \overline{\alpha} & . & . & . \\ . & . & \alpha & 1 \\ . & . & \overline{\alpha} & .\end{bmatrix}\cdot\begin{bmatrix}\alpha & . & 1 & . \\ . & \alpha & . & 1 \\ \overline{\alpha} & . & . & . \\ . & \overline{\alpha} & . & .\end{bmatrix}\right)\cdot\mathbf{m}_2 \\
&\quad + \left(m_1(a)\cdot\begin{bmatrix}\alpha & . & 1 & . \\ . & \alpha & . & 1 \\ \overline{\alpha} & . & . & . \\ . & \overline{\alpha} & . & .\end{bmatrix}\right)\cdot\mathbf{m}_2 + \left(m_1(b)\cdot\begin{bmatrix}\alpha & 1 & . & . \\ \overline{\alpha} & . & . & . \\ . & . & \alpha & 1 \\ . & . & \overline{\alpha} & .\end{bmatrix}\right)\cdot\mathbf{m}_2 \\
&\quad + m_1(\Omega)\cdot\mathbf{m}_2 \\
&= m_1(\emptyset)\cdot\begin{bmatrix}\alpha^2 & \alpha & \alpha & 1 \\ \alpha\overline{\alpha} & . & \overline{\alpha} & . \\ \alpha\overline{\alpha} & \overline{\alpha} & . & . \\ \overline{\alpha}^2 & . & . & .\end{bmatrix}\cdot\mathbf{m}_2 + m_1(a)\cdot\begin{bmatrix}\alpha & . & 1 & . \\ . & \alpha & . & 1 \\ \overline{\alpha} & . & . & . \\ . & \overline{\alpha} & . & .\end{bmatrix}\cdot\mathbf{m}_2 \\
&\quad + m_1(b)\cdot\begin{bmatrix}\alpha & 1 & . & . \\ \overline{\alpha} & . & . & . \\ . & . & \alpha & 1 \\ . & . & \overline{\alpha} & .\end{bmatrix}\cdot\mathbf{m}_2 + m_1(\Omega)\cdot\mathbf{m}_2.
\end{aligned}$$

6.2.2 Case $\mathbf{m}_{vac} = \mathbf{m}_\emptyset$

The definition of the matrices $\mathbf{K}_X$ that satisfy the above requirements when $\mathbf{m}_{vac} = \mathbf{m}_\emptyset$ is the following.

$$\begin{aligned}
\mathbf{K}_\emptyset &= \mathbf{I}, \\
\mathbf{K}_X &= \prod_{x\in X}\mathbf{K}_{\{x\}}, \quad \forall X \in 2^\Omega \setminus \{\emptyset\},
\end{aligned}$$

where

$$\mathbf{K}_{\{x\}} = [k_x(A,B)], \quad \forall x \in \Omega,$$

with

$$k_x(A,B) = \begin{cases} 1 & \text{if } x \notin B, \quad A = B \cup \{x\}, \\ \alpha & \text{if } x \in B, \quad B = A, \\ 1-\alpha & \text{if } x \notin A, \quad B = A \cup \{x\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in [0,1]$ and is constant for all $\mathbf{K}_X$.

When $\mathbf{m}_{vac} = \mathbf{m}_\emptyset$, the matrix $\mathbf{K}_m$ is noted $\mathbf{K}_m^{\cup,\alpha}$. The index $\cup$ is used because when $\alpha = 1$, we have $\mathbf{K}_{m_1}^{\cup,1}\cdot\mathbf{m}_2 = \mathbf{m}_{1 \bigcirc\!\!\!\!\cup\, 2}$. Furthermore, when $\mathbf{m}_{vac} = \mathbf{m}_\emptyset$, an α-junction is referred to as an α-disjunction since $\mathbf{m}_\emptyset$ is the neutral element of the disjunction [87, p. 144]; we denote an α-disjunctive rule by $\circledcup^{\alpha}$. Note that we will give in the next chapter another motivation for using the term α-disjunction when $\mathbf{m}_{vac} = \mathbf{m}_{\emptyset}$.

The case $\alpha = 0$ corresponds to the combination rule noted $\underline{\circledcup}$ and defined by

$$m_{1\underline{\circledcup}2}(A) \quad = \sum_{A=B\underline{\cup}C} m_1(B)\, m_2(C), \quad \forall A \subseteq \Omega, \tag{6.5}$$

where $\underline{\cup}$ is the exclusive OR (XOR), i.e., $B\underline{\cup}C = \big(B \cap \overline{C}\big) \cup \big(\overline{B} \cap C\big)$ for all $B, C \subseteq \Omega$. This rule corresponds to the situation where it is known that exactly one of the sources of information tells the truth, but it is not known which one [87]. This rule is called the exclusive disjunctive rule. In the same vein, the rule $\underline{\circledcap}$ may be called the exclusive conjunctive rule since it corresponds to the XAND logical operator[2].

Note that, besides using the notation $\mathbf{K}_m^{\cup,\alpha}$ and $\mathbf{K}_m^{\cap,\alpha}$ in order to enhance the difference between the cases $\mathbf{m}_{vac} = \mathbf{m}_{\emptyset}$ and $\mathbf{m}_{vac} = \mathbf{m}_{\Omega}$, we will also write $\mathbf{K}_X^{\cup,\alpha}$ and $\mathbf{K}_X^{\cap,\alpha}$ to make the distinction between the two possible sets of matrices $\mathbf{K}_X$.

Finally, we have, for any $\alpha \in [0,1]$ [91, Theorem 12.2]:

$$\begin{aligned} \overline{m_1 \circledcup^{\alpha} m_2} &= \overline{m_1} \circledcap^{\alpha} \overline{m_2} \\ \overline{m_1 \circledcap^{\alpha} m_2} &= \overline{m_1} \circledcup^{\alpha} \overline{m_2} \end{aligned} \tag{6.6}$$

i.e., α-conjunctive rules and α-disjunctive rules are linked by De Morgan laws. In particular, the De Morgan duality between the TBM conjunctive and TBM disjunctive rules (Equation (1.6)) is recovered by setting $\alpha = 1$ in (6.6).

Figure 6.1 presents an overview of the α-junctions.

6.3 Interpretation

In [91], Smets stated that the meaning of the α-junctions is unclear when $\alpha \in (0,1)$. In this section, an interpretation for the α-junctions is proposed. In order to derive this interpretation in Section 6.3.2, a new expression for the α-junctions is first unveiled in Section 6.3.1.

6.3.1 A new expression for the α-junctions

It was stressed in Section 1.3.1 that the TBM conjunctive rule may be seen as a generalization of the conditioning operation, since the conjunctive combination can be expressed using the unnormalized Dempster's rule of conditioning (see Equation (1.8)). Interestingly, Equation (1.8) may be generalized in the context of the α-conjunctions, as shown by the following proposition.

[2] The XAND logical operator is equivalent to the XNOR logical operator, and thus the rule $\underline{\circledcap}$ may also be called the exclusive non disjunctive rule. However, we will use the appellation exclusive conjunctive rule, since we feel this name better reflects the fact that the rule $\underline{\circledcap}$ has the vacuous BBA m_{Ω} as neutral element.

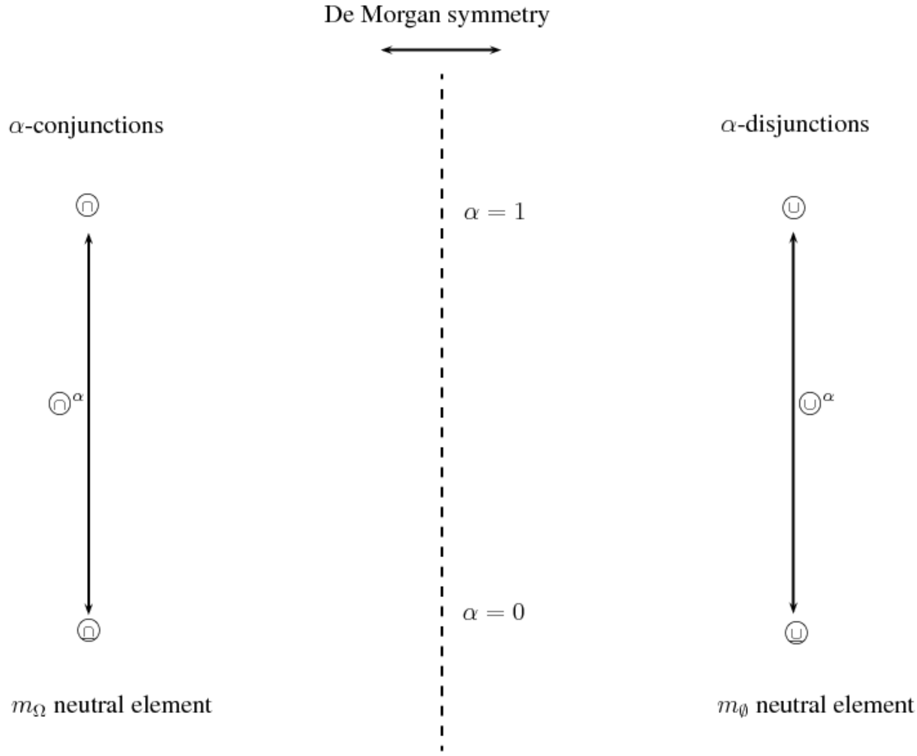

Figure 6.1: The α-junctions and the De Morgan duality.

Proposition 6.1. *Let m_1 and m_2 be two BBAs. Let $m_1[B]^\alpha$ denote $m_1 \bigcirc\!\!\!\!\cap^\alpha m_B$, where m_B is a categorical BBA focused on $B \subseteq \Omega$. We have*

$$m_{1 \bigcirc\!\!\!\!\cap^\alpha 2}(A) = \sum_{B \subseteq \Omega} m_1 [B]^\alpha (A) \, m_2 (B), \quad \forall A \subseteq \Omega. \tag{6.7}$$

Proof. We have

$$\mathbf{m}_{1 \bigcirc\!\!\!\!\cap^\alpha 2} = \mathbf{K}^{\cap,\alpha}_{m_1} \cdot \mathbf{m}_2.$$

Hence, we have

$$\mathbf{m}_{1 \bigcirc\!\!\!\!\cap^\alpha 2}(A) = \sum_{B \subseteq \Omega} K^{\cap,\alpha}_{m_1}(A, B) \cdot m_2(B), \quad \forall A \subseteq \Omega. \tag{6.8}$$

Furthermore, the column B of $\mathbf{K}^{\cap,\alpha}_{m_1}$ is equal to the column vector $\mathbf{m}_1 \bigcirc\!\!\!\!\cap^\alpha \mathbf{m}_B$ since $\mathbf{m}_1 \bigcirc\!\!\!\!\cap^\alpha \mathbf{m}_B = \mathbf{K}^{\cap,\alpha}_{m_1} \cdot \mathbf{m}_B$ and $\mathbf{K}^{\cap,\alpha}_{m_1} \cdot \mathbf{m}_B$ is equal to the column B of $\mathbf{K}^{\cap,\alpha}_{m_1}$. Hence, Equation (6.8) can be rewritten

$$\mathbf{m}_{1 \bigcirc\!\!\!\!\cap^\alpha 2}(A) = \sum_{B \subseteq \Omega} m_1 \bigcirc\!\!\!\!\cap^\alpha m_B(A) \cdot m_2(B), \quad \forall A \subseteq \Omega.$$

□

Note that, when $\alpha = 1$, Equation (6.7) becomes equivalent to (1.8); this is the reason why (6.7) may be seen as a generalization of (1.8).

Proposition 6.1 gives us another expression for the α-conjunctions using what may be called "α-conditioning", which may be formally defined as follows.

Definition 6.1. *The α-conditioning of a BBA by a subset $B \subseteq \Omega$ is equal to the α-conjunction of this BBA with a categorical BBA focused on B.*

The result of the α-conditioning operation on a BBA m given a subset $B \subseteq \Omega$, is noted $m[B]^\alpha$ and is, by definition, equal to $m \textcircled{\cap}^\alpha m_B$, where m_B is a categorical BBA focused on B. We use the term "α-conditioning" for $m[B]^\alpha = m \textcircled{\cap}^\alpha m_B$ because $m[B]^\alpha = m[B]$ when $\alpha = 1$.

The following proposition provides an expression for the α-conditioning operation.

Proposition 6.2. *Let $B \subseteq \Omega$. We have*

$$m[B]^\alpha(X) = \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X} m(A)\, m_{\alpha,\cap}(C), \quad \forall X \subseteq \Omega,$$

where $m_{\alpha,\cap}$ is a BBA such that $m_{\alpha,\cap}(A) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$.

Proof. See Appendix F.1. □

Using Propositions 6.1 and 6.2, we are now able to show the following proposition, which gives us a new expression for the α-conjunctions. This expression will be useful in the next section to derive an interpretation for these operators.

Proposition 6.3. *Let m_1 and m_2 be two BBAs. We have*

$$m_{1\textcircled{\cap}^\alpha 2}(X) = \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X} m_1(A)\, m_2(B)\, m_{\alpha,\cap}(C), \quad \forall X \subseteq \Omega, \tag{6.9}$$

where $m_{\alpha,\cap}(A) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$.

Proof. From Proposition 6.1, we have, for all $X \subseteq \Omega$

$$\begin{aligned} m_{1\textcircled{\cap}^\alpha 2}(X) &= \sum_{B\subseteq\Omega} m_1[B]^\alpha(X)\, m_2(B) \\ &= \sum_{B\subseteq\Omega} \left(\sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X} m_1(A)\, m_{\alpha,\cap}(C) \right) m_2(B) \quad \text{(from Proposition 6.2)} \\ &= \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X} m_1(A)\, m_2(B)\, m_{\alpha,\cap}(C). \end{aligned}$$

□

Remark 6.1. *If $\alpha = 0$, then the BBA $m_{\alpha,\cap}$ of Proposition 6.3 is such that $m_{\alpha,\cap}(\Omega) = 1$, and thus the term on the right side of (6.9) reduces to*

$$\begin{aligned}\sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap\Omega)=X} m_1(A)\, m_2(B)\, m_{\alpha,\cap}(\Omega) &= \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B})=X} m_1(A)\, m_2(B) \\ &= m_1 \circledcirc m_2(X),\end{aligned}$$

as expected.

If $\alpha = 1$, then $m_{\alpha,\cap}(\emptyset) = 1$ and thus the term on the right side of (6.9) reduces to

$$\begin{aligned}\sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap\emptyset)=X} m_1(A)\, m_2(B)\, m_{\alpha,\cap}(\emptyset) &= \sum_{(A\cap B)=X} m_1(A)\, m_2(B) \\ &= m_1 \textcircled{\cap} m_2(X),\end{aligned}$$

as expected.

Proposition 6.4. *Let m_1 and m_2 be two BBAs. We have*

$$m_{1\textcircled{\cup}^{\alpha}2}(X) = \sum_{(A\cap\overline{B})\cup(\overline{A}\cap B)\cup(A\cap B\cap C)=X} m_1(A)\, m_2(B)\, m_{\alpha,\cup}(C), \quad \forall X \subseteq \Omega, \tag{6.10}$$

where $m_{\alpha,\cup}$ is a BBA such that $m_{\alpha,\cup}(A) = \alpha^{|A|}\overline{\alpha}^{|\overline{A}|}$, for all $A \subseteq \Omega$.

Proof. See Appendix F.2. □

Remark 6.2. *If $\alpha = 0$, then the BBA $m_{\alpha,\cup}$ of Proposition 6.4 is such that $m_{\alpha,\cup}(\emptyset) = 1$, and thus the term on the right side of (6.10) reduces to*

$$\begin{aligned}\sum_{(A\cap\overline{B})\cup(\overline{A}\cap B)\cup(A\cap B\cap\emptyset)=X} m_1(A)\, m_2(B)\, m_{\alpha,\cup}(\emptyset) &= \sum_{(A\cap\overline{B})\cup(\overline{A}\cap B)=X} m_1(A)\, m_2(B) \\ &= m_1 \textcircled{\underline{\cup}} m_2(X),\end{aligned}$$

as expected.

If $\alpha = 1$, then $m_{\alpha,\cup}(\Omega) = 1$ and thus the term on the right side of (6.10) reduces, as expected, to $m_1 \textcircled{\cup} m_2(X)$ since we have

$$\begin{aligned}\sum_{(A\cap\overline{B})\cup(\overline{A}\cap B)\cup(A\cap B\cap\Omega)=X} m_1(A)\, m_2(B)\, m_{\alpha,\cup}(\Omega) &= \sum_{(A\cap\overline{B})\cup(\overline{A}\cap B)\cup(A\cap B)=X} m_1(A)\, m_2(B) \\ &= \sum_{A\cup B=X} m_1(A)\, m_2(B),\end{aligned}$$

as

$$\begin{aligned}(A\cap\overline{B})\cup(\overline{A}\cap B)\cup(A\cap B) &= (A\cap(B\cup\overline{B}))\cup(\overline{A}\cap B) \\ &= A\cup(\overline{A}\cap B) \\ &= (A\cup\overline{A})\cap(A\cup B) \\ &= A\cup B.\end{aligned}$$

6.3.2 Truthfulness of the sources

Let $\boldsymbol{\omega}$ be a variable, which takes its values in a frame Ω. Suppose an agent who does not know anything about the actual value ω_0 taken by $\boldsymbol{\omega}$. Suppose a source $S1$ that tells the agent that the actual value ω_0 is in $A \subseteq \Omega$, i.e., $\omega_0 \in A$. If the source tells the truth or, equivalently, is truthful, then the agent believes $\omega_0 \in A$. If the source does not tell the truth, i.e., it lies or, equivalently, "tells the false" [87], then the agent believes $\omega_0 \in \overline{A}$.

Let $\boldsymbol{\tau}$ be a variable taking its values in a frame $T = \{t, f\}$. We use $\boldsymbol{\tau}$ to denote the truthfulness of the source. The information $\omega_0 \in A$ provided by $S1$ can be modeled by a BBA m_1^Ω such that $m_1^\Omega(A) = 1$. The information *when the source tells the truth, ω_0 must be in A, and when the source does not tell the truth, ω_0 must be in $\overline{A}$*, may be modeled by a BBA noted $m_{1'}^{\Omega\times T}$ and defined on the product space $\Omega \times T$ by

$$m_{1'}^{\Omega\times T}(A \times \{t\} \cup \overline{A} \times \{f\}) = 1. \tag{6.11}$$

Note that we use the index $1'$ in $m_{1'}^{\Omega\times T}$, i.e., the source number followed by the prime symbol, to highlight that the BBA $m_{1'}^{\Omega\times T}$ is obtained from the source $S1$, as is the case of the BBA m_1^Ω, but that it conveys a different information from the BBA m_1^Ω.

One verifies that the BBA $m_{1'}^{\Omega\times T}$ is appropriate to model the information available in this scenario since

- combining $m_{1'}^{\Omega\times T}$ with a BBA m_t^T defined on T by $m_t^T(t) = 1$, and then marginalizing on Ω, yields a BBA m_{Ag}^Ω such that $m_{Ag}^\Omega(A) = 1$, i.e., if the agent believes that the source tells the truth, then the agent believes $\omega_0 \in A$;
- combining $m_{1'}^{\Omega\times T}$ with a BBA m_f^T defined on T by $m_f^T(f) = 1$, and then marginalizing on Ω, yields a BBA m_{Ag}^Ω such that $m_{Ag}^\Omega(\overline{A}) = 1$, i.e., if the agent believes that the source does not tell the truth, then the agent believes $\omega_0 \in \overline{A}$.

We may further remark that

$$(m_{1'}^{\Omega\times T} \circledcap m_t^{T\uparrow\Omega\times T})^{\downarrow\Omega} = m_1^\Omega \tag{6.12}$$

and

$$(m_{1'}^{\Omega\times T} \circledcap m_f^{T\uparrow\Omega\times T})^{\downarrow\Omega} = \overline{m_1^\Omega}, \tag{6.13}$$

which is sound since $\overline{m}$ represents the BBA that would be induced if the agent knows that the source providing a BBA m is not telling the truth [87], as mentioned in Section 1.2.

This reasoning may be generalized when the source produces an information in the form of a BBA rather than a set, in which case the BBA $m_{1'}^{\Omega\times T}$ is such that

$$m_{1'}^{\Omega\times T}(A \times \{t\} \cup \overline{A} \times \{f\}) = m_1^\Omega(A), \quad \forall A \subseteq \Omega. \tag{6.14}$$

Here again, Equations (6.12) and (6.13) are verified, which means that, as expected, the agent's beliefs are equated to what the source says if the source tells the truth, and the agent's beliefs are equal to the negation of what the source says if the source does not tell the truth.

Using the BBA $m_{1'}^{\Omega\times T}$, as defined by (6.14), to represent the agent's beliefs when she receives a BBA m_1^Ω from a source $S1$, we may now derive an interpretation for the α-junctions.

Interpretation of the α-conjunctions

Suppose two distinct sources $S1$ and $S2$ that induce two BBAs m_1^Ω and m_2^Ω on Ω. Let $T1 = \{t1, f1\}$ and $T2 = \{t2, f2\}$; these two frames will be used to model beliefs on the truthfulness of $S1$ and $S2$, respectively. Suppose we want to quantify the agent's beliefs on Ω given m_1^Ω, m_2^Ω and the following distinct pieces of evidence.

- A piece of evidence stating that both or none of the sources tell the truth. This piece of evidence may be modeled by a BBA $m_{xand}^{T1\times T2}$ defined by

$$m_{xand}^{T1\times T2}(\{(t1,t2),(f1,f2)\}) = 1. \tag{6.15}$$

- Distinct items of evidence for all $x \in \Omega$ of the form

$$pl^{T1\times T2}[x](\{(f1,f2)\}) = 1-\alpha, \tag{6.16}$$

  indicating that if $\omega_0 = x$, then it is plausible with strength $1-\alpha$ that both sources do not tell the truth.

To compute the agent's beliefs on Ω given these distinct pieces of evidence, the items of evidence of the form given by (6.16), must be transformed into BBAs. In the TBM, this may done using the LCP. The least committed BBA $m^{T1\times T2}[x]$ corresponding to (6.16) is the SBBA $m^{T1\times T2}[x] = \{(t1,t2),(f1,t2),(t1,f2)\}^{\overline{\alpha}}$. Using all these distinct items of evidence, the agent's belief m_{Ag}^Ω on Ω is then equal to

$$\begin{aligned} m_{Ag}^\Omega \;=\; & (m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \circledcap m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \circledcap m_{xand}^{T1\times T2\uparrow\Omega\times T1\times T2} \\ & \circledcap (\circledcap_{x\in\Omega} m^{T1\times T2}[x]^{\Uparrow\Omega\times T1\times T2}))^{\downarrow\Omega}, \end{aligned}$$

with

$$m_{i'}^{\Omega\times Ti}(A\times\{ti\}\cup\overline{A}\times\{fi\}) \;=\; m_i^\Omega(A), \quad \forall A\subseteq\Omega, \quad i=1,2,$$

and

$$m^{T1\times T2}[x] \;=\; \{(t1,t2),(f1,t2),(t1,f2)\}^{\overline{\alpha}}, \quad \forall x\in\Omega,$$

and

$$m_{xand}^{T1\times T2}(\{(t1,t2),(f1,f2)\}) = 1.$$

Theorem 6.1. *Let m_1 and m_2 be two BBAs. We have*

$$\begin{aligned} m_1 \circledcap^\alpha m_2 \;=\; & (m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \circledcap m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \circledcap m_{xand}^{T1\times T2\uparrow\Omega\times T1\times T2} \\ & \circledcap (\circledcap_{x\in\Omega} m^{T1\times T2}[x]^{\Uparrow\Omega\times T1\times T2}))^{\downarrow\Omega}, \end{aligned} \tag{6.17}$$

with $m_{i'}^{\Omega\times Ti}$, $i=1,2$, $m_{xand}^{T1\times T2}$ and $m^{T1\times T2}[x]$ as defined immediately above.

Proof. See Appendix F.3. □

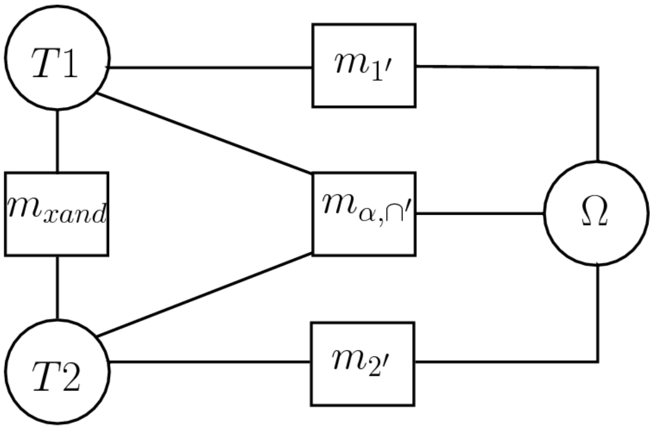

Figure 6.2: Valuation network for the α-conjunction of two BBAs m_1 and m_2. In the network, the term $(\circledcirc_{x\in\Omega} m^{T1\times T2}[x]^{\Uparrow\Omega\times T1\times T2})$ appearing in Theorem 6.1, is replaced by a BBA $m_{\alpha,\cap'}$ defined on $\Omega \times T1 \times T2$.

This theorem may be illustrated with a simple valuation network (Figure 6.2).

As shown by Theorem 6.1, an α-conjunction is equivalent to the pooling by the TBM conjunctive rule of some simple pieces of evidence, which can all be interpreted and that are, moreover, all related to the truthfulness of the sources. In particular, the parameter α involved in the α-conjunctions can be interpreted in terms of the plausibility, given $\omega_0 = x$, that the sources lie, since this plausibility is equal to $1-\alpha$. Note that since the BBA m_{xand} excludes the fact that one and only one source tells the truth, we clearly see, from the interpretation given to α, that we pass from the TBM conjunctive rule to the exclusive conjunctive rule as α varies from 1 to 0. Finally, we may remark that since (6.16) is logically equivalent to

$$bel^{T1\times T2}[x](\{(t1,t2),(f1,t2),(t1,f2)\}) = \alpha,$$

then the parameter α involved in the α-conjunctions is equal to the belief, given $\omega_0 = x$, that at least one of the sources tells the truth.

Interpretation of the α-disjunctions

The α-disjunctions can be interpreted in a similar way. Suppose two distinct sources $S1$ and $S2$ that induce two BBAs m_1^Ω and m_2^Ω on Ω. Suppose we want to compute the agent's beliefs on Ω given m_1^Ω, m_2^Ω and the following distinct pieces of evidence.

- A piece of evidence stating that the sources don't lie simultaneously. This piece of evidence may be modeled by a BBA $m_{or}^{T1\times T2}$ defined by

$$m_{or}^{T1\times T2}(\{(t1,t2),(t1,f2),(f1,t2)\}) = 1. \tag{6.18}$$

- Distinct items of evidence for all $x \in \Omega$ of the form

$$pl^{T1\times T2}[x](\{(t1,t2)\}) = \alpha, \tag{6.19}$$

indicating that if $\omega_0 = x$, then it is plausible with strength α that both sources tell the truth.

The least committed BBA $m^{T1\times T2}[x]$ corresponding to (6.19) is the SBBA $m^{T1\times T2}[x] = \{(f1,t2),(t1,f2),(f1,f2)\}^{\alpha}$. Using all these distinct items of evidence, the agent's belief m^{Ω}_{Ag} on Ω is then equal to

$$\begin{aligned} m^{\Omega}_{Ag} &= (m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \circledcap m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \circledcap m_{or}^{T1\times T2\uparrow\Omega\times T1\times T2} \\ &\quad \circledcap(\circledcap_{x\in\Omega} m^{T1\times T2}[x]^{\Uparrow\Omega\times T1\times T2}))^{\downarrow\Omega}, \end{aligned} \tag{6.20}$$

with

$$m_{i'}^{\Omega\times Ti}(A\times\{ti\}\cup\overline{A}\times\{fi\}) = m_i^{\Omega}(A), \quad \forall A\subseteq\Omega, \quad i=1,2,$$

and

$$m^{T1\times T2}[x] = \{(f1,t2),(t1,f2),(f1,f2)\}^{\alpha}, \quad \forall x\in\Omega,$$

and

$$m_{or}^{T1\times T2}(\{(t1,t2),(t1,f2),(f1,t2)\}) = 1.$$

Theorem 6.2. *Let m_1 and m_2 be two BBAs. We have*

$$\begin{aligned} m_1 \circledcup^{\alpha} m_2 &= (m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \circledcap m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \circledcap m_{or}^{T1\times T2\uparrow\Omega\times T1\times T2} \\ &\quad \circledcap(\circledcap_{x\in\Omega} m^{T1\times T2}[x]^{\Uparrow\Omega\times T1\times T2}))^{\downarrow\Omega}, \end{aligned}$$

with $m_{i'}^{\Omega\times Ti}$, $i=1,2$, $m_{or}^{T1\times T2}$ and $m^{T1\times T2}[x]$ as defined immediately above.

Proof. The proof is similar to the proof of Theorem 6.1. □

As shown by Theorem 6.2, an α-disjunction is equivalent to the pooling by the TBM conjunctive rule of some simple pieces of evidence. In particular, the parameter α involved in the α-disjunctions is equal to the plausibility that the sources tell the truth given $\omega_0 = x$. Note that since the BBA m_{or} excludes the fact that both sources lie, we clearly see, from the interpretation given to α, that we pass from the TBM disjunctive rule to the exclusive disjunctive rule as α varies from 1 to 0.

To complete this section on the interpretation of the α-junctions, we may note that the idea of recovering the TBM disjunctive rule and the exclusive disjunctive rule through the use of the TBM conjunctive rule and BBAs defined on product spaces was investigated by Haenni in [41]. The difference between Haenni's approach and ours is that Haenni used the notion of the reliability of the sources, rather than their truthfulness. In particular, he claimed that the exclusive disjunctive rule corresponds to the situation where exactly one of the sources is reliable. In Appendix D, we show that this claim is wrong, and we provide a short discussion on the implications of making the difference between a reliable source and a truthful source.

6.4 Computation

In addition to lacking an interpretation, the α-junctions suffered in Smets's paper [87] from another limitation: they were hard to compute. Indeed, as remarked by Smets [91] and as can be seen with Example 6.1 where Ω contains only two elements, the definitions of the matrices underlying the α-junctions are quite laborious and thus using an α-junctive rule looks like a complicated task. It seems thus interesting to have simpler and more efficient mechanisms to perform a combination by an α-junctive rule. This is the topic of this section.

As the reader may have noticed, we have already brought to light some new means to perform a combination by an α-conjunctive rule through either:

1. the α-conditioning operation (see Proposition 6.1);
2. or a "classical[3]" expression (see Proposition 6.3);
3. or the TBM conjunctive rule and BBAs defined on product spaces (see Theorem 6.1).

In Sections 6.4.1 and 6.4.2, we propose yet another technique. In Section 6.4.3, the computations involved in the use of each of these new means, will be illustrated with an example. Furthermore, these new means will also be compared with one another in Section 6.4.3.

6.4.1 The α-commonality function

In this section, a simple way to compute an α-conjunction is proposed. It is based on the eigendecomposition of the matrix $\mathbf{K}_m$.

The eigendecomposition of the matrices $\mathbf{K}_X^{\cap,\alpha}$, given in [91], is

$$\mathbf{K}_X^{\cap,\alpha} = (\mathbf{G}^{\cap,\alpha})^{-1} \cdot \mathbf{V}_X^{\cap,\alpha} \cdot \mathbf{G}^{\cap,\alpha},$$

where

$$\begin{aligned} \mathbf{V}_\Omega^{\cap,\alpha} &= \mathbf{I}, \\ \mathbf{V}_X^{\cap,\alpha} &= \prod_{x \notin X} \mathbf{V}_{\{\overline{x}\}}^{\cap,\alpha}, \quad \forall X \subseteq \Omega, \\ \mathbf{V}_{\{\overline{x}\}}^{\cap,\alpha} &= \left[v_{\overline{x}}^{\cap,\alpha}(A,B) \right], \quad \forall x \in \Omega, \end{aligned} \tag{6.21}$$

with

$$v_{\overline{x}}^{\cap,\alpha}(A,B) = \begin{cases} 1 & \text{if } x \notin A, \quad A = B, \\ \alpha - 1 & \text{if } x \in A, \quad A = B, \\ 0 & \text{if } A \neq B, \end{cases}$$

where $\alpha \in [0,1]$ and is constant for all $\mathbf{V}_X^{\cap,\alpha}$.

[3] We use the term *classical* since the expression in Proposition 6.3 is a generalization of the classical, or most often encountered, definition of the TBM conjunctive rule given by Equation (1.4).

The X column of the $2^{|\Omega|} \times 2^{|\Omega|}$ $\mathbf{G}^{\cap,\alpha}$ matrix is $\mathbf{V}_X^{\cap,\alpha} \cdot \mathbf{1}$. Furthermore, all matrices $\mathbf{K}_X^{\cap,\alpha}$ for $X \subseteq \Omega$ share the same left and right eigenvectors [87], hence the matrix $\mathbf{G}^{\cap,\alpha}$ does not depend on X [91]. We may thus obtain the following expression for $\mathbf{K}_m^{\cap,\alpha}$:

$$\begin{aligned}\mathbf{K}_m^{\cap,\alpha} &= \sum_{X \subseteq \Omega} m(X) \cdot \mathbf{K}_X^{\cap,\alpha} \\ &= \sum_{X \subseteq \Omega} m(X) \cdot (\mathbf{G}^{\cap,\alpha})^{-1} \cdot \mathbf{V}_X^{\cap,\alpha} \cdot \mathbf{G}^{\cap,\alpha} \\ &= (\mathbf{G}^{\cap,\alpha})^{-1} \cdot \left(\sum_{X \subseteq \Omega} m(X) \cdot \mathbf{V}_X^{\cap,\alpha} \right) \cdot \mathbf{G}^{\cap,\alpha}.\end{aligned}$$

Hence,

$$\mathbf{G}^{\cap,\alpha} \cdot \mathbf{K}_m^{\cap,\alpha} = \left(\sum_{X \subseteq \Omega} m(X) \cdot \mathbf{V}_X^{\cap,\alpha} \right) \cdot \mathbf{G}^{\cap,\alpha}. \tag{6.22}$$

In this last equation, we recognize an eigendecomposition: the elements on the diagonal of the diagonal matrix $\sum_{X \subseteq \Omega} m(X) \cdot \mathbf{V}_X^{\cap,\alpha}$ are the eigenvalues of $\mathbf{K}_m^{\cap,\alpha}$ (the vector $\left(\sum_{X \subseteq \Omega} m(X) \cdot \mathbf{V}_X^{\cap,\alpha}\right) \cdot \mathbf{1}$ is thus the vector of eigenvalues of $\mathbf{K}_m^{\cap,\alpha}$), and the rows of the matrix $\mathbf{G}^{\cap,\alpha}$ are the corresponding left eigenvectors of $\mathbf{K}_m^{\cap,\alpha}$ [87].

Let $\mathbf{g}^{\cap,\alpha} = \mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}$. From (6.22), we may obtain:

$$\begin{aligned}\mathbf{G}^{\cap,\alpha} \cdot \mathbf{K}_m^{\cap,\alpha} \cdot \mathbf{1}_\Omega &= \left(\sum_{X \subseteq \Omega} m(X) \cdot \mathbf{V}_X^{\cap,\alpha} \right) \cdot \mathbf{G}^{\cap,\alpha} \cdot \mathbf{1}_\Omega \\ \mathbf{G}^{\cap,\alpha} \cdot \mathbf{m} &= \left(\sum_{X \subseteq \Omega} m(X) \cdot \mathbf{V}_X^{\cap,\alpha} \right) \cdot \mathbf{1},\end{aligned}$$

as $\mathbf{1}_\Omega$ is the neutral element for an α-conjunction, and as $G^{\cap,\alpha}(A, \Omega) = 1$ for all $A \subseteq \Omega$. Hence, the vector $\mathbf{g}^{\cap,\alpha}$ is the vector of eigenvalues of $\mathbf{K}_m^{\cap,\alpha}$ and is the analogous of the commonality function within the generalized context of the α-conjunction [87]. However, let us note that, contrary to the commonality function, these eigenvalues are not necessarily on the diagonal of $\mathbf{K}_m^{\cap,\alpha}$ as shown by Example 6.2.

Example 6.2. *Let $\Omega = \{a, b\}$ be a frame of discernment. Let m be a BBA defined on Ω by $m(\{b\}) = 0.8$ and $m(\Omega) = 0.2$. Let $\alpha = 0.6$ and $m_{vac} = m_\Omega$. The function $g^{\cap,0.6}$ corresponding to this BBA is the following:*

$$\begin{aligned}g^{\cap,0.6}(\emptyset) &= 1 \\ g^{\cap,0.6}(\{a\}) &= -0.12 \\ g^{\cap,0.6}(\{b\}) &= 1 \\ g^{\cap,0.6}(\Omega) &= -0.12\end{aligned}$$

The matrix $\mathbf{K}_m^{\cap,0.6}$ *associated to* m *when* $\alpha = 0.6$ *and* $m_{vac} = m_\Omega$ *is the following:*

$$\begin{aligned}
\mathbf{K}_m^{\cap,0.6} &= \sum_{X\subseteq\Omega} m(X)\cdot\mathbf{K}_X^{\cap,0.6}\\
&= m(b)\cdot\mathbf{K}_b^{\cap,0.6} + m(\Omega)\cdot\mathbf{I}\\
&= \begin{bmatrix} m(b)\alpha + m(\Omega) & m(b) & \cdot & \cdot \\ m(b)\overline{\alpha} & m(\Omega) & \cdot & \cdot \\ \cdot & \cdot & m(b)\alpha + m(\Omega) & m(b) \\ \cdot & \cdot & m(b)\overline{\alpha} & m(\Omega) \end{bmatrix}\\
&= \begin{bmatrix} 0.68 & 0.8 & \cdot & \cdot \\ 0.32 & 0.2 & \cdot & \cdot \\ \cdot & \cdot & 0.68 & 0.8 \\ \cdot & \cdot & 0.32 & 0.2 \end{bmatrix}
\end{aligned} \tag{6.23}$$

We can see that the elements on the diagonal of $\mathbf{K}_m^{\cap,0.6}$ *are not its eigenvalues, since the eigenvalues are given by* $g^{\cap,0.6}$.

Let $m_1 \textcircled{\cap}^\alpha m_2 = m_{1\textcircled{\cap}^\alpha 2}$ and let $\mathbf{g}_{1\textcircled{\cap}^\alpha 2}^{\cap,\alpha} = \mathbf{G}^{\cap,\alpha}\cdot\mathbf{m}_{1\textcircled{\cap}^\alpha 2}$. We have $g_{1\textcircled{\cap}^\alpha 2}^{\cap,\alpha} = g_1^{\cap,\alpha}\cdot g_2^{\cap,\alpha}$ (using the counterpart of [87, Theorem 12] for the α-conjunctions), from which we obtain

$$\mathbf{m}_{1\textcircled{\cap}^\alpha 2} = (\mathbf{G}^{\cap,\alpha})^{-1}\cdot\mathbf{Diag}(\mathbf{g}_1^{\cap,\alpha})\cdot\mathbf{g}_2^{\cap,\alpha}. \tag{6.24}$$

Hence, the combination of two BBAs m_1 and m_2 by an α-conjunctive rule can be simply expressed as the pointwise product of the functions $g_1^{\cap,\alpha}$ and $g_2^{\cap,\alpha}$ associated, respectively, to m_1 and m_2. This is a first step in the simplification of the computation by an α-conjunction. However, one may note that the definition of the matrix $\mathbf{G}^{\cap,\alpha}$ is as tedious as the definition of the matrix $\mathbf{K}_m^{\cap,\alpha}$. Fortunately, Theorem 6.3 shows that it is possible to obtain the matrix $\mathbf{G}^{\cap,\alpha}$ in a simple manner, similar to the way the matrices $\mathbf{B}$ and $\mathbf{Q}$ can be obtained, i.e., using Kronecker multiplication and a particular building block.

Theorem 6.3. *The matrix* $\mathbf{G}^{\cap,\alpha}$ *can be obtained by Kronecker multiplication using the building block:*

$$\begin{bmatrix} 1 & 1 \\ \alpha - 1 & 1 \end{bmatrix} \tag{6.25}$$

Proof. See Appendix F.4. □

Corollary 6.1. *The matrix* $(\mathbf{G}^{\cap,\alpha})^{-1}$ *can be obtained by Kronecker multiplication using the building block:*

$$\left(\begin{bmatrix} 1 & 1 \\ \alpha - 1 & 1 \end{bmatrix}\right)^{-1} = \frac{1}{2-\alpha}\cdot\begin{bmatrix} 1 & -1 \\ 1-\alpha & 1 \end{bmatrix}. \tag{6.26}$$

Proof. This corollary may easily be shown using the following property of the Kronecker product. Let $\mathbf{A}$ be a $m \times n$ matrix and let $\mathbf{B}$ a $p \times q$ matrix. $\mathbf{Kron}(\mathbf{A}, \mathbf{B})$ is invertible if and only if $\mathbf{A}$ and $\mathbf{B}$ are invertible, in which case the inverse of $\mathbf{Kron}(\mathbf{A}, \mathbf{B})$ is given by

$$(\mathbf{Kron}(\mathbf{A}, \mathbf{B}))^{-1} = \mathbf{Kron}(\mathbf{A}^{-1}, \mathbf{B}^{-1}).$$

□

Let us remark that Equation (6.24) together with Theorem 6.3 and Corollary 6.1 make the computation of an α-conjunction really simple. This simple new way of computing an α-conjunction will be illustrated with an example in Section 6.4.3.

Two consequences of Theorem 6.3 are the following.

First, we can easily see that the $\mathbf{G}^{\cap,\alpha}$ matrix generalizes the $\mathbf{Q}$ matrix in that we have $\mathbf{G}^{\cap,1} = \mathbf{Q}$ and thus $g^{\cap,1} = q$. This is interesting because we will see in the next section that the matrix $\mathbf{G}^{\cup,\alpha}$, whose role is similar to that of the $\mathbf{G}^{\cap,\alpha}$ matrix in the context of the α-disjunctions, does not generalize the matrix $\mathbf{B}$ as one might have expected. The fact that the function $g^{\cap,\alpha}$ generalizes the commonality function can also be used to call this function the *α-commonality function* associated to a BBA m.

Second, we may easily show the following proposition, which will be useful in the next chapter.

Proposition 6.5. *For all $\alpha \in [0, 1]$, we have*

$$g^{\cap,\alpha}(\emptyset) = 1. \tag{6.27}$$

Proof. In order to show (6.27), one merely needs to show that

$$G^{\cap,\alpha}(\emptyset, A) = 1, \quad \forall A \subseteq \Omega, \tag{6.28}$$

holds, since $\sum_{A\subseteq\Omega} m(A) = 1$. The fact that (6.28) holds may easily be shown from the fact that the matrix $\mathbf{G}^{\cap,\alpha}$ is based on the building block given in Theorem 6.3. □

6.4.2 Alternative definition of the α-implicability function

In this section, we review Smets's presentation of the eigendecomposition of $\mathbf{K}_m^{\cup,\alpha}$ and we propose a slight technical modification to a part of his presentation.

The eigendecomposition of the matrices $\mathbf{K}_X^{\cup,\alpha}$ is given by

$$\mathbf{K}_X^{\cup,\alpha} = (\mathbf{G}^{\cup,\alpha})^{-1} \cdot \mathbf{V}_X^{\cup,\alpha} \cdot \mathbf{G}^{\cup,\alpha},$$

where

$$\begin{aligned}
\mathbf{V}_\Omega^{\cup,\alpha} &= \mathbf{I}, \\
\mathbf{V}_X^{\cup,\alpha} &= \prod_{x\in X} \mathbf{V}_{\{x\}}^{\cup,\alpha}, \quad \forall X \subseteq \Omega, \\
\mathbf{V}_{\{x\}}^{\cup,\alpha} &= \left[v_x^{\cup,\alpha}(A, B)\right], \quad \forall x \in \Omega,
\end{aligned}$$

with

$$v_x^{\cup,\alpha}(A,B) = \begin{cases} 1 & \text{if } x \notin A, \quad A = B, \\ \alpha - 1 & \text{if } x \in A, \quad A = B, \\ 0 & \text{if } A \neq B, \end{cases}$$

where $\alpha \in [0,1]$ and is constant for all $\mathbf{V}_X^{\cup,\alpha}$. The X column of the $\mathbf{G}^{\cup,\alpha}$ matrix is $\mathbf{V}_X^{\cup,\alpha} \cdot \mathbf{1}$.

In a similar way as was done in the previous section, it may be shown that $\sum_{X \subseteq \Omega} m(X) \cdot \mathbf{V}_X^{\cup,\alpha}$ is a diagonal matrix, whose diagonal elements are the eigenvalues of $\mathbf{K}_m^{\cup,\alpha}$ and that the rows of $\mathbf{G}^{\cup,\alpha}$ are the corresponding left eigenvectors. Furthermore, the vector $\mathbf{g}^{\cup,\alpha} = \mathbf{G}^{\cup,\alpha} \cdot \mathbf{m}$ is the vector of eigenvalues of $\mathbf{K}_m^{\cup,\alpha}$. We can also show that

$$\mathbf{m}_{1 \textcircled{\cup}^{\alpha} 2} = (\mathbf{G}^{\cup,\alpha})^{-1} \cdot \mathbf{Diag}(\mathbf{g}_1^{\cup,\alpha}) \cdot \mathbf{g}_2^{\cup,\alpha} \tag{6.29}$$

holds. Hence, the combination of two BBAs m_1 and m_2 by an α-disjunctive rule can be simply expressed as the pointwise product of the functions $g_1^{\cup,\alpha}$ and $g_2^{\cup,\alpha}$ associated, respectively, to m_1 and m_2. In particular, we have

$$\mathbf{m}_{1 \textcircled{\cup} 2} = (\mathbf{G}^{\cup,1})^{-1} \cdot \mathbf{Diag}(\mathbf{g}_1^{\cup,1}) \cdot \mathbf{g}_2^{\cup,1}. \tag{6.30}$$

However, as shown below, we do not have $\mathbf{G}^{\cup,1} = \mathbf{B}$ (and thus we also do not have $g^{\cup,1} = b$) as could be expected from the comparison of (6.30) with $\mathbf{m}_{1 \textcircled{\cup} 2} = \mathbf{B}^{-1} \cdot \mathbf{Diag}(\mathbf{b}_1) \cdot \mathbf{b}_2$, and from the fact that the matrix $\mathbf{G}^{\cap,\alpha}$ generalized the matrix $\mathbf{Q}$ when $\alpha = 1$. Indeed, let $\Omega = \{a, b\}$. The matrix $\mathbf{G}^{\cup,\alpha}$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\overline{\alpha} & 1 & -\overline{\alpha} \\ 1 & 1 & -\overline{\alpha} & -\overline{\alpha} \\ 1 & -\overline{\alpha} & -\overline{\alpha} & -\overline{\alpha}^2 \end{bmatrix},$$

whereas the matrix $\mathbf{B}$ is

$$\begin{bmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & . & 1 & . \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Clearly, $\mathbf{G}^{\cup,\alpha} \neq \mathbf{B}$ when $\alpha = 1$.

We have seen in Section 1.4.2 that the rows of the matrix $\mathbf{Q}$ are the left eigenvectors of the Dempsterian specialization matrix $\mathbf{S}_m$. Furthermore, it was explained that there exist $2^{|\Omega|}!$ ways to order those left eigenvectors in a matrix and thus there exist $2^{|\Omega|}!$ $\mathbf{Q}$-like matrices. The reason why we do not have $\mathbf{G}^{\cup,\alpha} = \mathbf{B}$ when $\alpha = 1$ is that the matrix $\mathbf{G}^{\cup,\alpha}$ is actually a permutation of another matrix of left eigenvectors of $\mathbf{K}_m^{\cup,\alpha}$, which does generalize $\mathbf{B}$ when $\alpha = 1$. We note this other matrix $\mathbf{G}_{new}^{\cup,\alpha}$ and the matrix $\mathbf{G}^{\cup,\alpha}$ is noted $\mathbf{G}_{smets}^{\cup,\alpha}$.

Proposition 6.6. *The matrix $\mathbf{G}_{new}^{\cup,\alpha}$ of left eigenvectors of $\mathbf{K}_m^{\cup,\alpha}$, which generalizes the $\mathbf{B}$ matrix, is defined by $\mathbf{G}_{new}^{\cup,\alpha} = \mathbf{J} \cdot \mathbf{G}_{smets}^{\cup,\alpha}$. It may be obtained using Kronecker multiplication and the building block*

$$\begin{bmatrix} 1 & \alpha - 1 \\ 1 & 1 \end{bmatrix}.$$

Proof. See Appendix F.5. □

The function $g_{new}^{\cup,\alpha}$ defined by $\mathbf{g}_{new}^{\cup,\alpha} = \mathbf{G}_{new}^{\cup,\alpha} \cdot \mathbf{m}$ will be called the α-implicability function associated to a BBA m, since it generalizes the implicability function.

There exists a relation between the matrices $\mathbf{Q}$ and $\mathbf{B}$: $\mathbf{Q} = \mathbf{J} \cdot \mathbf{B} \cdot \mathbf{J}$. Proposition 6.7 shows that a similar property holds for the matrices $\mathbf{G}_{new}^{\cup,\alpha}$ and $\mathbf{G}^{\cap,\alpha}$, which are generalizations of the matrices $\mathbf{B}$ and $\mathbf{Q}$, respectively.

Proposition 6.7. $\mathbf{G}^{\cap,\alpha} = \mathbf{J} \cdot \mathbf{G}_{new}^{\cup,\alpha} \cdot \mathbf{J}$.

Proof. We show this proposition by induction and using the fact that matrices $\mathbf{G}^{\cap,\alpha}$ and $\mathbf{G}_{new}^{\cup,\alpha}$ are based on Kronecker product.

In this proof, let $\mathbf{G}^{\cap,\alpha,n}$ and $\mathbf{G}_{new}^{\cup,\alpha,n}$ denote, respectively, the $2^n \times 2^n$ matrices $\mathbf{G}^{\cap,\alpha}$ and $\mathbf{G}_{new}^{\cup,\alpha}$ when Ω has cardinality n.

The base case of this proof by induction, i.e., $\mathbf{G}^{\cap,\alpha,1} = \mathbf{J} \cdot \mathbf{G}_{new}^{\cup,\alpha,1} \cdot \mathbf{J}$, clearly holds since we have

$$\begin{bmatrix} 1 & 1 \\ \alpha - 1 & 1 \end{bmatrix} = \mathbf{J} \cdot \begin{bmatrix} 1 & \alpha - 1 \\ 1 & 1 \end{bmatrix} \cdot \mathbf{J}.$$

Suppose

$$\mathbf{G}^{\cap,\alpha,n} = \mathbf{J}^n \cdot \mathbf{G}_{new}^{\cup,\alpha,n} \cdot \mathbf{J}^n \tag{6.31}$$

holds. We must show that $\mathbf{G}^{\cap,\alpha,n+1} = \mathbf{J}^{n+1} \cdot \mathbf{G}_{new}^{\cup,\alpha,n+1} \cdot \mathbf{J}^{n+1}$ holds. We have

$$\begin{aligned}
\mathbf{G}^{\cap,\alpha,n+1} &= \begin{bmatrix} \mathbf{G}^{\cap,\alpha,n} & \mathbf{G}^{\cap,\alpha,n} \\ (\alpha-1)\mathbf{G}^{\cap,\alpha,n} & \mathbf{G}^{\cap,\alpha,n} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{J}^n \cdot \mathbf{G}_{new}^{\cup,\alpha,n} \cdot \mathbf{J}^n & \mathbf{J}^n \cdot \mathbf{G}_{new}^{\cup,\alpha,n} \cdot \mathbf{J}^n \\ (\alpha-1)\mathbf{J}^n \cdot \mathbf{G}_{new}^{\cup,\alpha,n} \cdot \mathbf{J}^n & \mathbf{J}^n \cdot \mathbf{G}_{new}^{\cup,\alpha,n} \cdot \mathbf{J}^n \end{bmatrix} \quad \text{(from (6.31))} \\
&= \mathbf{J}^{n+1} \cdot \begin{bmatrix} \mathbf{G}_{new}^{\cup,\alpha,n} & (\alpha-1)\mathbf{G}_{new}^{\cup,\alpha,n} \\ \mathbf{G}_{new}^{\cup,\alpha,n} & \mathbf{G}_{new}^{\cup,\alpha,n} \end{bmatrix} \cdot \mathbf{J}^{n+1} \quad \text{(from Lemma F.7 in Appendix F.5)} \\
&= \mathbf{J}^{n+1} \cdot \mathbf{G}_{new}^{\cup,\alpha,n+1} \cdot \mathbf{J}^{n+1}. \quad \text{(from (F.38))}
\end{aligned}$$

□

Proposition 6.8 shows the counterpart to Proposition 6.5, which will be useful in the next chapter.

Proposition 6.8. *For all $\alpha \in [0, 1]$, we have*

$$g_{new}^{\cup,\alpha}(\Omega) = 1. \tag{6.32}$$

Proof. The proof is similar to the proof of Proposition 6.5. □

Finally, it was explained at the beginning of this section that the matrices $\mathbf{K}_X^{\cup,\alpha}$ for $X \subseteq \Omega$ admit an eigendecomposition of the form $\mathbf{K}_X^{\cup,\alpha} = (\mathbf{G}_{smets}^{\cup,\alpha})^{-1} \cdot \mathbf{V}_X^{\cup,\alpha} \cdot \mathbf{G}_{smets}^{\cup,\alpha}$. For completeness of the presentation of the new matrix $\mathbf{G}_{new}^{\cup,\alpha}$, Proposition 6.9 provides the expression of the corresponding diagonal matrices, noted $\mathbf{V}_{X,new}^{\cup,\alpha}$, which store the eigenvalues of the matrices $\mathbf{K}_X^{\cup,\alpha}$.

Proposition 6.9. *For all $X \subseteq \Omega$, the matrix $\mathbf{K}_X^{\cup,\alpha}$ admits the expression*

$$\mathbf{K}_X^{\cup,\alpha} = (\mathbf{G}_{new}^{\cup,\alpha})^{-1} \cdot \mathbf{V}_{X,new}^{\cup,\alpha} \cdot \mathbf{G}_{new}^{\cup,\alpha},$$

where $\mathbf{V}_{X,new}^{\cup,\alpha} = \mathbf{J} \cdot \mathbf{V}_X^{\cup,\alpha}\mathbf{J}$.

Proof.

$$\begin{aligned}
\mathbf{K}_X^{\cup,\alpha} &= (\mathbf{G}_{Smets}^{\cup,\alpha})^{-1} \cdot \mathbf{V}_X^{\cup,\alpha} \cdot \mathbf{G}_{Smets}^{\cup,\alpha} \\
&= (\mathbf{G}_{Smets}^{\cup,\alpha})^{-1} \cdot \mathbf{J} \cdot \mathbf{J} \cdot \mathbf{V}_X^{\cup,\alpha} \cdot \mathbf{J} \cdot \mathbf{J} \cdot \mathbf{G}_{Smets}^{\cup,\alpha} \\
&= (\mathbf{G}_{Smets}^{\cup,\alpha})^{-1} \cdot \mathbf{J}^{-1} \cdot \mathbf{J} \cdot \mathbf{V}_X^{\cup,\alpha} \cdot \mathbf{J} \cdot \mathbf{G}_{new}^{\cup,\alpha} \\
&= (\mathbf{J} \cdot \mathbf{G}_{Smets}^{\cup,\alpha})^{-1} \cdot \mathbf{J} \cdot \mathbf{V}_X^{\cup,\alpha} \cdot \mathbf{J} \cdot \mathbf{G}_{new}^{\cup,\alpha} \\
&= (\mathbf{G}_{new}^{\cup,\alpha})^{-1} \cdot \mathbf{J} \cdot \mathbf{V}_X^{\cup,\alpha} \cdot \mathbf{J} \cdot \mathbf{G}_{new}^{\cup,\alpha} \\
&= (\mathbf{G}_{new}^{\cup,\alpha})^{-1} \cdot \mathbf{V}_{X,new}^{\cup,\alpha} \cdot \mathbf{G}_{new}^{\cup,\alpha},
\end{aligned}$$

with $\mathbf{V}_{X,new}^{\cup,\alpha} = \mathbf{J} \cdot \mathbf{V}_X^{\cup,\alpha} \cdot \mathbf{J}$. □

6.4.3 Comparison and illustration of the new computation methods

In this section, the new means proposed for the computation of the combination by an α-conjunctive rule, are first illustrated with a simple example and then compared with one another.

We have laid bare four new ways of performing such a combination:

1. using the α-conditioning operation (see Proposition 6.1);
2. using a "classical" expression (see Proposition 6.3);
3. using the TBM conjunctive rule and BBAs defined on product spaces (see Theorem 6.1);
4. using the α-commonality function obtained using Kronecker product (see Equation (6.24), Theorem 6.3 and Corollary 6.1)[4].

These four new means are illustrated, respectively, by Examples 6.3, 6.4, 6.5 and 6.6 below. In these examples, we compute the α-conjunction of a BBA m_1 with another BBA m_2, with m_1 and m_2 defined on $\Omega = \{a, b\}$ by, respectively, $m_1(\{b\}) = 1$ and $m_2(\{b\}) = 0.8$, $m_2(\Omega) = 0.2$, and where $\alpha = 0.6$.

Example 6.3 (Computation of an α-conjunction using the α-conditioning operation)**.** *For the reader's convenience, let us recall that we have from Proposition 6.1:*

$$m_{1 \circledcap^{\alpha} 2}(X) = \sum_{B \subseteq \Omega} m_1 [B]^{\alpha} (X) \, m_2 (B), \quad \forall X \subseteq \Omega. \tag{6.33}$$

[4] Note that, as will be shown in the next chapter, this fourth way of computing an α-conjunction may lead us to view an α-conjunction as the combination by the TBM conjunctive rule of signed belief functions.

Furthermore, from Proposition 6.2, we have, for all $X, B \subseteq \Omega$

$$m[B]^\alpha(X) = \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X} m(A)\, m_{\alpha,\cap}(C), \tag{6.34}$$

where $m_{\alpha,\cap}(X) = \alpha^{|\overline{X}|}\overline{\alpha}^{|X|}$*, for all* $X \subseteq \Omega$ *and where* $\overline{\alpha} = 1-\alpha$.

To compute $m_{1 \textcircled{\cap}^\alpha 2}$ *using (6.33), we need first to compute* $m_1[B]^\alpha$*, for all* $B \subseteq \Omega$*, using (6.34).*

Let $B = \emptyset$*. We find*

$$\begin{aligned} m_1[\emptyset]^\alpha(\emptyset) &= m_{\alpha,\cap}(\emptyset) + m_{\alpha,\cap}(\{b\}) \\ &= \alpha^2 + \alpha\overline{\alpha} = \alpha = 0.6, \end{aligned}$$

$$\begin{aligned} m_1[\emptyset]^\alpha(\{a\}) &= m_{\alpha,\cap}(\{a\}) + m_{\alpha,\cap}(\Omega) \\ &= \alpha\overline{\alpha} + \overline{\alpha}^2 + = \overline{\alpha} = 0.4, \end{aligned}$$

and $m_1[\emptyset]^\alpha(\{b\}) = m_1[\emptyset]^\alpha(\Omega) = 0$.

Let $B = \{a\}$*. We have*

$$m_1[\{a\}]^\alpha(\emptyset) = m_{\alpha,\cap}(\emptyset) + m_{\alpha,\cap}(\{a\}) + m_{\alpha,\cap}(\{b\}) + m_{\alpha,\cap}(\Omega) = 1$$

and $m_1[\{a\}]^\alpha(\{a\}) = m_1[\{a\}]^\alpha(\{b\}) = m_1[\{a\}]^\alpha(\Omega) = 0$.

Let $B = \{b\}$*. We find*

$$\begin{aligned} m_1[\{b\}]^\alpha(\{b\}) &= m_{\alpha,\cap}(\emptyset) + m_{\alpha,\cap}(\{b\}) \\ &= 0.6, \end{aligned}$$

$$\begin{aligned} m_1[\{b\}]^\alpha(\Omega) &= m_{\alpha,\cap}(\{a\}) + m_{\alpha,\cap}(\Omega) \\ &= 0.4, \end{aligned}$$

and $m_1[\{b\}]^\alpha(\{a\}) = m_1[\{b\}]^\alpha(\emptyset) = 0$.

Let $B = \Omega$*. We have*

$$m_1[\Omega]^\alpha(\Omega) = m_{\alpha,\cap}(\emptyset) + m_{\alpha,\cap}(\{a\}) + m_{\alpha,\cap}(\{b\}) + m_{\alpha,\cap}(\Omega) = 1$$

and $m_1[\Omega]^\alpha(\{a\}) = m_1[\Omega]^\alpha(\{b\}) = m_1[\Omega]^\alpha(\emptyset) = 0$.

Having computed the α*-conditioning of* m_1 *by all* $B \subseteq \Omega$*, we may now compute* $m_{1 \textcircled{\cap}^\alpha 2}$ *using (6.33). We find*

$$\begin{aligned} m_{1 \textcircled{\cap}^\alpha 2}(\emptyset) &= m_1[\emptyset]^\alpha(\emptyset)m_2(\emptyset) + m_1[\{a\}]^\alpha(\emptyset)m_2(\{a\}) \\ &\quad + m_1[\{b\}]^\alpha(\emptyset)m_2(\{b\}) + m_1[\Omega]^\alpha(\emptyset)m_2(\Omega) = 0, \end{aligned}$$

$$\begin{aligned} m_{1 \textcircled{\cap}^\alpha 2}(\{a\}) &= m_1[\emptyset]^\alpha(\{a\})m_2(\emptyset) + m_1[\{a\}]^\alpha(\{a\})m_2(\{a\}) \\ &\quad + m_1[\{b\}]^\alpha(\{a\})m_2(\{b\}) + m_1[\Omega]^\alpha(\{a\})m_2(\Omega) = 0, \end{aligned}$$

$$\begin{aligned} m_{1 \circledcap^{\alpha} 2}(\{b\}) &= m_1 [\emptyset]^{\alpha} (\{b\}) m_2(\emptyset) + m_1 [\{a\}]^{\alpha} (\{b\}) m_2(\{a\}) \\ &\quad + m_1 [\{b\}]^{\alpha} (\{b\}) m_2(\{b\}) + m_1 [\Omega]^{\alpha} (\{b\}) m_2(\Omega) \\ &= 0.6 \cdot 0.8 + 1 \cdot 0.2 = 0.68, \end{aligned}$$

and

$$\begin{aligned} m_{1 \circledcap^{\alpha} 2}(\Omega) &= m_1 [\emptyset]^{\alpha} (\Omega) m_2(\emptyset) + m_1 [\{a\}]^{\alpha} (\Omega) m_2(\{a\}) \\ &\quad + m_1 [\{b\}]^{\alpha} (\Omega) m_2(\{b\}) + m_1 [\Omega]^{\alpha} (\Omega) m_2(\Omega) \\ &= 0.4 \cdot 0.8 = 0.32. \end{aligned}$$

Example 6.4 (Computation of an α-conjunction using a classical expression)**.** *Recall that we have from Proposition 6.3:*

$$m_{1 \circledcap^{\alpha} 2} (X) = \sum_{(A \cap B) \cup (\overline{A} \cap \overline{B} \cap C) = X} m_1 (A) \, m_2 (B) \, m_{\alpha, \cap} (C), \quad \forall X \subseteq \Omega, \tag{6.35}$$

where $m_{\alpha,\cap}(X) = \alpha^{|\overline{X}|} \overline{\alpha}^{|X|}$, *for all* $X \subseteq \Omega$. *Applying (6.35), we find*

$$\begin{aligned} m_{1 \circledcap^{\alpha} 2}(\{b\}) &= m_1(\{b\}) m_2(\{b\}) m_{\alpha,\cap}(\emptyset) + m_1(\{b\}) m_2(\{b\}) m_{\alpha,\cap}(\{b\}) \\ &\quad + m_1(\{b\}) m_2(\Omega) m_{\alpha,\cap}(\emptyset) + m_1(\{b\}) m_2(\Omega) m_{\alpha,\cap}(\{a\}) \\ &\quad + m_1(\{b\}) m_2(\Omega) m_{\alpha,\cap}(\{b\}) + m_1(\{b\}) m_2(\Omega) m_{\alpha,\cap}(\Omega) \\ &= m_2(\{b\}) m_{\alpha,\cap}(\emptyset) + m_2(\{b\}) m_{\alpha,\cap}(\{b\}) + m_2(\Omega) \\ &= 0.8 \cdot 0.36 + 0.8 \cdot 0.24 + 0.2 = 0.68, \end{aligned}$$

$$\begin{aligned} m_{1 \circledcap^{\alpha} 2}(\Omega) &= m_1(\{b\}) m_2(\{b\}) m_{\alpha,\cap}(\{a\}) + m_1(\{b\}) m_2(\{b\}) m_{\alpha,\cap}(\Omega) \\ &= 0.8 \cdot 0.24 + 0.8 \cdot 0.16 = 0.32 \end{aligned}$$

and $m_{1 \circledcap^{\alpha} 2}(\emptyset) = m_{1 \circledcap^{\alpha} 2}(\{a\}) = 0$.

Example 6.5 (Computation of an α-conjunction using the TBM conjunctive rule and BBAs defined on product spaces)**.** *Recall that we have from Theorem 6.1:*

$$\begin{aligned} m_{1 \circledcap^{\alpha} 2} &= (m_{1'}^{\Omega \times T1 \uparrow \Omega \times T1 \times T2} \circledcap m_{2'}^{\Omega \times T2 \uparrow \Omega \times T1 \times T2} \circledcap m_{xand}^{T1 \times T2 \uparrow \Omega \times T1 \times T2} \\ &\quad \circledcap (\circledcap_{x \in \Omega} m^{T1 \times T2}[x]^{\Uparrow \Omega \times T1 \times T2}))^{\downarrow \Omega}, \end{aligned} \tag{6.36}$$

with

$$m_{i'}^{\Omega \times Ti}(A \times \{ti\} \cup \overline{A} \times \{fi\}) = m_i^{\Omega}(A), \quad \forall A \subseteq \Omega, \quad i = 1, 2, \tag{6.37}$$

and

$$m^{T1 \times T2}[x] = \{(t1, t2), (f1, t2), (t1, f2)\}^{\overline{\alpha}}, \quad \forall x \in \Omega,$$

and

$$m_{xand}^{T1 \times T2}(\{(t1, t2), (f1, f2)\}) = 1.$$

Using Lemma F.6, Equation (6.36) may be more simply rewritten

$$m_{1\textcircled{\cap}^{\alpha}2} = (m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \textcircled{\cap} m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \textcircled{\cap} m)^{\downarrow\Omega}, \quad (6.38)$$

with m a BBA defined on $\Omega \times T1 \times T2$ by

$$m((A \times (f1, f2)) \cup (\Omega \times (t1, t2))) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}, \quad \forall A \subseteq \Omega. \quad (6.39)$$

The BBA m defined by (6.39) is thus such that

$$m(A) = \begin{cases} 0.36 & \textit{if } A = \{(a,t1,t2),(b,t1,t2)\}, \\ 0.16 & \textit{if } A = \{(a,f1,f2),(b,f1,f2),(a,t1,t2),(b,t1,t2)\}, \\ 0.24 & \textit{if } A = \{(a,f1,f2),(a,t1,t2),(b,t1,t2)\}, \\ 0.24 & \textit{if } A = \{(b,f1,f2),(a,t1,t2),(b,t1,t2)\}, \\ 0 & \textit{otherwise.} \end{cases}$$

From (6.37), we obtain

$$m_{1'}^{\Omega\times T1}(A) = \begin{cases} 1 & \textit{if } A = \{(b,t1),(a,f1)\}, \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$m_{2'}^{\Omega\times T2}(A) = \begin{cases} 0.8 & \textit{if } A = \{(b,t2),(a,f2)\}, \\ 0.2 & \textit{if } A = \{(a,t2),(b,t2)\}, \\ 0 & \textit{otherwise,} \end{cases}$$

The vacuous extensions on $\Omega \times T1 \times T2$ of the BBAs $m_{1'}^{\Omega\times T1}$ and $m_{2'}^{\Omega\times T2}$ are

$$m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2}(A) = \begin{cases} 1 & \textit{if } A = \{(b,t1,t2),(b,t1,f2),(a,f1,t2),(a,f1,f2)\}, \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2}(A) = \begin{cases} 0.8 & \textit{if } A = \{(b,t1,t2),(b,f1,t2),(a,t1,f2),(a,f1,f2)\}, \\ 0.2 & \textit{if } A = \{(a,t1,t2),(a,f1,t2),(b,t1,t2),(b,f1,t2)\}, \\ 0 & \textit{otherwise,} \end{cases}$$

Let $m_{12}^{\Omega\times T1\times T2}$ be a BBA defined on $\Omega \times T1 \times T2$ by

$$m_{12}^{\Omega\times T1\times T2} = m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \textcircled{\cap} m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \textcircled{\cap} m, \quad (6.40)$$

with m the BBA defined by (6.39). Hence, from (6.38), we get

$$m_{12}^{\Omega\times T1\times T2\downarrow\Omega} = m_{1\textcircled{\cap}^{\alpha}2}.$$

From (6.40), we find

$$m_{12}^{\Omega\times T1\times T2}(A) = \begin{cases} 0.36\cdot 0.8 + 0.36\cdot 0.2 + 0.16\cdot 0.2 & \\ \quad +0.24\cdot 0.2 + 0.24\cdot 0.8 + 0.24\cdot 0.2 & \textit{if } A = \{(b,t1,t2)\}, \\ 0.16\cdot 0.8 + 0.24\cdot 0.8 & \textit{if } A = \{(a,f1,f2),(b,t1,t2)\}, \\ 0 & \textit{otherwise.} \end{cases}$$

Marginalizing $m_{12}^{\Omega\times T1\times T2}$ on Ω, we obtain

$$m_{12}^{\Omega\times T1\times T2\downarrow\Omega}(A) = \begin{cases} 0.68 & \textit{if } A = \{b\}, \\ 0.32 & \textit{if } A = \Omega, \\ 0 & \textit{otherwise.} \end{cases}$$

Example 6.6 (Computation of an α-conjunction using the α-commonality function)**.** *Let us recall the computation of an α-conjunction using the α-commonality function. We have*

$$\mathbf{m}_{1 \circledcap^{\alpha} 2} = (\mathbf{G}^{\cap,\alpha})^{-1} \cdot \mathbf{Diag}(\mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}_1) \cdot \mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}_2, \tag{6.41}$$

with $\mathbf{G}^{\cap,\alpha}$ and $(\mathbf{G}^{\cap,\alpha})^{-1}$ two matrices of size $2^{|\Omega|} \times 2^{|\Omega|}$, that can be obtained, as shown by Theorem 6.3 and Corollary 6.1, by Kronecker multiplication using the building blocks:

$$\begin{bmatrix} 1 & 1 \\ \alpha - 1 & 1 \end{bmatrix},$$

and

$$\frac{1}{2-\alpha} \cdot \begin{bmatrix} 1 & -1 \\ 1-\alpha & 1 \end{bmatrix},$$

respectively. In particular, when $|\Omega| = 2$ and $\alpha = 0.6$, we have

$$\begin{aligned} \mathbf{G}^{\cap,\alpha} &= \mathbf{Kron}(\begin{bmatrix} 1 & 1 \\ \alpha - 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ \alpha - 1 & 1 \end{bmatrix}) \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha-1 & 1 & \alpha-1 & 1 \\ \alpha-1 & \alpha-1 & 1 & 1 \\ (\alpha-1)^2 & \alpha-1 & \alpha-1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ -0.4 & 1 & -0.4 & 1 \\ -0.4 & -0.4 & 1 & 1 \\ 0.16 & -0.4 & -0.4 & 1 \end{bmatrix}. \end{aligned}$$

and

$$\begin{aligned} (\mathbf{G}^{\cap,\alpha})^{-1} &= \mathbf{Kron}(\frac{1}{2-\alpha} \cdot \begin{bmatrix} 1 & -1 \\ 1-\alpha & 1 \end{bmatrix}, \frac{1}{2-\alpha} \cdot \begin{bmatrix} 1 & -1 \\ 1-\alpha & 1 \end{bmatrix}) \\ &= \begin{bmatrix} \frac{1}{(2-\alpha)^2} & -\frac{1}{(2-\alpha)^2} & -\frac{1}{(2-\alpha)^2} & \frac{1}{(2-\alpha)^2} \\ \frac{1-\alpha}{(2-\alpha)^2} & \frac{1}{(2-\alpha)^2} & -\frac{1-\alpha}{(2-\alpha)^2} & -\frac{1}{(2-\alpha)^2} \\ \frac{1-\alpha}{(2-\alpha)^2} & -\frac{1-\alpha}{(2-\alpha)^2} & \frac{1}{(2-\alpha)^2} & -\frac{1}{(2-\alpha)^2} \\ \frac{(1-\alpha)^2}{(2-\alpha)^2} & \frac{1-\alpha}{(2-\alpha)^2} & \frac{1-\alpha}{(2-\alpha)^2} & \frac{1}{(2-\alpha)^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{25}{49} & -\frac{25}{49} & -\frac{25}{49} & \frac{25}{49} \\ \frac{10}{49} & \frac{25}{49} & -\frac{10}{49} & -\frac{25}{49} \\ \frac{10}{49} & -\frac{10}{49} & \frac{25}{49} & -\frac{25}{49} \\ \frac{4}{49} & \frac{10}{49} & \frac{10}{49} & \frac{25}{49} \end{bmatrix}. \end{aligned}$$

We have

$$\mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}_1 = \begin{bmatrix} 1 \\ -0.4 \\ 1 \\ -0.4 \end{bmatrix},$$

$$\mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}_2 = \begin{bmatrix} 1 \\ -0.12 \\ 1 \\ -0.12 \end{bmatrix},$$

$$\mathbf{Diag}(\mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}_1) \cdot \mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}_2 = \begin{bmatrix} 1 \\ 0.048 \\ 1 \\ 0.048 \end{bmatrix}$$

and

$$(\mathbf{G}^{\cap,\alpha})^{-1} \cdot \mathbf{Diag}(\mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}_1) \cdot \mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}_2 = \begin{bmatrix} 0 \\ 0 \\ 0.68 \\ 0.32 \end{bmatrix}.$$

Each of the four new techniques that we have brought to light for the computation of an α-conjuntion, has some advantages and some drawbacks. Method 4, i.e., the computation of an α-conjunction using the α-commonality function, is arguably the most simple one to implement. As a matter of fact, this method was used to perform the experiments presented in Section 6.5. We provide below the MatLab code that builds the matrix $\mathbf{G}^{\cap,\alpha}$ needed in method 4, since Smets's MatLab code (see [91, p. 7]) for the construction of the matrix $\mathbf{B}$ cannot be extended in the context of the α-junctions, due to the fact that we no longer have a zero in the building block. In the code below, "`Gca`" stands for $\mathbf{G}^{\cap,\alpha}$.

```
Gca = [1];
GcaBuildingBlock = [1 1; alpha-1 1];
for i=1:cardinalOmega
  Gca = kron(GcaBuildingBlock,Gca);
end
```

The problem of method 4 is that it may rapidly become impossible to use if the frame of discernment Ω is too big, since this method requires computing matrices $\mathbf{G}^{\cap,\alpha}$ of size $2^{|\Omega|} \times 2^{|\Omega|}$, which are, in addition, not sparse, and it requires performing the pointwise product of vectors $\mathbf{g}^{\cap,\alpha}$ of size $2^{|\Omega|}$. Method 3 is also rather simple to implement, since we merely need to perform combinations by the TBM conjunctive rule. However, it requires working in the space $\Omega \times T1 \times T2$. Method 1 and 2 share the same characteristics: they are more efficient than method 4 when the frame is big, since they do not require to work with vectors of size $2^{|\Omega|}$ as m_1 and m_2 may have only a few focal sets, but they are harder to implement.

6.5 Application to a Classification Problem

The previous section has presented simple means to perform a combination by an α-junctive rule. This makes the practical use of these rules in applications easier. In this section, the usefulness of these rules in a classification application is investigated.

6.5.1 The α-junctions in the evidential K-nearest neighbor classification scheme

Recall Section 4.6.2: the TBM conjunctive rule in the evidential K-nearest neighbor classification scheme was replaced by a conjunctive t-rule, whose behavior was determined by a parameter θ. Then, two experiments were run. They showed that for certain values of θ and for certain data sets, it is possible to obtain better classification results than with the TBM conjunctive rule.

Here, we propose to replace the TBM conjunctive rule in the evidential K-nearest neighbor classification scheme by an α-conjunctive rule. We obtain then the following expression for the BBA $m[\mathbf{x}^s|K]$ quantifying our beliefs on the class of a new object $\mathbf{x}^s$ given its K nearest neighbors:

$$m[\mathbf{x}^s|K] \quad = \quad \bigcirc\!\!\!\!\cap^{\alpha}_{n\in S_K(\mathbf{x}^s)} m[\mathbf{x}|\mathbf{x}^n],$$

where, we recall, $S_K(\mathbf{x}^s)$ denotes the set of K nearest neighbors of $\mathbf{x}^s$.

6.5.2 Numerical experiments

Similar experiments to the two experiments reported in Section 4.6.2 were conducted with this other extended version of the original evidential K-nearest neighbor classification rule. Furthermore, in order to be able to compare results with the extension based on the conjunctive t-rules, we used the same settings for the parameters K, β, and γ_q, $q = 1, ..., Q$.

Experiment 1

For each of the data sets (Cleveland heart disease, mammographic mass, and vehicle silhouettes), we computed the LOO cross-validation error rate of the extended scheme for different values of α: $\alpha = 0, 0.1, ..., 1$. These LOO error rates are given in Figures 6.3 to 6.5. Let us stress that, in these figures, the figure shown for $\alpha = 1$ is the LOO error rate of the TBM conjunctive rule and thus one may verify that, as expected, the LOO error rates for $\alpha = 1$ in Figures 6.3 to 6.5 is equal to the LOO error rates for $\theta = 1$ in Figures 4.1 to 4.3.

As can be seen in Figures 6.4 and 6.5, it is not possible to find, for the mammographic mass and vehicle silhouettes data sets, an α-conjunctive rule that has a lower LOO error rate than the TBM conjunctive rule. However, on the Cleveland heart disease data set, there exists at least one α-conjunctive rule that performs better than the TBM conjunctive rule. This is an interesting result for two reasons. First and foremost, it shows that the α-conjunctive rules may be useful in a classification application. Second, it completes the experiments performed in Section 4.6.2, in that it was not possible to obtain a lower LOO error rate on this latter data set, with conjunctive t-rules intermediate between the TBM conjunctive rule and the cautious rule.

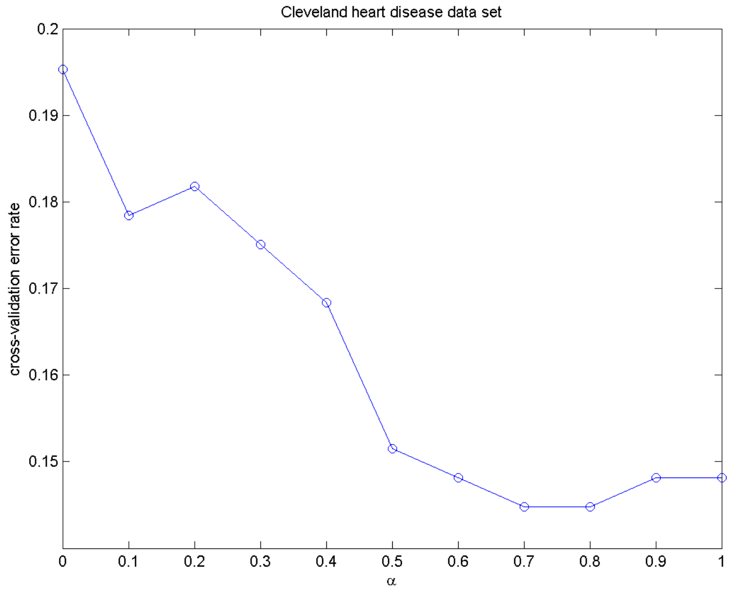

Figure 6.3: Cleveland heart disease data set.
Best performance obtained for $\alpha = 0.7$ and $\alpha = 0.8$.

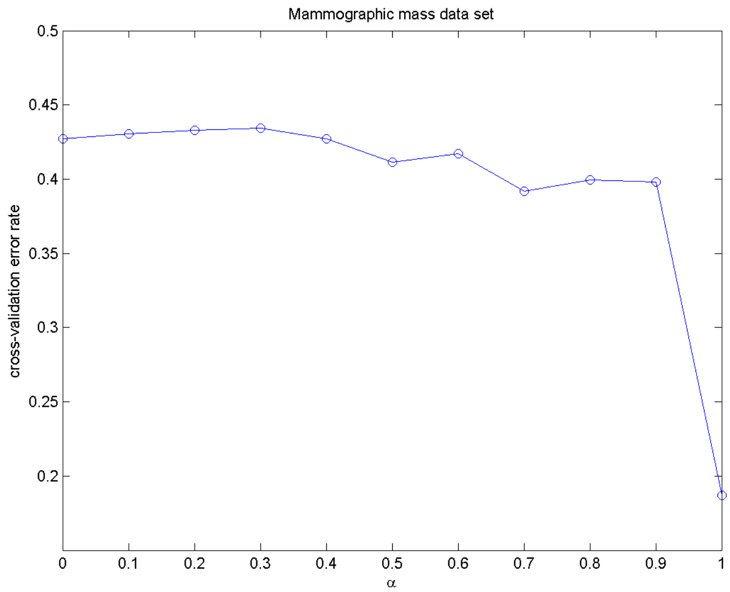

Figure 6.4: Mammographic mass data set. Best performance obtained for $\alpha = 1$.

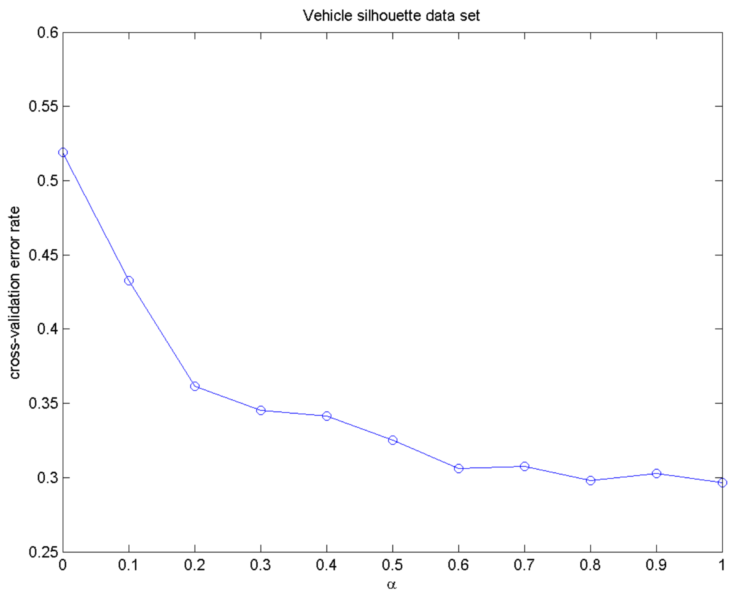

Figure 6.5: Vehicle silhouettes data set. Best performance obtained for $\alpha = 1$.

Experiment 2

In a similar fashion as Experiment 2 of Section 4.6.2, we learnt the value $\hat{\alpha}$ of the parameter α (the parameter space was restricted to $\alpha = 0, 0.1, ..., 1$) optimizing the LOO error rates on each of the training sets built from the ionosphere, liver disorders, wine, and segment data sets. Then, we computed the test error rates for the learnt α-conjunctive rule and the TBM conjunctive rule, together with 95% confidence intervals. The results are presented in Table 6.1, with the best results underlined.

Table 6.1: Error rates of the TBM conjunctive rule and the learnt α-conjunctive rule, together with 95% confidence intervals.

Data	TBM conjunctive rule	α-conjunctive rule
Ionosphere	$\underline{0.1466} \pm 0.0644$	$\underline{0.1466} \pm 0.0644$ ($\hat{\alpha} = 1$)
Liver disorders	0.3130 ± 0.0848	$\underline{0.2696} \pm 0.0811$ ($\hat{\alpha} = 0.8$)
Wine	0.0333 ± 0.0454	$\underline{0.0167} \pm 0.0324$ ($\hat{\alpha} = 0$)
Segment	$\underline{0.0870} \pm 0.0199$	$\underline{0.0870} \pm 0.0199$ ($\hat{\alpha} = 1$)

As was the case with the learnt conjunctive t-rules, we can first remark that the learnt α-conjunctive rules exhibit at worst the classification results of the TBM conjunctive rule and at best, better results than those of the TBM conjunctive rule. Furthermore, on the liver disorders and wine data sets, where the conjunctive t-rules did not outperform the TBM conjunctive rule, we note that the learnt α-conjunctive rules show lower error rates than those of the TBM conjunctive rule, which is an experimental verification of the validity of the α-conjunctive rules in this classification application. However, we must stress that this conclusion is less strong than the similar conclusion reached for the conjunctive t-rules, since the error rates here were not judged significantly different by a McNemar test at level 5%.

6.6 Conclusion

The α-junctions represent the set of associative, commutative and linear combination operators for belief functions. They include as particular cases familiar combination rules such as the TBM conjunctive rule and the TBM disjunctive rule. They have never been used in the literature due, most certainly, to two factors: in the original paper of Smets [87], they lacked (1) an interpretation and (2) simple means to compute them. This chapter has proposed solutions to these two issues.

It was shown that the α-junctions correspond to some particular form of knowledge about the truthfulness of the sources, making the α-junctions interesting for applications where such kind of knowledge may be available. This might for instance be the case when dealing with automatic deceiving agents, tampering with messages sent between sensors and a coordination center [94].

It was known that the α-junctions generalize standard combination rules of the TBM, in that these latter rules are particular cases of this family. We showed that various notions that can be used to perform the computation by these standard rules can be generalized to the α-junctions. In particular, we generalized the conditioning operation and the part of the matrix calculus for belief functions related to the computation of the commonality and implicability functions. This allowed us to uncover simple methods to perform a combination by an α-junctive rule. A modification to a technical part of [87] related to the α-disjunctions was also proposed. Eventually, the usefulness of this family of rules was investigated in a classification application. The results were encouraging, although not as good as the ones obtained in Chapter 4 with the conjunctive t-rules.

In addition to generalizing the TBM conjunctive and TBM disjunctive rules, the α-junctions also lead to a generalization of the conjunctive and disjunctive canonical decompositions of a belief function. This is the subject of the next chapter.

α-Conjunctive and α-Disjunctive Canonical Decompositions

Summary

In this chapter, it is shown that the generalization of the TBM conjunctive and TBM disjunctive rules by the α-junctions goes all the way up to the conjunctive and disjunctive canonical decompositions. Indeed, it is shown that one can find a canonical decomposition of a belief function such that the combination by an α-junctive rule be expressed as the pointwise product of some weight functions. Furthermore, this decomposition degenerates to the classical ones when $\alpha = 1$, i.e., it degenerates to the conjunctive or disjunctive canonical decompositions when the α-junction is equivalent to the TBM conjunctive rule or to the TBM disjunctive rule. However, finding this decomposition is not trivial since it requires finding the conjunctive and disjunctive canonical decompositions of a signed belief function. It is also shown in this chapter that the combination by an α-conjunction amounts to the combination by the TBM conjunctive rule of some signed belief functions.

Résumé

Dans ce chapitre, nous montrons que la généralisation par les α-jonctions de la règle conjonctive du MCT et de la règle disjonctive du MCT se vérifie aussi pour les décompositions canoniques conjonctive et disjonctive. En effet, nous montrons qu'il existe une décomposition canonique d'une fonction de croyance telle que la combinaison par une α-jonction s'exprime comme le produit terme à terme de deux fonctions de poids. Il est aussi intéressant de remarquer que cette décomposition se réduit aux décompositions classiques lorsque $\alpha = 1$, c'est-à-dire lorsque l'α-jonction est équivalente à la règle conjonctive du MCT ou à la règle disjonctive du MCT. Cependant, il faut remarquer que la mise à jour de cette généralisation de la décomposition canonique n'est pas triviale car elle implique de trouver les décompositions canoniques conjonctive et disjonctive d'une fonction de croyance signée. Nous montrons également dans ce chapitre que la combinaison par une α-conjonction revient à la combinaison par la règle conjonctive du MCT de fonctions de croyance signées.

7.1 Introduction

The preceding chapter has presented new results related to an interesting family of combination rules for belief functions called the α-junctions. This family generalizes the classical TBM conjunctive and TBM disjunctive rules in that it not only includes these two latter rules as particular cases, but also generalizes the mathematics related to their computation. Most importantly, this family provides some flexibility for the combination of belief functions. Indeed, it is possible, using this family, to account for less categorical knowledge on the truthfulness of the sources than conjunctive and disjunctive mergings represent.

In part II of this thesis, it was shown that it is possible to introduce some flexibility for conjunctive and disjunctive mergings using the conjunctive and disjunctive canonical decompositions. In [91, p.24], Smets mentions the existence of α-junctive canonical decompositions. It seems interesting to find these canonical decompositions, since it might then become possible to generalize the results of Part II to the α-junctions, resulting in two-dimensional flexibility for the combination of belief functions. It might also lead to a generalization of the cautious rule.

This chapter will show that the generalization of the TBM conjunctive and TBM disjunctive rules by the α-junctions goes indeed all the way up to the conjunctive and disjunctive canonical decompositions. As will be seen, one can find a canonical decomposition of a belief function such that the combination by an α-junctive rule be expressed as pointwise product of some weight functions. Furthermore, this decomposition degenerates to the classical ones when $\alpha = 1$. However, it is important to note that finding this decomposition is not trivial, since it is based on the conjunctive and disjunctive canonical decompositions of a signed belief function [52, 53].

It will also be shown that an α-conjunction and an α-disjunction actually amount, respectively, to a conjunction and a disjunction of signed belief functions. This result is interesting as the TBM conjunctive rule and its dual are appearing once again and as the use of the words "conjunction" and "disjunction" in the terms "α-conjunction" and "α-disjunction" finds a new motivation.

This chapter is organized as follows. In Section 7.2, the conjunctive and disjunctive canonical decompositions of a signed belief function are introduced after a brief summary of signed belief function theory. Those decompositions are then used in Section 7.3 to obtain what will be called the α-junctive canonical decomposition of a belief function.

7.2 Canonical Decompositions of Signed Belief Functions

In this section, it is shown that there exist conjunctive and disjunctive canonical decompositions of a signed belief function. Section 7.2.1 summarizes necessary notions on signed belief functions and also introduces new notions such as the negation of a signed belief function, the TBM disjunctive rule for signed belief functions, and the signed implicability function. Sections 7.2.2 and 7.2.3 present, respectively, the conjunctive and disjunctive canonical decompositions of a signed belief function.

7.2.1 Signed belief functions

Historically, signed belief functions were introduced as a solution to the "inversion problem" [53]. This problem may be formulated as follows. Suppose the state of belief of an agent is represented by a BBA m_1. Then, this agent receives a piece of evidence Ev_2 inducing a BBA m_2, which she/he combines with her/his previous beliefs represented by m_1. Some time later, the agent obtains a new piece of information saying that Ev_2 was completely irrelevant and must thus be removed from her/his knowledge base. This situation can be handled in two ways. On the one hand, one may use the inverse of the TBM conjunctive rule defined in Section 1.3, in which case we have $m_1 \circledcap m_2 \oslash m_2 = m_1$. On the other hand, if one wants to realize the cancellation of Ev_2 by an application of the rule $\circledcap$, then she/he must represent the information claiming that Ev_2 is irrelevant by the means of a BBA m_3 such that $m_1 \circledcap m_2 \circledcap m_3 = m_1$ [53]. This actually amounts to finding a BBA m_3 such that $m_2 \circledcap m_3 = m_\Omega$. Clearly, there exists no BBA m_3 that is such an inverse element to m_2. In fact, the only set function m_3 that verifies $m_2 \circledcap m_3 = m_\Omega$ is defined by $m_3 = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{1/w_2(A)}$. Indeed, we have, using (2.5) and (2.6):

$$\begin{aligned} m_\Omega &= m_2 \oslash m_2 \\ &= \left(\bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w_2(A)}\right) \oslash \left(\bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w_2(A)}\right) \\ &= \left(\bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w_2(A)}\right) \circledcap \left(\bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{1/w_2(A)}\right) \\ &= m_2 \circledcap m_3. \end{aligned}$$

It may easily be checked that m_3 is a function from 2^Ω to $\mathbb{R}$. Consequently, if one wants to find a solution to $m_2 \circledcap m_3 = m_\Omega$, one needs to enlarge the space of BBAs by some generalization of this notion [52]. This enlargement leads to the concept of a signed belief function.

Formally, a signed belief function and its associated *basic signed measure assignment* (BSMA) are defined as follows.

Definition 7.1 (Definition 11.3.1 of [53]). *Let Ω be a nonempty finite set. A BSMA is a mapping $sm : 2^\Omega \to [-\infty, +\infty]$ such that sm takes at most one of the infinite values $-\infty, \infty$. The signed belief function $sbel$ induced by sm is the mapping $sbel : 2^\Omega \to [-\infty, +\infty]$ defined by*

$$sbel\,(A) = \sum_{\emptyset \neq B \subseteq A} sm\,(B)\,,$$

for all $A \subseteq \Omega$. A BSMA sm is called finite, if $-\infty < sm(A) < \infty$ holds for each $A \subseteq \Omega$.

In this chapter, we work only with finite BSMAs. Hence, we will omit the term *finite* from now on to simplify the presentation. Let us also note that BSMAs have been referred to as real-valued set functions in [47].

Lemma 11.3.2 of [53] shows that if two BSMAs sm_1 and sm_2 are such that $sm_1(A) \neq sm_2(A)$, for some $A \subseteq \Omega$, then their associated signed belief functions are such that $sbel_1(B) \neq sbel_2(B)$, for some $B \subseteq \Omega$. Furthermore, using Lemma 2.3 of [77], it may easily be shown that sm can be computed from $sbel$ using the same equation that allows one to transform a belief function bel to it associated BBA m.

One may remark that a BBA is a particular kind of BSMA, and thus a BSMA is a generalization of a BBA. Similarly a signed belief function is a generalization of a belief function. Other functions used in belief function theory can be generalized, for instance the commonality and implicability functions.

Definition 7.2 (Based on Definition 4 of [52]). *Let sm be a BSMA defined on a finite set Ω. The signed commonality function induced by sm is the mapping $sq : 2^\Omega \to (-\infty, +\infty)$ defined by*

$$sq\left(A\right) = \sum_{B \supseteq A} sm\left(B\right),$$

Definition 7.3. *Let sm be a BSMA defined on a finite set Ω. The signed implicability function induced by sm is the mapping $sb : 2^\Omega \to (-\infty, +\infty)$ defined by*

$$sb\left(A\right) = \sum_{B \subseteq A} sm\left(B\right),$$

Using the proof of Theorem 2.4 of [77], it may be shown that a signed commonality function is in one-to-one relation with its associated signed belief function, and thus with its associated BSMA. It may also be shown that a signed implicability function is in one-to-one relation with its associated BSMA. Consequently, all equations relating the functions bel, m, q, b in belief function theory (see, e.g., [91, p. 4]) can be used for signed belief functions as well. Let us note that the conjunctive and disjunctive weight functions, which are other equivalent representations of a BBA, cannot be generalized so straightforwardly to signed belief functions because the proof of [85] does not work as it assumes commonalities to be positive.

The negation of a BBA can also be extended to BSMAs. The negation (or complement) $\overline{sm}$ of a BSMA sm is defined as the BSMA verifying $\overline{sm}(A) = sm(\overline{A})$, $\forall A \subseteq \Omega$, where $\overline{A}$ denotes the complement of A.

Proposition 7.1. *The signed implicability function $\overline{sb}$ associated to $\overline{sm}$ and the signed commonality function sq associated to sm are linked by the following relation:*

$$\overline{sb}(A) = sq(\overline{A}), \quad \forall A \subseteq \Omega.$$

Proof.

$$\begin{aligned}
\overline{sb}(A) &= \sum_{B\subseteq A} \overline{sm}(B) \\
&= \sum_{B\subseteq A} sm(\overline{B}) \\
&= \sum_{\overline{B}\subseteq A} sm(B) \\
&= \sum_{\overline{A}\subseteq B} sm(B) \\
&= sq(\overline{A})
\end{aligned}$$

□

The generalization of the TBM conjunctive rule to signed belief functions is defined as follows.

Definition 7.4 (Definition 3 of [52]). *Let sm_1 and sm_2 be two BSMAs. Let $sm_{1 \circledcap 2}$ be the result of their combination by the TBM conjunctive rule $\circledcap$. We have:*

$$sm_{1\circledcap 2}(A) = \sum_{B\cap C=A} sm_1(B)\, sm_2(C), \quad \forall A \subseteq \Omega.$$

Kramosil [53] shows that this rule is commutative, associative and that it admits m_Ω as neutral element.

The combination by $\circledcap$ of two BSMAs has a simple expression using signed commonalities. Indeed, it may be shown using Theorem 3.3 of [77] that we have:

$$sq_{1\circledcap 2}(A) = sq_1(A) \cdot sq_2(A), \quad \forall A \subseteq \Omega. \tag{7.1}$$

A simple BSMA (SBSMA) is defined as a BSMA which has at most two focal sets and, if it has two, Ω is one of those. A SBSMA sm such that $sm(A) = 1 - sw$ for some $A \neq \Omega$ and $sm(\Omega) = sw$, with $sw \in (-\infty, +\infty)$, can be noted A^{sw}. Let us remark that we have $A^{sw1} \circledcap A^{sw2} = A^{sw1 \cdot sw2}$.

The TBM disjunctive rule may also be generalized to BSMAs, and it may be shown that it can be simply expressed as pointwise multiplication of signed implicabilities. The neutral element of the TBM disjunctive rule $\circledcup$ is $m_\emptyset$.

Proposition 7.2 (De Morgan's Laws). *Let sm_1 and sm_2 be two BSMAs. We have*

$$\overline{sm_1 \circledcup sm_2} = \overline{sm_1} \circledcap \overline{sm_2} \tag{7.2}$$

$$\overline{sm_1 \circledcap sm_2} = \overline{sm_1} \circledcup \overline{sm_2} \tag{7.3}$$

Proof. This proof is based on a proof that works in belief function theory and that was given in [87, p. 137].

Let us prove (7.3). Let $\overline{sb}_{1 \textcircled{\cap} 2}$ denote the signed implicability function associated to $\overline{sm_1 \textcircled{\cap} sm_2}$. We have:

$$\begin{aligned} \overline{sb}_{1 \textcircled{\cap} 2}(A) &= sq_{1 \textcircled{\cap} 2}(\overline{A}) \\ &= sq_1(\overline{A}) \cdot sq_2(\overline{A}) \\ &= \overline{sb}_1(A) \cdot \overline{sb}_2(A) \end{aligned}$$

The proof of (7.2) is similar. □

7.2.2 Conjunctive canonical decomposition

Much as it is possible to find the conjunctive canonical decomposition of a particular kind of BBAs, called nondogmatic BBAs, this section will show that it is possible to provide a solution for the conjunctive canonical decomposition of a particular kind of BSMAs, called *invertible* BSMAs in [52] (such BSMAs are renamed sq-invertible BSMAs in this thesis).

Definition 7.5 (Definition 4 of [52]). *A BSMA sm is called sq-invertible if* $sq(A) \neq 0$ *for all* $A \subseteq \Omega$.

The notion of sq-invertibility may be seen as generalizing nondogmatism [52]. In particular, one may remark that a nondogmatic BBA, which, we recall, is a particular BSMA, is necessarily sq-invertible.

Let us define another particular kind of BSMA.

Definition 7.6 (Regular BSMA). *A BSMA sm is called regular if* $sq(\emptyset) = 1$ *or, equivalently,* $sb(\Omega) = 1$ *or, equivalently*

$$\sum_{A \subseteq \Omega} sm(A) = 1. \tag{7.4}$$

We may note that all BBAs are regular.

Theorem 7.1. *Let sm be a sq-invertible regular BSMA and with associated signed commonality function sq. Then,*

$$sm = \textcircled{\cap}_{A \subset \Omega} A^{sw(A)}, \tag{7.5}$$

with sw a function $2^{\Omega} \setminus \{\Omega\} \to (-\infty, +\infty) \setminus \{0\}$ *defined by:*

$$sw(A) = \prod_{B \supseteq A} sq(B)^{(-1)^{|B|-|A|+1}}, \tag{7.6}$$

for all $A \subset \Omega$.

Proof. See Appendix F.6. □

The function $sw : 2^{\Omega} \setminus \{\Omega\} \to (-\infty, +\infty) \setminus \{0\}$ is called the conjunctive signed weight function.

Remark 7.1. *Let m be a nondogmatic BBA (m is thus sq-invertible and regular). The conjunctive signed weight function associated to m and computed using Theorem 7.1 is equal to its associated conjunctive weight function.*

Proof. Direct from the comparison of (7.6) with (2.2). □

From this remark, it is clear that this theorem is a direct generalization of the theorem of [85]. Hence, in addition to the belief, mass, implicability and commonality functions, the conjunctive weight function of belief function theory has been generalized to signed belief function theory.

The next proposition shows that the conjunctive combination of two sq-invertible regular BSMAs has a simple expression using the function sw.

Proposition 7.3. *Let sm_1 and sm_2 be two sq-invertible regular BBAs and with associated conjunctive signed weight functions sw_1 and sw_2. We have*

$$sm_{1\circledcap 2} \quad = \quad \circledcap_{A\subset\Omega} A^{sw_1(A)\cdot sw_2(A)},$$

and, equivalently, $sw_{1\circledcap 2} = sw_1 \cdot sw_2$.

Proof.

$$\begin{aligned} sw_{1\circledcap 2}(A) &= \prod_{B\supseteq A} sq_{1\circledcap 2}(B)^{(-1)^{|B|-|A|+1}} \\ &= \prod_{B\supseteq A} (sq_1(B)\cdot sq_2(B))^{(-1)^{|B|-|A|+1}} \\ &= \left(\prod_{B\supseteq A} sq_1(B)^{(-1)^{|B|-|A|+1}}\right)\cdot\left(\prod_{B\supseteq A} sq_2(B)^{(-1)^{|B|-|A|+1}}\right) \\ &= sw_1(A)\cdot sw_2(A) \end{aligned}$$

□

7.2.3 Disjunctive canonical decomposition

This section extends the disjunctive canonical decomposition of belief functions introduced in [18] to signed belief functions. Since there is a little subtlety related to the notion of invertibility, demonstrations of [18] are fully rewritten here in the context of signed belief functions.

The disjunctive canonical decomposition is possible for a belief function only if this belief function is subnormal. The following definition generalizes the concept of subnormality to signed belief functions, in a similar way that the concept of nondogmatism was generalized to signed belief functions in the previous section.

Definition 7.7. *A BSMA sm is called sb-invertible if $sb(A) \neq 0$ for all $A \subseteq \Omega$.*

Let sm be a sb-invertible regular BSMA. Its complement $\overline{sm}$ is sq-invertible and regular since $sb(\overline{A}) = \overline{sq}(A)$. It can thus be decomposed as:

$$\overline{sm} = \circledcap_{A\subset\Omega} A^{\overline{sw}(A)}.$$

Consequently, sm can be written

$$\begin{aligned} sm &= \overline{\textcircled{\cap}_{A \subset \Omega} A^{\overline{sw}(A)}} \\ &= \textcircled{\cup}_{A \subset \Omega} \overline{A^{\overline{sw}(A)}}. \end{aligned}$$

We recall that $A^{\overline{sw}(A)}$ denotes the simple BSMA assigning a mass $\overline{sw}(A)$ to Ω and a mass $1 - \overline{sw}(A)$ to A. Its complement $\overline{A^{\overline{sw}(A)}}$ thus assigns a mass $\overline{sw}(A)$ to $\emptyset$ and a mass $1 - \overline{sw}(A)$ to $\overline{A}$. Such a mapping can be called a *negative* simple BSMA, and noted $\overline{A}_{sv(\overline{A})}$ with $sv(\overline{A}) = \overline{sw}(A)$. We can thus write:

$$\begin{aligned} sm &= \textcircled{\cup}_{A \subset \Omega} \overline{A}_{sv(\overline{A})} \\ &= \textcircled{\cup}_{A \neq \emptyset} A_{sv(A)}. \end{aligned}$$

Using the reasoning of [18] and Theorem 7.1, we have thus proved the following corollary.

Corollary 7.1. *Let sm be a sb-invertible regular BSMA. Then,*

$$sm = \textcircled{\cup}_{A \neq \emptyset} A_{sv(A)}, \tag{7.7}$$

with sv a function $2^\Omega \setminus \{\emptyset\} \to (-\infty, +\infty) \setminus \{0\}$ defined by:

$$sv(A) = \overline{sw}(\overline{A}), \quad \forall A \neq \emptyset, \tag{7.8}$$

where $\overline{sw}$ is the conjunctive signed weight function associated to the negation $\overline{sm}$ of sm.

The function sv is called the disjunctive signed weight function. This function generalizes the disjunctive weight function of belief function theory.

The disjunctive combination of two sb-invertible regular BSMAs has a simple expression using the function sv.

Proposition 7.4. *Let sm_1 and sm_2 be two sb-invertible regular BSMAs and with disjunctive signed weight functions sv_1 and sv_2. We have*

$$sm_{1 \textcircled{\cup} 2} = \textcircled{\cup}_{A \neq \emptyset} A_{sv_1(A) \cdot sv_2(A)},$$

and, equivalently, $sv_{1 \textcircled{\cup} 2} = sv_1 \cdot sv_2$.

Proof. The proof is similar to the one of Proposition 7.3 □

7.3 α-Junctive Canonical Decompositions

7.3.1 α-conjunctive weight function

In this section, we explain how the results of the preceding section can be used to obtain a canonical decomposition of a belief function such that the combination by an α-conjunctive rule be expressed as pointwise product of some weight functions.

The key to this decomposition is to see the α-commonality function $g^{\cap,\alpha}$ associated to a BBA m as a signed commonality function, noted sq^{α}, associated to a BSMA, noted sm^{α}. We may note that the idea of assimilating an α-commonality function to a signed commonality function comes from the fact an α-commonality function can take negative values as shown by Example 6.2 of Chapter 6.

For a fixed α, one may obtain sm^{α} from m in three steps.

1. Compute the α-commonality function $g^{\cap,\alpha}$ from m:

$$\mathbf{g}^{\cap,\alpha} = \mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}, \tag{7.9}$$

2. Let

$$sq^{\alpha} \quad = \quad g^{\cap,\alpha}, \tag{7.10}$$

3. Compute sm^{α} from sq^{α}:

$$\mathbf{sm}^{\alpha} \quad = \quad \mathbf{Q}^{-1} \cdot \mathbf{sq}^{\alpha}.$$

   Hence, from (7.9) and (7.10)

$$\mathbf{sm}^{\alpha} = \mathbf{Q}^{-1} \cdot \mathbf{G}^{\cap,\alpha} \cdot \mathbf{m}, \tag{7.11}$$

   and

$$\mathbf{m} = (\mathbf{G}^{\cap,\alpha})^{-1} \cdot \mathbf{Q} \cdot \mathbf{sm}^{\alpha}. \tag{7.12}$$

It is clear that $sq^{\alpha}(\emptyset) = 1$ since $g^{\cap,\alpha}(\emptyset) = 1$ by Proposition 6.5. Provided that sm^{α} is also sq-invertible, Theorem 7.1 can be used to rewrite (7.12) into:

$$\mathbf{m} = (\mathbf{G}^{\cap,\alpha})^{-1} \cdot \mathbf{Q} \cdot \textcircled{\cap}_{A\subset\Omega} A^{sw^{\alpha}(A)}. \tag{7.13}$$

The signed conjunctive weight function sw^{α} associated to sm^{α} is called the α-conjunctive weight function associated to m. Furthermore, this canonical decomposition is called the α-conjunctive canonical decomposition of m. If $\alpha = 1$, this decomposition reduces to the conjunctive canonical decomposition of m. Example 7.1 illustrates the α-conjunctive canonical decomposition of a BBA.

Example 7.1. *Let $\Omega = \{a, b, c\}$ be a frame of discernment, and m a BBA with associated α-commonality function $g^{\cap,\alpha}$ and α-conjunctive weight function sw^{α}. Those functions are shown in Table 7.1 for $\alpha = 0.5$. It may be checked that we indeed have:*

$$\mathbf{m} \quad = \quad (\mathbf{G}^{\cap,0.5})^{-1} \cdot \mathbf{Q} \cdot \left(\{b\}^{-0.8} \textcircled{\cap} \{a,b\}^{-0.5} \textcircled{\cap} \{b,c\}^{-0.5}\right)$$

The next example illustrates the fact that the α-conjunctive weights take their values in $(-\infty, +\infty) \setminus \{0\}$.

Table 7.1: An α-conjunctive weight function.

A	m	$g^{\cap,\alpha}$	sw^{α}
$\emptyset$	0	1	1
$\{a\}$	0	0.4	1
$\{b\}$	0	1	-0.8
$\{a,b\}$	0.4	0.4	-0.5
$\{c\}$	0	0.4	1
$\{a,c\}$	0	-0.2	1
$\{b,c\}$	0.4	0.4	-0.5
Ω	0.2	-0.2	

Table 7.2: A more complex α-conjunctive weight function (numbers are rounded to hundredths).

A	m	$g^{\cap,\alpha}$	sw^{α}
$\emptyset$	0	1	-0.51
$\{a\}$	0	-0.28	0.19
$\{b\}$	0.6	0.68	0.07
$\{a,b\}$	0	-0.09	-5.55
$\{c\}$	0.2	0.04	-4.25
$\{a,c\}$	0	0.3	1.65
$\{b,c\}$	0	-0.28	-1.74
Ω	0.2	0.49	

Example 7.2. *Let $\Omega = \{a, b, c\}$ be a frame of discernment, and m a BBA with associated α-commonality function $g^{\cap,\alpha}$ and α-conjunctive weight function sw^α. Those functions are shown in Table 7.2 for $\alpha = 0.4$.*

The α-conjunctive weight function can be found iff the BSMA sm^α associated to m is sq-invertible. One should note that a nondogmatic BBA m does not necessarily yield a sq-invertible BSMA sm^α as shown by the following example.

Example 7.3. *Let m be the BBA of example 6.2. Let $\alpha = 0.75$. We have:*

$$\begin{aligned} g^{\cap,0.75}(\emptyset) &= 1 \\ g^{\cap,0.75}(\{a\}) &= 0 \\ g^{\cap,0.75}(\{b\}) &= 1 \\ g^{\cap,0.75}(\Omega) &= 0 \end{aligned}$$

The BSMA $sm^{0.75}$ associated to m is thus not sq-invertible.

The following proposition shows that the combination by an α-conjunctive rule of two BBAs actually amounts to a combination by the TBM conjunctive rule of two BSMAs.

Proposition 7.5. *Let m_1 and m_2 be two BBAs. The result of their α-conjunction is noted $m_{1\textcircled{\cap}^\alpha 2}$. We have:*

$$\mathbf{m}_{1\textcircled{\cap}^\alpha 2} = (\mathbf{G}^{\cap,\alpha})^{-1} \cdot \mathbf{Q} \cdot sm_1^\alpha \textcircled{\cap} sm_2^\alpha, \tag{7.14}$$

where sm_1^α and sm_2^α are obtained from m_1 and m_2 using (7.11).

Proof. We have

$$\begin{aligned} g^{\cap,\alpha}_{1\textcircled{\cap}^\alpha 2} &= g_1^{\cap,\alpha} \cdot g_2^{\cap,\alpha} \\ &= sq_1^\alpha \cdot sq_2^\alpha, \end{aligned}$$

with sq_1^α and sq_2^α the signed commonality functions associated to sm_1^α and sm_2^α. □

Provided that both sm_1^α and sm_2^α are sq-invertible and since they verify $sq_1^\alpha(\emptyset) = 1$ and $sq_2^\alpha(\emptyset) = 1$, Proposition 7.3 can be used to rewrite (7.14) into:

$$\mathbf{m}_{1\textcircled{\cap}^\alpha 2} = (\mathbf{G}^{\cap,\alpha})^{-1} \cdot \mathbf{Q} \cdot \textcircled{\cap}_{A\subset\Omega} A^{sw_1^\alpha(A)\cdot sw_2^\alpha(A)}. \tag{7.15}$$

Hence, the α-conjunction of BBAs can be seen as being based on pointwise product of signed conjunctive weights (or α-conjunctive weights according to the terminology introduced above). Furthermore, note that when $\alpha = 1$, Equation (7.15) reduces to the classical pointwise product of conjunctive weights.

7.3.2 α-disjunctive weight function

For the sake of completeness, this section presents succinctly the results corresponding to the previous ones for α-disjunctions.

The function $g^{\cup,\alpha}$ may be seen as the signed implicability function sb^{α} of a BSMA sm^{α}. Providing that sm^{α} is sb-invertible, and since $sb^{\alpha}(\Omega) = 1$, we may use Corollary 7.1 to obtain, in a similar manner as is done in Section 7.3.1, the α-disjunctive weight function associated to a BBA m. We note this function sv^{α}.

The counterpart of Proposition 7.5 shows that the combination by an α-disjunctive rule of two BBAs m_1 and m_2 actually amounts to a combination by the TBM disjunctive rule of two BSMAs sm_1^{α} and sm_2^{α}. Hence, provided that both sm_1^{α} and sm_2^{α} are sb-invertible, we may obtain the following expression (using Proposition 7.4):

$$\mathbf{m}_{1 \circledcup^{\alpha} 2} = (\mathbf{G}_{new}^{\cup,\alpha})^{-1} \cdot \mathbf{B} \cdot \circledcup_{A \neq \emptyset} A_{sv_1^{\alpha}(A) \cdot sv_2^{\alpha}(A)}. \tag{7.16}$$

Consequently, the α-disjunction of BBAs can be seen as being based on pointwise product of signed disjunctive weights (or α-disjunctive weights). In addition, if $\alpha = 1$, then (7.16) reduces to the classical pointwise product of disjunctive weights.

7.3.3 Discussion

Alternative canonical decompositions

Although Smets [91] mentioned the existence of α-junctive canonical decompositions, he did not explain how one may obtain such decompositions and, in particular, he did not mention the need to use signed belief function theory to find these decompositions. Hence, he may have had in mind different decompositions than the ones proposed in this chapter. Let us note that since our α-junctive decompositions are based on Theorem 7.1, which is a generalization of Theorem 1 of [85] that lead to the conjunctive and disjunctive canonical decompositions, it seems reasonable to believe that Smets' decompositions, if different from ours, would have had to be based on a completely different reasoning than the one followed in this chapter.

Futhermore, one may wonder whether it is possible to find a canonical decomposition of a BBA m, which would be based on an α-conjunctive rule $\circledcap^{\alpha}$, i.e., an expression of the form

$$m = \circledcap_{A \subset \Omega}^{\alpha} A^{z(A)}, \tag{7.17}$$

where z is some set function. It may well be that such a canonical decomposition exists. However, let us remark that the nice expression $A^{w_1} \circledcap A^{w_2} = A^{w_1 \cdot w_2}$ for all $A \subset \Omega$ and all $w_1, w_2 \in (-\infty, +\infty) \setminus \{0\}$ does not generalize well to α-conjunctions. Indeed, let $\Omega = \{a, b\}$, we have for instance, for $\alpha \neq 1$ and $w_1, w_2 \in [0, 1]$:

$$\{a\}^{w_1} \circledcap^{\alpha} \{a\}^{w_2} \quad = \quad \{a\}^{w_1 \cdot w_2 + (1-w_1) \cdot \overline{\alpha} \cdot (1-w_2)},$$

and

$$\{\emptyset\}^{w_1} \circledcap^{\alpha} \{\emptyset\}^{w_2} \quad = \quad m,$$

where m is *not* a simple BBA and is defined by

$$\begin{aligned}
m(\emptyset) &= (1-w_1)\cdot\alpha^2\cdot(1-w_2)+(1-w_1)\cdot w_2+w_1\cdot(1-w_2),\\
m(\{a\}) &= (1-w_1)\cdot\alpha\cdot\overline{\alpha}\cdot(1-w_2),\\
m(\{b\}) &= (1-w_1)\cdot\alpha\cdot\overline{\alpha}\cdot(1-w_2),\\
m(\Omega) &= (1-w_1)\cdot\overline{\alpha}^2\cdot(1-w_2)+w_1\cdot w_2.
\end{aligned}$$

Hence, even if a decomposition of the form given by (7.17) exists, it seems that there is no hope to express an α-conjunction as a pointwise product of weights z obtained using (7.17). In other words, a decomposition based on an α-conjunctive rule, if it exists, does not seem interesting since it does not yield the convenient expression that we have with the other canonical decompositions. Note that a similar conclusion can be reached for the α-disjunctions.

Rules based on α-junctive weights

As explained in the introduction of this chapter, the motivation behind finding the α-junctive canonical decompositions is that it might be possible to generalize part II of this thesis or the cautious rule to the α-junctions. Unfortunately, it has not been possible during the course of this thesis to find such generalizations.

The fact that the α-junctive weights take their values in $(-\infty,+\infty)\setminus\{0\}$ makes these generalizations hard. In particular, Lemma 2.1, which states that it is possible to decrease weights and that is essential for Part II and to derive the cautious rule, does not hold for α-junctive weights as shown by Example 7.4. There exists perhaps a property similar to the one given by Lemma 2.1 for α-junctive weights, e.g., it is possible to decrease the weights on $(0,+\infty)$ and increase them on $(-\infty,0)$, but we have not been able to find it. Some insight to find such a property can perhaps be gained by looking at the behavior of the product on $(-\infty,+\infty)\setminus\{0\}$.

Example 7.4. *Let $\Omega=\{a,b\}$ be a frame of discernment. Let m_1 and m_2 be two BBAs as given in Table 7.3. Let sw_1^α and sw_2^α be their associated α-conjunctive weight functions, for $\alpha=0.5$. Let $sw_{1\wedge2}^\alpha$ denote the minimum of the α-conjunctive weight functions sw_1^α and sw_2^α, and let $m_{1\wedge2}^\alpha$ be the BSMA associated to $sw_{1\wedge2}^\alpha$. As can be seen in Table 7.3, $m_{1\wedge2}^\alpha$ is not a BBA.*

Table 7.3: Minimum of α-conjunctive weights (numbers are rounded to hundredths).

A	m_1	m_2	sw_1^α	sw_2^α	$sw_{1\wedge2}^\alpha$	$m_{1\wedge2}^\alpha$
$\emptyset$	0.2	0.25	-0.8	0.11	-0.8	-0.05
$\{a\}$	0.4	0.24	-0.13	2.38	-0.13	0.65
$\{b\}$	0.2	0.5	0.5	-1.12	-1.12	0.12
$\{a,b\}$	0.2	0.01	1	1	1	0.28

7.4 Conclusion

In this chapter, we have shown that there exist an α-conjunctive and an α-disjunctive canonical decomposition of a belief function that generalize the classical conjunctive and disjunctive canonical decompositions, in the same manner as the α-commonality and α-implicability functions generalize the commonality and implicability functions. Indeed, when $\alpha \neq 1$, we have seen in the preceding chapter that an α-conjunction, for instance, can be expressed as pointwise product of α-commonalities, thus generalizing the classical expression of pointwise product of commonalities. And, when $\alpha = 1$, the generalization degenerates to the classical expression. Similarly, this chapter has shown that an α-conjunction can be expressed as pointwise product of α-conjunctive weights, thus generalizing the classical expression of pointwise product of conjunctive weights. And, when $\alpha = 1$, the generalization reduces to the classical expression.

Conclusion

Summary of Contributions

This thesis has tackled the problem of the lack of flexibility for the combination of belief functions, by introducing or revisiting infinite families of combination rules. Two main contributions were exposed in this thesis.

The first one is the introduction of four infinite families of combination rules based on weight functions and extended t-norms or uninorms. More precisely, there exist two t-norm-based families that are based, respectively, on the conjunctive and disjunctive weight functions. There exist also two uninorm-based families that are based, respectively, on the conjunctive and disjunctive weight functions. It was also shown that t-norm-based conjunctive and disjunctive rules, as well as uninorm-based conjunctive and disjunctive rules, are related by De Morgan laws. Numerical experiments showed that the t-norm-based conjunctive rules may improve the performances in some classification applications. The existence of such families of rules suggests that the TBM is not poorer than possibility theory in terms of conjunctive and disjunctive fusion operators.

Of particular interest is that the four basic rules - the TBM conjunctive, TBM disjunctive, cautious, and bold rules - occupy a special position in each of their respective family: the $\bigcirc\!\!\!\!\!\cap$ and $\bigcirc\!\!\!\!\!\wedge$ rules are the least committed elements, whereas the $\bigcirc\!\!\!\!\!\cup$ and $\bigcirc\!\!\!\!\!\vee$ rules are the most committed elements. The fact that the TBM conjunctive rule is the least committed element among the rules based on conjunctive weights and that have the vacuous belief function as neutral element, was also used in this thesis to propose a new justification for this rule.

Computational aspects of the uninorm-based conjunctive rules in problems involving multiple variables were also investigated. It was shown that, except the TBM conjunctive rule, these rules cannot benefit from the valuation algebra framework in order to perform inference efficiently. This other singular property of the TBM conjunctive rule in its family of rules, may be seen as yet another argument in favor of this rule, but also as a limitation to the breadth of problems that can be tackled by the uninorm-based conjunctive rules.

Our second main contribution is a set of results making an infinite and purely formal family of rules, called the α-junctions, of practical interest. Those operators, which generalize the TBM conjunctive, TBM disjunctive, exclusive conjunctive and exclusive disjunctive rules, were shown to correspond to some particular knowledge about the truthfulness of the sources. The α-junctions becomes thus suitable as flexible combination rules that allows one to take into account some particular knowledge about the sources. Several simple ways of computing a combination by an α-junction were also proposed, making the practical use of the α-junctions in applications possible. These new means are based on generalizations of mechanisms that can be used to compute the combinations by the TBM conjunctive and TBM disjunctive rules. In particular, the conditioning operation and the matrices that permit the easy computation of the commonality and implicability functions associated to a belief function, have been generalized in the context of the α-junctions. In the same vein, it was revealed that the conjunctive and disjunctive canonical decompositions can be generalized, yielding the α-junctive canonical decomposition of a belief function. Finding this decomposition was not trivial since it required the use of the canonical decompositions of signed belief functions.

Perspectives

The work presented in this thesis may be continued in many directions. In the following paragraphs, we sketch a few of them.

On a theoretical level, the problem of finding infinite family of rules based on α-junctive weights must be tackled. This problem is interesting since it might lead to a two-dimensional flexibility for the combination of belief functions and also to a generalization of the cautious rule. Additionally, it seems also interesting to check whether the α-junctions verify the axioms of the valuation algebra framework. The answer to this problem might be found using the fact that an α-conjunction corresponds to the combination by the TBM conjunctive rule of signed belief functions. Another line of potentially interesting research is to work on the difference between a truthful source and a reliable source, as touched upon in Appendix D, which shows that this difference is not always clearly made in the literature although it is important. In addition, it must be explained why the cautious rule does not reduce to the minimum rule of the possibility distributions when the belief functions to be combined are consonant.

On a practical level, it seems worthwhile to experimentally verify that the α-junctions lead to better performances in applications. In particular, experiments are under way to check whether replacing the TBM conjunctive rule by the α-junctions when fusing classifiers, improves classification results. Finally, we should also pursue the investigation on the usefulness of the families of rules based on extended t-norms and uninorms, with a focus on information fusion problems involving nonseparable belief functions.

Part IV

Appendices

Inverse of Dempster's Rule: Historical Remark

Let m be a normal BBA defined on a binary frame of discernment $\Omega = \{p, \neg p\}$ as

$$\begin{aligned} m(p) &= a, \\ m(\neg p) &= b, \\ m(\Omega) &= 1 - a - b, \end{aligned}$$

with $0 \leq a, b < 1$, $a + b < 1$, hence m is nondogmatic. The BBA m can be equivalently noted using a pair (a, b), called a Dempster pair or d-pair in [44]. The vacuous BBA m_Ω can thus be noted using the pair $(0, 0)$. Note that the d-pairs $(0, 1)$ and $(1, 0)$, called extremal in [44], are excluded of our definition of m.

In [44, p. 31], Hájek and Valdes note that Ginsberg [38] claims that Dempster's rule, which we note $\oplus$, is invertible. They further remark that this is false in the set of (non extremal) d-pairs as there exists no BBA m' such that $m \oplus m' = m_\Omega$ with $m \neq m_\Omega$, i.e., m does not have an inverse element with respect to $\oplus$ in the set of (non extremal) d-pairs. In this sense, it may indeed be said that $\oplus$ is not invertible.

The operator proposed by Ginsberg [38, p. 127, equation (7)], which we note $\ominus$, is nonetheless historically the first operator, to our knowledge, to realize an inverse operation to the Dempster's operation, i.e., it is an operator such that $m_1 \oplus m_2 \ominus m_2 = m_1$, for m_1 and m_2 two nondogmatic BBAs defined on a binary frame of discernment. Let us recall the definition of the $\ominus$ operator of Ginsberg:

$$(a, b) \ominus (c, d) = \left(\frac{\bar{c}\left(a\bar{d} - \bar{b}c\right)}{\bar{c}\bar{d} - \bar{b}c\bar{c} - \bar{a}d\bar{d}}, \frac{\bar{d}\left(b\bar{c} - \bar{a}d\right)}{\bar{c}\bar{d} - \bar{b}c\bar{c} - \bar{a}d\bar{d}} \right), \tag{A.1}$$

with $\bar{a} = 1 - a$.

Let us verify that the operator $\ominus$ of Ginsberg allows one to retract previous evidence. Let $c = a$ and $d = b$, we have:

$$\begin{aligned} a\bar{d} - \bar{b}c = a\bar{b} - \bar{b}a &= 0, \\ b\bar{c} - \bar{a}d = b\bar{a} - \bar{a}b &= 0. \end{aligned}$$

Further, we have

$$\begin{aligned} \bar{c}\bar{d} - \bar{b}c\bar{c} - \bar{a}d\bar{d} &= \bar{a}\bar{b} - \bar{b}a\bar{a} - \bar{a}b\bar{b} \\ &= \bar{a}\bar{b}\left(1 - a - b\right) \neq 0. \end{aligned}$$

Hence $(a, b) \ominus (a, b) = (0, 0)$.

Interestingly, Proposition A.1 shows that the operator $\ominus$ defined by (A.1) is just the equivalent on d-pairs to the operator $\oslash\!\!\!\bigcirc$ defined by (1.9) followed by normalization using (1.1).

Proposition A.1. *Let m_1 and m_2 be two nondogmatic BBAs associated to the d-pairs (a, b) and (c, d), respectively. We have*

$$m_1 \ominus m_2 = k \cdot m_1 \oslash\!\!\!\bigcirc m_2,$$

with $k = \left(1 - m_{1\oslash\!\!\!\bigcirc 2}(\emptyset)\right)^{-1}$

Proof. The commonality functions associated to those BBAs are the following:

$$q_1(p) = 1 - b,\ q_1(\neg p) = 1 - a,\ q_1(\Omega) = 1 - a - b,\ q_1(\emptyset) = 1,$$

and

$$q_2(p) = 1 - d,\ q_2(\neg p) = 1 - c,\ q_2(\Omega) = 1 - c - d,\ q_2(\emptyset) = 1.$$

We have

$$\begin{aligned} q_{1\oslash\!\!\!\bigcirc 2}(p) &= \frac{1-b}{1-d} \quad , \\ q_{1\oslash\!\!\!\bigcirc 2}(\neg p) &= \frac{1-a}{1-c} = \frac{\bar{a}}{\bar{c}} \quad , \\ q_{1\oslash\!\!\!\bigcirc 2}(\Omega) &= \frac{1-a-b}{1-c-d} \quad , \\ q_{1\oslash\!\!\!\bigcirc 2}(\emptyset) &= 1 \quad . \end{aligned}$$

Hence

$$\begin{aligned} m_{1\oslash\!\!\!\bigcirc 2}(p) &= q_{1\oslash\!\!\!\bigcirc 2}(p) - q_{1\oslash\!\!\!\bigcirc 2}(\Omega) \\ &= \frac{1-b}{1-d} - \frac{1-a-b}{1-c-d} \\ &= \frac{\bar{b}\,(1-c-d)}{\bar{d}\,(1-c-d)} - \frac{\bar{d}\,(1-a-b)}{\bar{d}\,(1-c-d)} \\ &= \frac{\bar{b} - \bar{b}c - \bar{b}d - \bar{d} + a\bar{d} + b\bar{d}}{\bar{d}\,(1-c-d)} \quad . \end{aligned}$$

Since $\bar{b} - \bar{b}d - \bar{d} + b\bar{d} = 0$, we have

$$\begin{aligned} m_{1\oslash\!\!\!\bigcirc 2}(p) &= \frac{a\bar{d} - \bar{b}c}{\bar{d}\,(1-c-d)} \\ &= \frac{\bar{c}\left(a\bar{d} - \bar{b}c\right)}{\bar{c}\bar{d}\,(1-c-d)} \\ &= \frac{\bar{c}\left(a\bar{d} - \bar{b}c\right)}{\bar{c}\bar{d} - \bar{d}c\bar{c} - \bar{c}d\bar{d}} \quad . \end{aligned}$$

Furthermore, we have

$$\begin{aligned} 1 - m_{1\circledslash 2}(\emptyset) &= 1 - q_{1\circledslash 2}(\emptyset) + q_{1\circledslash 2}(p) + q_{1\circledslash 2}(\neg p) - q_{1\circledslash 2}(\Omega) \\ &= m_{1\circledslash 2}(p) + q_{1\circledslash 2}(\neg p) \quad . \end{aligned}$$

Let $m^*_{1\circledslash 2}$ denote the normalized BBA obtained from $m_{1\circledslash 2}$ using (1.1). We have

$$\begin{aligned} m^*_{1\circledslash 2}(p) &= \frac{m_{1\circledslash 2}(p)}{m_{1\circledslash 2}(p) + q_{1\circledslash 2}(\neg p)} \\ &= \frac{\frac{\bar{c}\left(a\bar{d}-\bar{b}c\right)}{\bar{c}\bar{d}(1-c-d)}}{\frac{\bar{c}\left(a\bar{d}-\bar{b}c\right)}{\bar{c}\bar{d}(1-c-d)} + q_{1\circledslash 2}(\neg p)} \\ &= \frac{\frac{\bar{c}\left(a\bar{d}-\bar{b}c\right)}{\bar{c}\bar{d}(1-c-d)}}{\frac{\bar{c}\left(a\bar{d}-\bar{b}c\right)}{\bar{c}\bar{d}(1-c-d)} + \frac{\bar{a}}{\bar{c}} \cdot \frac{\bar{d}(1-c-d)}{\bar{d}(1-c-d)}} \\ &= \frac{\bar{c}\left(a\bar{d}-\bar{b}c\right)}{\bar{c}\left(a\bar{d}-\bar{b}c\right) + \bar{a}\bar{d}\left(1-c-d\right)} \\ &= \frac{\bar{c}\left(a\bar{d}-\bar{b}c\right)}{a\bar{d}\bar{c} - \bar{a}\bar{d}c + \bar{a}\bar{d} - \bar{b}c\bar{c} - \bar{a}\bar{d}d} \\ &= \frac{\bar{c}\left(a\bar{d}-\bar{b}c\right)}{(1-\bar{a})\bar{d}\bar{c} - \bar{a}\bar{d}c + \bar{a}\bar{d} - \bar{b}c\bar{c} - \bar{a}\bar{d}d} \\ &= \frac{\bar{c}\left(a\bar{d}-\bar{b}c\right)}{\bar{c}\bar{d} + \bar{a}\bar{d}(1-\bar{c}-c) - \bar{b}c\bar{c} - \bar{a}\bar{d}d} \\ &= \frac{\bar{c}\left(a\bar{d}-\bar{b}c\right)}{\bar{c}\bar{d} - \bar{b}c\bar{c} - \bar{a}\bar{d}d} \quad . \end{aligned} \tag{A.2}$$

We can see that the left term of the right side of (A.1) is equal to (A.2). The proof is similar for the right term. □

Right and Left Eigenvectors

This appendix provides necessary notions on right and left eigenvectors. This appendix is based on [45, 101, 102, 103].

B.1 Basic Definitions

Let $\mathbf{A}$ be a $n \times n$ square matrix. Consider the following equation

$$\mathbf{A} \cdot \mathbf{r} = \lambda_r \cdot \mathbf{r},$$

where λ_r is a scalar and $\mathbf{r}$ is a column vector. A value of λ_r for which this equation has a solution $\mathbf{r} \neq \mathbf{0}$ is called a right eigenvalue of the matrix $\mathbf{A}$ and $\mathbf{r}$ is a corresponding right eigenvector (right eigenvectors are the usual eigenvectors encountered in linear algebra).

Consider now the following equation

$$\mathbf{l} \cdot \mathbf{A} = \lambda_l \cdot \mathbf{l},$$

where λ_l is a scalar and $\mathbf{l}$ is a row vector. A value of λ_l for which this equation has a solution $\mathbf{l} \neq \mathbf{0}$ is called a left eigenvalue of the matrix $\mathbf{A}$ and $\mathbf{l}$ is a corresponding left eigenvector.

It may be shown that the right eigenvalues of $\mathbf{A}$ are also its left eigenvalues [102].

B.2 Relation between Left and Right Eigenvectors

Assume $\mathbf{A}$ has eigenvalues λ_1, λ_2, ...,λ_n and corresponding right eigenvectors $\mathbf{r}_1, \mathbf{r}_2$, ...,$\mathbf{r}_n$ and corresponding left eigenvectors $\mathbf{l}_1, \mathbf{l}_2, ..., \mathbf{l}_n$. Let $\mathbf{\Lambda}$ denote the $n \times n$ diagonal matrix whose diagonal element Λ_{ii} is λ_i. Let $\mathbf{R}$ be the $n \times n$ matrix whose ith column is $\mathbf{r}_i$ and let $\mathbf{L}$ be the $n \times n$ matrix whose ith row is $\mathbf{l}_i$. From the eigen decomposition theorem [101], we obtain:

$$\mathbf{A} = \mathbf{R} \cdot \mathbf{\Lambda} \cdot \mathbf{R}^{-1}.$$

Hence, we have

$$\mathbf{A} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{\Lambda}, \tag{B.1}$$

and

$$\mathbf{R}^{-1} \cdot \mathbf{A} = \mathbf{\Lambda} \cdot \mathbf{R}^{-1}. \tag{B.2}$$

It may also be shown (the proof is similar to the proof of (B.1) given in [103]) that

$$\mathbf{L} \cdot \mathbf{A} = \mathbf{\Lambda} \cdot \mathbf{L}, \tag{B.3}$$

holds. Hence, from (B.2) and (B.3), it is clear that any row of $\mathbf{R}^{-1}$ satisfies the properties of a left eigenvector [45]. Similarly, any column of L^{-1} satisfies the properties of a right eigenvector.

On Latent Belief Structures

C.1 Introduction

It was explained in Section 2.2.2 that a nondogmatic BBA m can be decomposed into two u-separable BBAs noted m^c and m^d, and called, respectively, the confidence and diffidence component of m. Furthermore, Smets [85] proposed to interpret m^c as representing positive evidence, i.e., good reasons to believe in various propositions $A \subseteq \Omega$, and m^d as representing negative evidence, i.e., good reasons not to believe in the same propositions. From this decomposition, Smets [85] defined a structure to allow the representation of belief states, where both confidence and diffidence are involved. He called this structure a Latent Belief Structure (LBS). Formally, a LBS may be defined as follows.

Definition C.1 (Latent Belief Structure)**.** *A latent belief structure is defined as a pair of u-separable nondogmatic BBAs m^c and m^d called respectively the confidence and diffidence components. A LBS is noted using a upper-case L.*

He also defined the concept of Apparent Belief Structure (ABS).

Definition C.2 (Apparent Belief Structure)**.** *The apparent belief structure associated with a LBS $L = (m^c, m^d)$ is the BSMA[1] sm obtained from the decombination $m^c \oslash m^d$ of the confidence and diffidence components of L.*

The properties linking these definitions are the following. By definition, the apparent belief structure associated to a LBS may or may not be a belief function. Furthermore, an infinity of LBSs correspond to the same apparent belief structure.

In [85], Smets extends the combination of belief functions by the TBM conjunctive rule to LBSs. In this appendix, we propose an exploratory work, which extends to LBSs some notions of belief function theory such as combination rules, informational comparison, and transformation to a probability measure.

This appendix is organized as follows. In Section C.2, some combination rules and partial orderings allowing the informational comparison of LBSs are defined. A transformation of a LBS to a probability measure is then defined in Section C.3.

This work was published in English in [63] and in French in [67].

[1] BSMA stands for Basic Signed Measure Assignment, see Section 7.2 for material on BSMAs.

C.2 Combination Rules for LBSs

This section studies mathematical operations on LBSs. Let us first express two known operations of belief function theory using LBSs.

Let $\left(m_1^c, m_1^d\right)$ and $\left(m_2^c, m_2^d\right)$ be the LBSs associated with two nondogmatic BBAs m_1 and m_2. Then $\left(m_1^c \textcircled{\cap} m_2^c, m_1^d \textcircled{\cap} m_2^d\right)$ is a LBS associated with $m_1 \textcircled{\cap} m_2$. This lead Smets to define the conjunctive combination of two LBSs as follows.

Definition C.3. *The conjunctive combination of two LBSs L_1 and L_2 is a LBS noted $L_{1\textcircled{\cap}2}$. It is defined by the weight functions (C.1) and (C.2):*

$$w^c_{1\textcircled{\cap}2}(A) = w^c_1(A) \cdot w^c_2(A), \tag{C.1}$$
$$w^d_{1\textcircled{\cap}2}(A) = w^d_1(A) \cdot w^d_2(A). \tag{C.2}$$

This rule is commutative and associative. Furthermore, the LBS associated with the vacuous BBA m_Ω, which we note L_Ω, is a neutral element for $\textcircled{\cap}$, i.e. $L \textcircled{\cap} L_\Omega = L$ for all LBSs L. The cautious rule of combination [18] can also be expressed in terms of LBSs.

Definition C.4. *([18, Proposition 6]) The cautious combination of two LBSs L_1 and L_2 is a LBS noted $L_{1\textcircled{\wedge}2}$. It is defined by the following weight functions:*

$$w^c_{1\textcircled{\wedge}2}(A) = w^c_1(A) \wedge w^c_2(A), \tag{C.3}$$
$$w^d_{1\textcircled{\wedge}2}(A) = w^d_1(A) \vee w^d_2(A). \tag{C.4}$$

This rule is commutative, associative and idempotent.

C.2.1 Informational comparison of LBSs

It is clear that the TBM conjunctive and cautious rules belong to different families of combination rules. Indeed, we have

$$m^c_{1\textcircled{\cap}2} \sqsubseteq_w m^c_i,$$
$$m^d_{1\textcircled{\cap}2} \sqsubseteq_w m^d_i,$$

and

$$m^c_{1\textcircled{\wedge}2} \sqsubseteq_w m^c_i,$$
$$m^d_{1\textcircled{\wedge}2} \sqsupseteq_w m^d_i,$$

where $i \in \{1, 2\}$. In other words, the $\textcircled{\cap}$ rule produces a LBS $L_{1\textcircled{\cap}2}$ which is both w-more committed in confidence and in diffidence than the LBSs L_1 and L_2, whereas the $\textcircled{\wedge}$ rule produces a LBS $L_{1\textcircled{\wedge}2}$ which is w-more committed in confidence and w-less committed in diffidence than the LBSs L_1 and L_2.

The behaviors of the TBM conjunctive and cautious rules suggest two partial orderings for LBSs, which are formally defined in Definitions C.5 and C.6 below.

Definition C.5. *A LBS L which is both w-more committed in confidence and in diffidence than a LBS L' is said to be l-more committed than L' (l stands for* latent*), which is noted $L \sqsubseteq_l L'$.*

Definition C.6. *A LBS L which is w-more committed in confidence and w-less committed in diffidence than a LBS L' is said to be a-more committed than L' (a stands for* apparent*), which is noted $L \sqsubseteq_a L'$.*

We may remark that the $\sqsubseteq_a$ ordering implies the order of the apparent belief structures, i.e., if the ABS of L is a BBA m and the ABS of L' is a BBA m' and if $L \sqsubseteq_a L'$, then $m \sqsubseteq_w m'$. In this respect, the a-ordering seems reasonable. The l-ordering does not verify such a property. However, this latter ordering does seem to make sense if one reasons in terms of the semantics of the LBSs. Indeed, the l-ordering considers that a LBS is more committed than another one if it has *more* positive evidence and *more* negative evidence.

C.2.2 Cautious merging technique applied to LBSs

Equipped with these two partial orderings for LBSs, we may study what the cautious merging technique (see Section 2.5.1) applied to LBSs yields.

Suppose that an agent receives two LBSs L_1 and L_2. Upon receiving those two pieces of information, the agent's state of belief should be represented by a LBS L_{12} more informative than L_1 and L_2. Let $\mathcal{S}_x(L)$ be the set of LBSs L' such that $L' \sqsubseteq_x L$, with $x \in \{a, l\}$. Hence $L_{12} \in \mathcal{S}_x(L_1)$ and $L_{12} \in \mathcal{S}_x(L_2)$ or, equivalently, $L_{12} \in \mathcal{S}_x(L_1) \cap \mathcal{S}_x(L_2)$, with $x \in \{a, l\}$. According to the LCP, the x-least committed LBS should be chosen in $\mathcal{S}_x(L_1) \cap \mathcal{S}_x(L_2)$. This defines a combination rule if the x-least committed LBS exists and is unique. Interestingly, this is the case for both orderings $\sqsubset_a$ and $\sqsubset_l$. Indeed, the a-least committed element in $\mathcal{S}_a(L_1) \cap \mathcal{S}_a(L_2)$ is clearly given by the cautious rule. The l-least committed element in $\mathcal{S}_l(L_1) \cap \mathcal{S}_l(L_2)$ is given by Proposition C.1.

Proposition C.1. *Let L_1 and L_2 be two LBSs. The l-least committed element in $\mathcal{S}_l(L_1) \cap \mathcal{S}_l(L_2)$ exists and is unique. It is defined by the following confidence and diffidence weight functions:*

$$w^c_{1\circledwedge 2}(A) = w^c_1(A) \wedge w^c_2(A), \quad A \in 2^\Omega \setminus \{\Omega\}, \tag{C.5}$$

$$w^d_{1\circledwedge 2}(A) = w^d_1(A) \wedge w^d_2(A), \quad A \in 2^\Omega \setminus \{\Omega\}. \tag{C.6}$$

Proof. Direct using Proposition 4 of [18]. □

Proposition C.1 allows us to define a combination rule for LBSs as follows.

Definition C.7 (Weak Rule)**.** *Let L_1 and L_2 be two LBSs. Their combination with the weak rule is defined as the LBS, noted $L_{1\circledwedge 2}$, which weight functions are given by (C.5) and (C.6).*

This rule is commutative, associative and idempotent. In addition, $\textcircled{\cap}$ is distributive with respect to $\textcircled{\wedge\wedge}$. Those properties originate from the properties of the $\textcircled{\wedge}$ rule [18] since there is a connection between the partial orderings on which those two rules are built. We can thus see that the combination by the $\textcircled{\wedge\wedge}$ rule consists in combining the confidence and diffidence components by the $\textcircled{\wedge}$ rule.

The $\textcircled{\wedge\wedge}$ rule exhibits other properties: L_Ω is a neutral element and if $L_1 \sqsubseteq_l L_2$, the result of the least committed combination of those LBSs is $L_1 \textcircled{\wedge\wedge} L_2 = L_1$. Further, using the l-ordering in the derivation of the rule allows the construction of a 'weaker', or l-less committed, version of the TBM conjunctive rule, i.e. $L_1 \textcircled{\cap} L_2 \sqsubseteq_l L_1 \textcircled{\wedge\wedge} L_2$.

Note that the apparent form of a LBS $L_{1\textcircled{\wedge\wedge}2}$, produced by the $\textcircled{\wedge\wedge}$ combination of two LBSs L_1 and L_2 obtained from two nondogmatic BBAs m_1 and m_2, may not be a BBA. However, if m_1 and m_2 are separable BBAs then the apparent form of the LBS $L_{1\textcircled{\wedge\wedge}2}$ is a BBA since a separable BBA yields a LBS whose diffidence component is vacuous and the $\textcircled{\wedge\wedge}$ combination consists in combining the confidence component of L_1 and L_2 by the $\textcircled{\wedge}$ rule. Furthermore, it may easily be shown that the $\textcircled{\wedge\wedge}$ rule applied to two LBSs obtained from two consonant BBAs will yield a LBS whose apparent form is a separable BBA which is not necessarily consonant.

Example C.1. *Table C.1 gives two LBS $L_1 = (m_1^c, m_1^d)$ and $L_2 = (m_2^c, m_2^d)$ together with their associated weight functions. Tables C.2 shows the weight functions resulting from the weak ($\textcircled{\wedge\wedge}$), the TBM conjunctive ($\textcircled{\cap}$), and the cautious ($\textcircled{\wedge}$) combinations of the LBSs given by Table C.1.*

Table C.1: Two LBS together with their associated weight functions.

A	$m_1^c(A)$	$m_1^d(A)$	$w_1^c(A)$	$w_1^d(A)$	$m_2^c(A)$	$m_2^d(A)$	$w_2^c(A)$	$w_2^d(A)$
$\emptyset$	0	0	1	1	0	0	1	1
$\{a\}$	0	0	1	1	0	0	1	1
$\{b\}$	4/9	4/9	1	5/9	0	0	1	1
$\{a,b\}$	2/9	0	1/3	1	0	0	1	1
$\{c\}$	0	0	1	1	4/9	4/9	1	5/9
$\{a,c\}$	0	0	1	1	2/9	0	1/3	1
$\{b,c\}$	2/9	0	1/3	1	2/9	0	1/3	1
Ω	1/9	5/9			1/9	5/9		

Finally, we may remark that, in the same vein as Chapter 4, it is possible to define infinite families of conjunctive combination rules for LBSs. The $\textcircled{\cap}$ and $\textcircled{\wedge\wedge}$ rules are then merely instances of these families. This extension is based on the observation that the $\textcircled{\cap}$ rule uses the product, whereas the $\textcircled{\wedge\wedge}$ rule uses the minimum of weights belonging to the unit interval. These two operations on this interval are triangular norms. Replacing them by any positive t-norm $\top$ yields operators, noted $\textcircled{\top}$, which possess the following properties: commutativity, associativity, neutral element L_Ω and monotonicity with respect to $\sqsubseteq_l$, i.e. $\forall L_1, L_2$ and L_3, $L_1 \sqsubseteq_l L_2 \Rightarrow L_1 \textcircled{\top} L_3 \sqsubseteq_l L_2 \textcircled{\top} L_3$. Only the $\textcircled{\wedge\wedge}$ rule is idempotent.

Table C.2: Weight functions obtained from different combinations.

A	$w^c_{1\circledwedge 2}(A)$	$w^d_{1\circledwedge 2}(A)$	$w^c_{1\bigcirc\!\!\!\!\cap 2}(A)$	$w^d_{1\bigcirc\!\!\!\!\cap 2}(A)$	$w^c_{1\circledwedge 2}(A)$	$w^d_{1\circledwedge 2}(A)$
$\emptyset$	1	1	1	1	1	1
$\{a\}$	1	1	1	1	1	1
$\{b\}$	1	5/9	1	5/9	1	1
$\{a, b\}$	1/3	1	1/3	1	1/3	1
$\{c\}$	1	5/9	1	5/9	1	1
$\{a, c\}$	1/3	1	1/3	1	1/3	1
$\{b, c\}$	1/3	1	1/9	1	1/3	1

C.3 Transformation to a Probability Measure

This section provides a means to transform a LBS into a probability measure. The plausibility transformation is of particular interest here due to one of its properties: it is invariant with respect to the decombination by $\circledcirc$ (see Proposition 1.3, Section 1.5).

Using this property, a LBS $L = \left(m^c, m^d\right)$ can be transformed into a probability measure as follows:

$$PlP_L = PlP_{m^c} \oslash PlP_{m^d} \ .$$

Example C.2. *Table C.3 illustrates the computation of the plausibility transformation of the LBSs obtained from different combinations in Example C.1.*

Table C.3: Plausibility transformations of the LBSs obtained in Example C.1.

A	$PlP_{1\circledwedge 2}$	$PlP_{1\bigcirc\!\!\!\!\cap 2}$	$PlP_{1\circledwedge 2}$
$\{a\}$	9/19	0.23	1/3
$\{b\}$	5/19	0.385	1/3
$\{c\}$	5/19	0.385	1/3

The plausibility transformation of a LBS has different interesting properties.

Let $\overset{PlP}{\sim}$ denote the equivalence relation between LBSs defined by $L_1 \overset{PlP}{\sim} L_2$ iff $PlP_{L_1}(\{\omega_k\}) = PlP_{L_2}(\{\omega_k\})$, $\forall \omega_k \in \Omega$.

Proposition C.2. $\bar{A}^\alpha \overset{PlP}{\sim} A^{\frac{1}{\alpha}}$*, for* $\alpha \in (0, 1]$.

Proof. $\forall \omega_k \in A, A \subset \Omega$,

$$PlP_{\bar{A}^\alpha}(\{\omega_k\}) = \frac{\alpha}{|\bar{A}| + |A|\,\alpha} \ , \tag{C.7}$$

$$PlP_{A^{1/\alpha}}(\{\omega_k\}) = \frac{1}{|A| + |\bar{A}|\,\frac{1}{\alpha}} \ . \tag{C.8}$$

(C.7) and (C.8) are equal. $\square$ $\square$

Proposition C.2 shows that two ways of modeling negative statements become equivalent when PlP is used. Indeed, according to Smets's terminology [85], for $A \subset \Omega$, *having good reasons to believe in not* A is equivalent, in terms of probability measure generated using PlP, to *having good reasons not to believe in* A.

Finally, let us note that Propositions 1.3 and C.2 allow us to define equivalence classes with respect to the plausibility transformation in which there is at least one u-separable BBA; for instance we have: $\left(\bar{A}^{0.6}, A^{0.5}\right) \overset{PlP}{\sim} \left(\bar{A}^{0.3}, \Omega\right)$. Note also that the combination by $\circledcap$ of any two LBSs belonging to two different equivalence classes always falls in the same equivalence class, for instance if $L_1 \overset{PlP}{\sim} L_2$ and $L_3 \overset{PlP}{\sim} L_4$, then, e.g., $L_1 \circledcap L_3 \overset{PlP}{\sim} L_2 \circledcap L_4$. It can easily be shown that this is not true for the $\circledwedge$ and $\circledwedge\!\!\!\wedge$ rules.

C.4 Conclusion

From the canonical decomposition of a belief function, Smets defined a construct called a latent belief structure. As shown in this appendix, it is possible to extend to LBSs some notions that apply to belief functions. Clearly, those extensions are only sound at a mathematical level and further work on the interpretation and the justification of the LBSs and the operations defined is needed.

To conclude, we may note that Smets is not the only author to have been interested in the idea of positive and negative information. Indeed, Dubois et al. [32] and Labreuche et al. [54] have also investigated in belief function theory the existence of positive and negative information, which is usually coined under the term *bipolarity*. An in-depth comparison of those various models would be interesting since it is not clear yet what their relationships are. It does not seem, for instance, that the negative information as understood in [33] corresponds to the negative information of Smets' LBS. Indeed, the negative information in the former model is represented by a BBA, which may be canonically decomposed into a confidence and a diffidence component, i.e., into positive and negative information as understood in Smets' model.

Reliability Versus Truthfulness

In this appendix, a short discussion on the implications of making the difference between a reliable source and a truthful source, is provided. First, we follow in Section D.1, a similar reasoning to the one of Section 6.3.2, placing the focus on the reliability of a source rather than its truthfulness. Then, we show in Section D.2 that the exclusive disjunctive rule does not correspond to the situation where it is known that exactly one of the sources is reliable, as wrongly stated by [41, Theorem 3.3]. We then conclude this appendix with some further comments related to the reliability and the truthfulness of a source.

D.1 Reliability of a Source

Let $\boldsymbol{\omega}$ be a variable, which takes its values in a frame Ω. Suppose an agent who does not know anything about the actual value ω_0 taken by $\boldsymbol{\omega}$. Suppose a source $S1$ that tells the agent that the actual value ω_0 is in $A \subseteq \Omega$, i.e., $\omega_0 \in A$. If the source is reliable, then the agent believes $\omega_0 \in A$. If the source is not reliable, then the agent believes $\omega_0 \in \Omega$.

Let $\boldsymbol{\rho}$ be a variable taking its values in a frame $R = \{r, \overline{r}\}$. We use $\boldsymbol{\rho}$ to denote the reliability of the source. The information $\omega_0 \in A$ provided by $S1$ can be modeled by a BBA m_1^Ω such that $m_1^\Omega(A) = 1$. The information *when the source is reliable, ω_0 must be in A, and when the source is not reliable, ω_0 must be in* Ω, may be modeled by a BBA noted $m_{1''}^{\Omega \times R}$ and defined on the product space $\Omega \times R$ by

$$m_{1''}^{\Omega \times R}(A \times \{r\} \cup \Omega \times \{\overline{r}\}) = 1. \tag{D.1}$$

Note that we use the index $1''$ in $m_{1''}^{\Omega \times R}$, i.e., the source number followed by two prime symbols, to highlight that the BBA $m_{1''}^{\Omega \times R}$ is obtained from the source $S1$, as is the case of the BBA $m_{1'}^{\Omega \times T}$ (see Equation (6.11)), but that it conveys a different information from the BBA $m_{1'}^{\Omega \times T}$.

One verifies that the BBA $m_{1''}^{\Omega \times R}$ is appropriate to model the information available in this scenario since

- combining $m_{1''}^{\Omega \times R}$ with a BBA m_r^R defined on R by $m_r^R(r) = 1$, and then marginalizing on Ω, yields a BBA m_{Ag}^Ω such that $m_{Ag}^\Omega(A) = 1$, i.e., if the agent believes that the source is reliable, then the agent believes $\omega_0 \in A$;
- combining $m_{1''}^{\Omega \times R}$ with a BBA $m_{\overline{r}}^R$ defined on R by $m_{\overline{r}}^R(\overline{r}) = 1$, and then marginalizing on Ω, yields a BBA m_{Ag}^Ω such that $m_{Ag}^\Omega(\Omega) = 1$, i.e., if the agent believes that the source is not reliable, then the agent believes $\omega_0 \in \Omega$.

We may further remark that

$$(m_{1''}^{\Omega\times R} \circledcirc m_r^{R\uparrow\Omega\times R})^{\downarrow\Omega} = m_1^\Omega \tag{D.2}$$

and

$$(m_{1''}^{\Omega\times R} \circledcirc m_{\overline{r}}^{R\uparrow\Omega\times R})^{\downarrow\Omega} = m_\Omega^\Omega, \tag{D.3}$$

where m_Ω^Ω is the vacuous BBA on Ω, which is sound since m_Ω^Ω represents the BBA that would be induced on Ω if the agent knows that the source providing a BBA m_1^Ω is not reliable.

This reasoning may be generalized when the source produces an information in the form of a BBA rather than a set, in which case the BBA $m_{1''}^{\Omega\times R}$ is such that

$$m_{1''}^{\Omega\times R}(A \times \{r\} \cup \Omega \times \{\overline{r}\}) = m_1^\Omega(A), \quad \forall A \subseteq \Omega. \tag{D.4}$$

Here again, Equations (D.2) and (D.3) are verified, which means that, as expected, the agent's beliefs are equated to what the source says if the source is reliable, and the agent's beliefs are vacuous if the source is not reliable.

Note that the BBA $m_{1''}^{\Omega\times R}$ as defined by (D.4) is equal to the BBA obtained by a combination by $\circledcirc$ of the ballooning extensions on $\Omega \times R$ of the BBAs $m^\Omega[r]$ and $m^\Omega[\overline{r}]$ defined by $m^\Omega[r](A) = m_1(A)$, for all $A \subseteq \Omega$, and by $m^\Omega[\overline{r}] = m_\Omega$, where m_Ω denotes the vacuous BBA – the BBA $m^\Omega[r]$ represents the beliefs of the agent given that the source is reliable, and the BBA $m^\Omega[\overline{r}]$ represents the beliefs of the agent given that the source is not reliable. Furthermore, note that since the ballooning extension of the BBA $m^\Omega[\overline{r}]$ on $\Omega \times R$ is the vacuous BBA on $\Omega \times R$, we also have that $m_{1''}^{\Omega\times R}$ is merely equal to the BBA obtained when performing the ballooning extension on $\Omega \times R$ of the BBA $m^\Omega[r]$. Hence, we may use interchangeably the BBA $m_{1''}^{\Omega\times R}$ or the ballooning extension of $m^\Omega[r]$. This is important since BBAs of the kind of $m^\Omega[r]$, rather than $m_{1''}^{\Omega\times R}$, are used in the next section.

D.2 Haenni's Exclusive Disjunction

Theorem 3.3 of [41] states without proof that the exclusive disjunctive rule corresponds to the situation where exactly one of the sources is reliable, without knowing which one. We show in this section that this theorem is wrong.

Let m_1^Ω and m_2^Ω be two BBAs provided by two sources of information $S1$ and $S2$. Let $R1 = \{r1, \overline{r1}\}$ and $R2 = \{r2, \overline{r2}\}$; these frames are used to represent the reliability of $S1$ and $S2$, respectively.

Suppose we want to build a BBA $m_{1\odot 2}^\Omega$ quantifying our belief on Ω based on the BBAs m_1^Ω and m_2^Ω, and based on a piece of evidence stating that exactly one of the sources is reliable, but we do not know which one. We have thus

$$m_{1\odot 2}^\Omega \;=\; (m_{1''}^{\Omega\times R1\uparrow\Omega\times R1\times R2} \circledcirc m_{2''}^{\Omega\times R2\uparrow\Omega\times R1\times R2} \circledcirc m_{xor}^{R1\times R2\uparrow\Omega\times R1\times R2})^{\downarrow\Omega},$$

with

$$m_{i''}^{\Omega\times Ri}(A \times \{ri\} \cup \Omega \times \{\overline{ri}\}) \;=\; m_i^\Omega(A), \quad \forall A \subseteq \Omega, \quad i = 1, 2,$$

and

$$m_{xor}^{R1\times R2}\left(\left\{\left(r1,\overline{r2}\right),\left(\overline{r1},r2\right)\right\}\right) = 1. \tag{D.5}$$

Or, equivalently,

$$m_{1\odot 2}^{\Omega} = \left(\left(m^{\Omega}[r1]^{\Uparrow R1\times\Omega}\right)^{\uparrow R1\times R2\times\Omega} \cap\!\!\!\!\bigcirc \left(m^{\Omega}[r2]^{\Uparrow R2\times\Omega}\right)^{\uparrow R1\times R2\times\Omega} \cap\!\!\!\!\bigcirc m_{xor}^{R1\times R2\uparrow R1\times R2\times\Omega}\right)^{\downarrow\Omega}, \tag{D.6}$$

where $m^{\Omega}[r1](A) = m_1^{\Omega}(A)$ and $m^{\Omega}[r2](A) = m_2^{\Omega}(A)$ for all $A \subseteq \Omega$.

According to [41, Theorem 3.3], the BBA $m_{1\odot 2}^{\Omega}$ defined by (D.6) is supposedly equal to the BBA $m_{1\underline{\cup}2}^{\Omega}$ obtained from the combination by the exclusive disjunctive rule $\underline{\cup}$ of the BBAs m_1^{Ω} and m_2^{Ω}. Example D.1 shows that one can find two BBAs m_1^{Ω} and m_2^{Ω} such that $m_{1\odot 2}^{\Omega} \neq m_{1\underline{\cup}2}^{\Omega}$, i.e., Theorem 3.3 of [41] is wrong.

Example D.1. *Let $\Omega = \{a,b,c\}$ and $m_1^{\Omega}(\{a,b\}) = 1$ and $m_2^{\Omega}(\{b,c\}) = 1$. Using (6.5), we find*

$$m_{1\underline{\cup}2}^{\Omega}(\{a,c\}) = 1. \tag{D.7}$$

Let us now compute $m_{1\odot 2}^{\Omega}$ using (D.6). We have $m^{\Omega}[r1](\{a,b\}) = 1$ and thus

$$m^{\Omega}[r1]^{\Uparrow R1\times\Omega}\left(\left\{(a,r1),(b,r1),(a,\overline{r1}),(b,\overline{r1}),(c,\overline{r1})\right\}\right) = 1.$$

Hence, $\left(m^{\Omega}[r1]^{\Uparrow R1\times\Omega}\right)^{\uparrow R1\times R2\times\Omega}(B) = 1$ with

$$\begin{aligned} B = \; & \{(a,r1,r2),(a,r1,\overline{r2}),(b,r1,r2),(b,r1,\overline{r2}),(a,\overline{r1},r2),(a,\overline{r1},\overline{r2}),\\ & (b,\overline{r1},r2),(b,\overline{r1},\overline{r2}),(c,\overline{r1},r2),(c,\overline{r1},\overline{r2})\}. \end{aligned}$$

Furthermore, we have $m^{\Omega}[r2](\{b,c\}) = 1$ and thus

$$m^{\Omega}[r2]^{\Uparrow R2\times\Omega}\left(\left\{(b,r2),(c,r2),(a,\overline{r2}),(b,\overline{r2}),(c,\overline{r2})\right\}\right) = 1.$$

Hence, $\left(m^{\Omega}[r2]^{\Uparrow R2\times\Omega}\right)^{\uparrow R1\times R2\times\Omega}(C) = 1$ with

$$\begin{aligned} C = \; & \{(b,r1,r2),(b,\overline{r1},r2),(c,r1,r2),(c,\overline{r1},r2),(a,r1,\overline{r2}),(a,\overline{r1},\overline{r2}),\\ & (b,r1,\overline{r2}),(b,\overline{r1},\overline{r2}),(c,r1,\overline{r2}),(c,\overline{r1},\overline{r2})\}. \end{aligned}$$

We also have $m_{xor}^{R1\times R2\uparrow R1\times R2\times\Omega}(D) = 1$ with

$$D = \left\{\left(a,r1,\overline{r2}\right),\left(a,\overline{r1},r2\right),\left(b,r1,\overline{r2}\right),\left(b,\overline{r1},r2\right),\left(c,r1,\overline{r2}\right),\left(c,\overline{r1},r2\right)\right\}.$$

Combining $\left(m^{\Omega}[r1]^{\Uparrow R1\times\Omega}\right)^{\uparrow R1\times R2\times\Omega}$ and $\left(m^{\Omega}[r2]^{\Uparrow R2\times\Omega}\right)^{\uparrow R1\times R2\times\Omega}$ and $m_{xor}^{R1\times R2\uparrow R1\times R2\times\Omega}$ by the rule $\cap\!\!\!\!\bigcirc$ yields a BBA defined on $R1\times R2\times\Omega$, which has only one focal set: $B\cap C\cap D = \left\{(a,r1,\overline{r2}),(b,r1,\overline{r2}),(b,\overline{r1},r2),(c,\overline{r1},r2)\right\}$. Marginalizing this latter BBA on Ω yields $m_{1\odot 2}^{\Omega}(\{a,b,c\}) = 1$, which is different from (D.7).

D.3 Some Comments

As shown in the previous section, the BBA $m_{1\odot 2}$ computed using (D.6), is not equal to the BBA $m^{\Omega}_{1\textcircled{$\underline{\cup}$}2}$. This means that the exclusive disjunctive rule does not correspond to the situation where exactly one of the sources is known to be reliable, without knowing which one. As a matter of fact, the following proposition holds.

Proposition D.1. *Let m_1^{Ω} and m_2^{Ω} be two BBAs. We have*

$$m^{\Omega}_{1\textcircled{$\cup$}2} = \left(\left(m^{\Omega}[r1]^{\Uparrow R1\times\Omega} \right)^{\uparrow R1\times R2\times\Omega} \textcircled{$\cap$} \left(m^{\Omega}[r2]^{\Uparrow R2\times\Omega} \right)^{\uparrow R1\times R2\times\Omega} \textcircled{$\cap$} m_{xor}^{R1\times R2\uparrow R1\times R2\times\Omega} \right)^{\downarrow\Omega},$$

and

$$m^{\Omega}_{1\textcircled{$\cup$}2} = \left(\left(m^{\Omega}[r1]^{\Uparrow R1\times\Omega} \right)^{\uparrow R1\times R2\times\Omega} \textcircled{$\cap$} \left(m^{\Omega}[r2]^{\Uparrow R2\times\Omega} \right)^{\uparrow R1\times R2\times\Omega} \textcircled{$\cap$} m_{or}^{R1\times R2\uparrow R1\times R2\times\Omega} \right)^{\downarrow\Omega},$$

where $m^{\Omega}[r1](A) = m_1^{\Omega}(A)$ and $m^{\Omega}[r2](A) = m_2^{\Omega}(A)$ for all $A \subseteq \Omega$ and

$$m_{xor}^{R1\times R2}\left(\left\{\left(r1,\overline{r2}\right),\left(\overline{r1},r2\right)\right\}\right) = 1,$$

and

$$m_{or}^{R1\times R2}\left(\left\{\left(r1,r2\right),\left(r1,\overline{r2}\right),\left(\overline{r1},r2\right)\right\}\right) = 1.$$

Proof. See Appendix F.7 □

This proposition means that the TBM disjunctive rule corresponds to the situation where at least one source is known to be reliable, but it also corresponds to the situation where it is known that one and only one of the sources is reliable, without knowing which one. Additionally, let us note that, as stated in [87] and as shown by Proposition D.2 below, the TBM disjunctive rule fits *also* with the situation where it is known that at least one of the sources tells the truth, whereas the exclusive disjunctive rule fits with the situation where one and only one of the sources tells the truth, without knowing which one.

Proposition D.2. *Let m_1^{Ω} and m_2^{Ω} be two BBAs. We have*

$$m_1 \textcircled{$\cup$} m_2 = (m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \textcircled{$\cap$} m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \textcircled{$\cap$} m_{or}^{T1\times T2\uparrow\Omega\times T1\times T2})^{\downarrow\Omega}$$

and

$$m_1 \textcircled{$\underline{\cup}$} m_2 = (m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \textcircled{$\cap$} m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \textcircled{$\cap$} m_{xor}^{T1\times T2\uparrow\Omega\times T1\times T2})^{\downarrow\Omega}$$

with

$$m_{i'}^{\Omega\times Ti}(A \times \{ti\} \cup \overline{A} \times \{fi\}) = m_i^{\Omega}(A), \quad \forall A \subseteq \Omega, \quad i = 1, 2,$$

and

$$m_{xor}^{T1\times T2}\left(\{(t1,f2),(f1,t2)\}\right) = 1,$$

and

$$m_{or}^{T1\times T2}\left(\{(t1,t2),(t1,f2),(f1,t2)\}\right) = 1.$$

Proof. See Appendix F.8. □

In order to enhance the difference between a reliable source and a truthful source, we may informally look at the reason why the BBA $m_{1\odot 2}$ computed using (D.6), is equal to $m_{1\bigcirc\!\!\!\!\cup\, 2}$ and is thus different from $m_{1\bigcirc\!\!\!\!\underline{\cup}\, 2}$. Suppose two sources $S1$ and $S2$. $S1$ states $\omega_0 \in A$ and $S2$ states $\omega_0 \in B$. Suppose further that exactly one of the sources is reliable, without knowing which one, i.e., either $S1$ is reliable and $S2$ is not, or $S1$ is not reliable and $S2$ is. As explained in Section D.1, if a source states $\omega_0 \in C$ and the source is reliable, then the agent believes $\omega_0 \in C$, and if the source is not reliable, then the agent believes $\omega_0 \in \Omega$. Hence, we may conclude $\omega_0 \in (A \cap \Omega)$ when $S1$ is reliable and $S2$ is not, and we may deduce $(\Omega \cap B)$ when $S1$ is not reliable and $S2$ is. Since exactly one of the sources is reliable, without knowing which one, i.e., either $S1$ is reliable and $S2$ is not, or $S1$ is not reliable and $S2$ is, we must have $\omega_0 \in (A \cap \Omega) \cup (\Omega \cap B)$, hence $\omega_0 \in A \cup B$. Now, suppose that exactly one of the sources tells the truth, without knowing which one. Then, based on the fact that if a source states $\omega_0 \in C$, then the agent believes $\omega_0 \in C$ when the source tells the truth, and the agent believes $\omega_0 \in \overline{C}$ when the source does not tell the truth, we conclude that $\omega_0 \in (A \cap \overline{B}) \cup (\overline{A} \cap B)$ when exactly one of the sources tells the truth, which is in general different from $A \cup B$. One may verify that $\omega_0 \in (A \cap \overline{B}) \cup (\overline{A} \cap B)$ is the conclusion reached, when combining by the exclusive disjunctive rule, the categorical BBAs (the sets A and B) provided by the sources $S1$ and $S2$. Finally, we may note that if it is known that at least one of the sources tells the truth, then we must have $\omega_0 \in (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (A \cap B)$[1], which, as we know from Remark 6.2, is equivalent to $\omega_0 \in A \cup B$.

As a final comment, we may add that it would be interesting to study the behavior of a discounting-like operation, based on the truthfulness of a source rather than its reliability. Such an operation might be useful when dealing with an automatic deceiving agent, tampering with messages sent between a sensor and a coordination center [94].

[1]We have $\omega_0 \in (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (A \cap B)$ because either S1 is telling the truth and S2 is not and thus we believe $\omega_0 \in A \cap \overline{B}$, or S1 is not telling the truth and S2 is and thus we believe $\omega_0 \in \overline{A} \cap B$, or S1 and S2 are telling the truth and thus we believe $\omega_0 \in A \cap B$.

Weight-Based Combination Rules: Proofs

E.1 Proof of Theorem 3.1

The proof of Theorem 3.1 requires the two following technical lemmas (Lemmas E.1 and E.2).

Lemma E.1. *Let m be a BBA. For $B \subset \Omega$, the following equality holds:*

$$\sum_{A \subseteq B} (-1)^{|A|} q(A) = \sum_{A \cap B = \emptyset} m(A).$$

Proof. Let m_B denote a categorical BBA focused on $B \subset \Omega$. Let m be a BBA and $m' = m \textcircled{\scriptsize$\cap$} m_B$. We have

$$m'(\emptyset) = \sum_{A \cap B = \emptyset} m(A)$$

Let q_B denote the commonality function associated to m_B.

$$q_B(A) = \begin{cases} 1 & \text{if } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases}$$

Let q' and q denote the commonality functions associated to m' and m, respectively. We have:

$$q'(A) = q(A) \cdot q_B(A) \quad \forall A \subseteq \Omega$$

Hence

$$q'(A) = \begin{cases} q(A) & \text{if } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, using (1.2), we have

$$\begin{aligned} m'(\emptyset) &= \sum_{C \subseteq \Omega} (-1)^{|C|} q'(C) \\ &= \sum_{A \subseteq B} (-1)^{|A|} q(A), \end{aligned}$$

which completes the proof. □

Lemma E.2. *Let m be a normal, nondogmatic BBA and such that $m(C) > 0$, for a proper subset $C \subset \Omega$. Let w be the conjunctive weight function associated to m. Further, let $m' = m \circledcirc B^{w(B)} \circledcap B^{w(B)+\varepsilon}$, with $B \subset \Omega$, $C \cap B = \emptyset$ and $\varepsilon > 0$. m' is not a BBA.*

Proof. The proof consists in showing that $m'(\emptyset) < 0$. Let B be a strict subset of Ω such that $C \cap B = \emptyset$. The following equality holds:

$$m'(\emptyset) = \sum_{A \subseteq \Omega} (-1)^{|A|} \frac{q(A)}{q_B(A)} q'_B(A),$$

where q, q_B and q'_B are the commonality functions associated to m, $B^{w(B)}$ and $B^{w(B)+\varepsilon}$, respectively. We have:

$$q_B(A) = \begin{cases} 1 & \text{if } A \subseteq B, \\ w(B) & \text{otherwise,} \end{cases} \tag{E.1}$$

$$q'_B(A) = \begin{cases} 1 & \text{if } A \subseteq B, \\ w(B) + \varepsilon & \text{otherwise.} \end{cases} \tag{E.2}$$

Using (E.1) and (E.2), one can obtain:

$$m'(\emptyset) = \sum_{A \subseteq B} (-1)^{|A|} q(A) + \sum_{A \nsubseteq B} (-1)^{|A|} q(A) \frac{w(B) + \varepsilon}{w(B)}.$$

As

$$\begin{aligned} m(\emptyset) &= \sum_{A \subseteq \Omega} (-1)^{|A|} q(A) \\ &= \sum_{A \subseteq B} (-1)^{|A|} q(A) + \sum_{A \nsubseteq B} (-1)^{|A|} q(A), \end{aligned}$$

then

$$m'(\emptyset) = m(\emptyset) + \frac{\varepsilon}{w(B)} \sum_{A \nsubseteq B} (-1)^{|A|} q(A).$$

We can thus remark that $m'(\emptyset)$ is equal to $m(\emptyset)$, which is itself equal to 0, plus another term. Let us prove that this term is always strictly smaller than 0. We have

$$\begin{aligned} \frac{\varepsilon}{w(B)} \sum_{A \nsubseteq B} (-1)^{|A|} q(A) &= \frac{\varepsilon}{w(B)} \left(m(\emptyset) - \sum_{A \subseteq B} (-1)^{|A|} q(A) \right) \\ &= -\frac{\varepsilon}{w(B)} \sum_{A \subseteq B} (-1)^{|A|} q(A). \end{aligned}$$

We thus have from Lemma E.1:

$$m'(\emptyset) = -\frac{\varepsilon}{w(B)} \sum_{A \cap B = \emptyset} m(A). \tag{E.3}$$

As $m(C) > 0$ for C such that $C \cap B = \emptyset$, the sum in the right-hand side of (E.3) is strictly greater than zero. Further we have $\varepsilon > 0$ and $w(B) > 0$. Hence $m'(\emptyset) < 0$, thus m' is not a BBA.

□

Theorem 3.1 can then be proved as follows.

Proof. Let x and y be any two numbers such that $x \circ y > xy$. Obviously, as 1 is assumed to be a neutral element of $\circ$, we have $x \neq 1$ and $y \neq 1$. Let $\varepsilon = x \circ y - xy > 0$. The proof consists in choosing two logically consistent BBAs m_1 and m_2, i.e., $m_{1 \textcircled{\cap} 2}(\emptyset) = 0$, such that:

- $\exists B \in 2^\Omega \backslash \{\Omega\}$ such that $w_1(B) = x$ and $w_2(B) = y$;
- $\forall A \in 2^\Omega \backslash \{\Omega, B\}$, $w_1(A) = 1$ or $w_2(A) = 1$;
- $\exists C \in 2^\Omega$ such that $m_{1 \textcircled{\cap} 2}(C) > 0$ and $C \cap B = \emptyset$.

For those BBAs, we thus have:

$$\begin{aligned} w_{1 \textcircled{\cap} 2}(B) &= w_1(B) \cdot w_2(B), \\ w_{1 \textcircled{\cap} 2}(A) &= \begin{cases} w_1(A) & \text{if } w_2(A) = 1, \\ w_2(A) & \text{otherwise,} \end{cases} \end{aligned}$$

for all $A \neq B$, and

$$\begin{aligned} w_1(B) \circ w_2(B) &= w_{1 \textcircled{\cap} 2}(B) + \varepsilon, \\ w_1(A) \circ w_2(A) &= w_{1 \textcircled{\cap} 2}(A), \end{aligned}$$

for all $A \neq B$.

Hence, we have:

$$\textcircled{\cap}_{A \subset \Omega} A^{w_1(A) \circ w_2(A)} = m_{1 \textcircled{\cap} 2} \textcircled{\varnothing} B^{w_{1 \textcircled{\cap} 2}(B)} \textcircled{\cap} B^{w_{1 \textcircled{\cap} 2}(B) + \varepsilon}, \tag{E.4}$$

and $\exists C \in 2^\Omega$ such that $m_{1 \textcircled{\cap} 2}(C) > 0$ and $C \cap B = \emptyset$. By Lemma E.2, (E.4) is not a BBA, hence $w_1 \circ w_2$ is not a conjunctive weight function of some nondogmatic BBA.

Let us now provide the BBAs m_1 and m_2 which verify the above scheme. Since the considered numbers x and y take their values in $(0, +\infty) \backslash \{1\}$, we consider in the remainder of this proof the following cases:

- Case 1: $x \vee y < 1$;
- Case 2: $x \wedge y > 1$;
- Case 3: $x \vee y > 1$ and $x \wedge y < 1$.

We must thus provide a pair of BBAs m_1 and m_2 verifying the above scheme for each of the three possible cases.

- Case 1:

  Let $\Omega = \{a, b, c\}$ and let m_1 and m_2 be two BBAs defined on Ω as follows, for $\alpha, \beta \in (0, 0.5)$:

$$m_1(A) = \begin{cases} \alpha & \text{if } A = \{a, b\} \text{ or } A = \{b, c\}, \\ 1 - 2\alpha & \text{if } A = \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

$$m_2(A) = \begin{cases} \beta & \text{if } A = \{a,c\} \text{ or } A = \{b,c\}, \\ 1-2\beta & \text{if } A = \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

The conjunctive weight functions associated to those BBAs are:

$$w_1(A) = \begin{cases} \frac{(1-2\alpha)}{(1-\alpha)} & \text{if } A = \{a,b\} \text{ or } A = \{b,c\}, \\ \frac{(1-\alpha)^2}{(1-2\alpha)} & \text{if } A = \{b\}, \\ 1 & \text{otherwise.} \end{cases}$$

$$w_2(A) = \begin{cases} \frac{(1-2\beta)}{(1-\beta)} & \text{if } A = \{a,c\} \text{ or } A = \{b,c\}, \\ \frac{(1-\beta)^2}{(1-2\beta)} & \text{if } A = \{c\}, \\ 1 & \text{otherwise.} \end{cases}$$

For those two BBAs, we have:

- $m_{1 \bigcirc\!\!\!\!\cap\, 2}(\emptyset) = 0$,
- $\exists B = \{b,c\}$ such that $w_1(B) = x, x \in (0,1)$ as $w_1(B) = f(\alpha)$ with f a surjective function from $(0,0.5)$ to $(0,1)$, and $w_2(B) = y, y \in (0,1)$, as $w_2(B) = g(\beta)$ with g a surjective function from $(0,0.5)$ to $(0,1)$.
- $\forall A \in 2^\Omega \setminus \{B,\Omega\}$, $w_1(A) = 1$ or $w_2(A) = 1$,
- $\exists C = \{a\}$ such that $m_{1 \bigcirc\!\!\!\!\cap\, 2}(C) > 0$ and $C \cap B = \emptyset$.

* Case 2:

Let $\Omega = \{a,b,c,d,e\}$ and let m_1 and m_2 be two BBAs defined on Ω as follows, for $\alpha \in (0,0.5)$ and $\beta \in (0,1/3)$:

$$m_1(A) = \begin{cases} \alpha & \text{if } A = \{a,b\} \text{ or } A = \{b,c\}, \\ 1-2\alpha & \text{if } A = \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

$$m_2(A) = \begin{cases} \beta & \text{if } A \in \{\{a,b,c\},\{a,c,e\},\{b,d,e\}\}, \\ 1-3\beta & \text{if } A = \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

The conjunctive weight functions associated to those BBAs are:

$$w_1(A) = \begin{cases} \frac{(1-2\alpha)}{(1-\alpha)} & \text{if } A = \{a,b\} \text{ or } A = \{b,c\}, \\ \frac{(1-\alpha)^2}{(1-2\alpha)} & \text{if } A = \{b\}, \\ 1 & \text{otherwise.} \end{cases}$$

$$w_2(A) = \begin{cases} \frac{1-3\beta}{1-2\beta} & \text{if } A \in \{\{a,b,c\},\{a,c,e\},\{b,d,e\}\}, \\ \frac{(1-\beta)^3(1-3\beta)}{(1-2\beta)^3} & \text{if } A = \{\emptyset\}, \\ \frac{(1-2\beta)^2}{(1-\beta)(1-3\beta)} & \text{if } A \in \{\{b\},\{e\},\{a,c\}\}, \\ 1 & \text{otherwise.} \end{cases}$$

For those two BBAs, we have:

- $m_{1\textcircled{\cap}2}(\emptyset) = 0$,
- $\exists B = \{b\}$ such that $w_1(B) = x, x \in (1, +\infty)$ as $w_1(B) = f(\alpha)$ with f a surjective function from $(0, 0.5)$ to $(1, +\infty)$, and $w_2(B) = y, y \in (1, +\infty)$, as $w_2(B) = g(\beta)$ with g a surjective function from $(0, 1/3)$ to $(1, +\infty)$.
- $\forall A \in 2^\Omega \setminus \{B, \Omega\}$, $w_1(A) = 1$ or $w_2(A) = 1$,
- $\exists C = \{a\}$ such that $m_{1\textcircled{\cap}2}(C) > 0$ and $C \cap B = \emptyset$.

- Case 3:

  Let $\Omega = \{a, b, c, d\}$ and let m_1 and m_2 be two BBAs defined on Ω as follows, for $\alpha \in (0, 0.5)$ and $\beta \in (0, 1/3)$:

$$m_1(A) = \begin{cases} \alpha & \text{if } A = \{a,b\} \text{ or } A = \{b,c\}, \\ 1-2\alpha & \text{if } A = \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

$$m_2(A) = \begin{cases} \beta & \text{if } A \in \{\{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}, \\ 1-3\beta & \text{if } A = \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

  The conjunctive weight functions associated to those BBAs are:

$$w_1(A) = \begin{cases} \frac{(1-2\alpha)}{(1-\alpha)} & \text{if } A = \{a,b\} \text{ or } A = \{b,c\}, \\ \frac{(1-\alpha)^2}{(1-2\alpha)} & \text{if } A = \{b\}, \\ 1 & \text{otherwise.} \end{cases}$$

$$w_2(A) = \begin{cases} \frac{1-3\beta}{1-2\beta} & \text{if } A \in \{\{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}, \\ \frac{(1-\beta)^3(1-3\beta)}{(1-2\beta)^3} & \text{if } A = \{a\}, \\ \frac{(1-2\beta)^2}{(1-\beta)(1-3\beta)} & \text{if } A \in \{\{a,b\}, \{a,c\}, \{a,d\}\}, \\ 1 & \text{otherwise.} \end{cases}$$

  For those two BBAs, we have:

  - $m_{1\textcircled{\cap}2}(\emptyset) = 0$,
  - $\exists B = \{a, b\}$ such that $w_1(B) = x, x \in (0, 1)$ as $w_1(B) = f(\alpha)$ with f a surjective function from $(0, 0.5)$ to $(0, 1)$, and $w_2(B) = y, y \in (1, +\infty)$, as $w_2(B) = g(\beta)$ with g a surjective function from $(0, 1/3)$ to $(1, +\infty)$.
  - $\forall A \in 2^\Omega \setminus \{B, \Omega\}$, $w_1(A) = 1$ or $w_2(A) = 1$,
  - $\exists C = \{c\}$ such that $m_{1\textcircled{\cap}2}(C) > 0$ and $C \cap B = \emptyset$.

$\square$

E.2 Proof of Proposition 4.2

Proof. The notation is simplified in this proof: the operator $\mathcal{U}_{(\top,\top')}$ defined by (4.3) is simply noted $\mathcal{U}$. The convention related to the use of the extended real line and adopted in the proof of Proposition 4.1 is also used in this proof.

The proof consists in showing that $\mathcal{U}$ satisfies the following properties

1. $x \,\mathcal{U}\, y = y \,\mathcal{U}\, x$, for all $x, y \in (0, +\infty]$, i.e., $\mathcal{U}$ is commutative;
2. $x \,\mathcal{U}\, 1 = 1 \,\mathcal{U}\, x = x$, for all $x \in (0, +\infty]$, i.e., $\mathcal{U}$ has 1 as neutral element;
3. $w \,\mathcal{U}\, x \geq y \,\mathcal{U}\, z$, for all $w, x, y, z \in (0, +\infty]$, such that $w \geq y$ and $x \geq z$, i.e., $\mathcal{U}$ is monotonic;
4. $(x \,\mathcal{U}\, y) \,\mathcal{U}\, z = x \,\mathcal{U}\, (y \,\mathcal{U}\, z)$, for all $x, y, z \in (0, +\infty]$, i.e., $\mathcal{U}$ is associative;
5. $x \,\mathcal{U}\, y > 0$ for all $x, y \in (0, +\infty]$.

The commutativity of $\mathcal{U}$ (Property 1) results from the commutativity of $\top$, $\top'$ and $\wedge$. In order to show Property 2, we merely need to show $x \,\mathcal{U}\, 1 = x$ since $\mathcal{U}$ is commutative. Suppose $x \leq 1$, then $x \,\mathcal{U}\, 1 = x \top 1 = 1$. Suppose $x \geq 1$, then $x \,\mathcal{U}\, 1 = ((1/x) \top' 1)^{-1} = 1$.

Let us now prove the monotonicity of $\mathcal{U}$ (Property 3).

- Suppose $w \,\mathcal{U}\, x = w \top x$. We then necessarily have $y \,\mathcal{U}\, z = y \top z$ since $w \geq y$ and $x \geq z$. By the monotonicity of $\top$, we have $w \top x \geq y \top z$.
- Suppose $w \,\mathcal{U}\, x = w \wedge x$. We then necessarily have $y \,\mathcal{U}\, z = y \top z$ or $y \,\mathcal{U}\, z = y \wedge z$ since when $w \,\mathcal{U}\, x = w \wedge x$, either w or x is strictly smaller than 1 and thus either y or z is necessarily also strictly smaller than 1.
  - Suppose $y \,\mathcal{U}\, z = y \wedge z$. By the monotonicity of $\wedge$, $w \wedge x \geq y \wedge z$.
  - Suppose $y \,\mathcal{U}\, z = y \top z$. Since $\wedge$ is the largest t-norm on $[0, 1]$, we have $y \top z \leq y \wedge z$. Without loss of generality, assume $w \wedge x = x$.
    - Suppose $y \leq z$. Then, $y \wedge z = y$. We have $x \geq z$, hence $x \geq y \wedge z$, and thus $w \wedge x \geq y \wedge z \geq y \top z$.
    - Suppose $y > z$. Then, $y \wedge z = z \geq y \top z$. Since $x \geq z$, we have $x \geq y \wedge z$, and thus $w \wedge x \geq y \wedge z \geq y \top z$.
- Suppose $w \,\mathcal{U}\, x = ((1/w) \top' (1/x))^{-1}$. In this case, we have $w \wedge x \geq 1$, and thus clearly $w \,\mathcal{U}\, x \geq 1$.
  - Suppose $y \,\mathcal{U}\, z = ((1/y) \top' (1/z))^{-1}$. By the monotonicity of $\top'$, we have

$$
\begin{aligned}
(1/w) \top' (1/x) &\leq (1/y) \top' (1/z) \\
((1/w) \top' (1/x))^{-1} &\geq ((1/y) \top' (1/z))^{-1} .
\end{aligned}
$$

  - Suppose $y \,\mathcal{U}\, z = y \wedge z$. Hence, either y or z is strictly smaller than 1. Thus, $y \wedge z < 1$. Hence, $w \,\mathcal{U}\, x \geq y \wedge z$, since $w \,\mathcal{U}\, x \geq 1$.

- Suppose $y \,\mathcal{U}\, z = y \top z$. Hence, $y \vee z \leq 1$ and thus $y \top z \leq 1$. We then have $w \,\mathcal{U}\, x \geq y \top z$, since $w \,\mathcal{U}\, x \geq 1$.

We prove the associativity of $\mathcal{U}$ (Property 4) as follows. Without loss of generality, suppose $x \geq y \geq z$, and $x, y, z \in (0, +\infty]$.

- If $x < 1$, then

$$\begin{aligned} x \,\mathcal{U}\, (y \,\mathcal{U}\, z) &= x \,\mathcal{U}\, (y \top z) \\ &= x \top (y \top z) \\ &= (x \top y) \top z \\ &= (x \,\mathcal{U}\, y) \,\mathcal{U}\, z; \end{aligned}$$

- If $z > 1$, then

$$\begin{aligned} x \,\mathcal{U}\, (y \,\mathcal{U}\, z) &= x \,\mathcal{U}\, \left((1/y) \top' (1/z)\right)^{-1} \\ &= \left(\frac{1}{x} \top' \frac{1}{\left((1/y) \top' (1/z)\right)^{-1}}\right)^{-1} \\ &= \left((1/x) \top' (1/y) \top' (1/z)\right)^{-1} \\ &= \left(\frac{1}{\left((1/x) \top' (1/y)\right)^{-1}} \top' \frac{1}{z}\right)^{-1} \\ &= \left((1/x) \top' (1/y)\right)^{-1} \,\mathcal{U}\, z \\ &= (x \,\mathcal{U}\, y) \,\mathcal{U}\, z; \end{aligned}$$

- If $x > 1 > y \geq z$, then

$$x \,\mathcal{U}\, (y \,\mathcal{U}\, z) = x \,\mathcal{U}\, (y \top z) = x \wedge (y \top z) = y \top z,$$

  and

$$(x \,\mathcal{U}\, y) \,\mathcal{U}\, z = (x \wedge y) \,\mathcal{U}\, z = y \,\mathcal{U}\, z = y \top z;$$

- if $x \geq y > 1 > z$, then

$$x \,\mathcal{U}\, (y \,\mathcal{U}\, z) = x \,\mathcal{U}\, (y \wedge z) = x \,\mathcal{U}\, z = x \wedge z = z,$$

  and

$$(x \,\mathcal{U}\, y) \,\mathcal{U}\, z = \left((1/x) \top' (1/y)\right)^{-1} \,\mathcal{U}\, z = \left((1/x) \top' (1/y)\right)^{-1} \wedge z = z;$$

- When either x, y or z equals 1, the proof of the associativity is easy since 1 is a neutral element of $\mathcal{U}$. For instance, let $x = 1$. Then, we have

$$x \,\mathcal{U}\, (y \,\mathcal{U}\, z) = 1 \,\mathcal{U}\, (y \,\mathcal{U}\, z) = y \,\mathcal{U}\, z,$$

  and

$$(x \,\mathcal{U}\, y) \,\mathcal{U}\, z = (1 \,\mathcal{U}\, y) \,\mathcal{U}\, z = y \,\mathcal{U}\, z.$$

Eventually, Property 5 can be shown in a similar manner as it was shown that the operator $\mathcal{T}_{(\top,\top')}$ defined by (4.1) is such that $x \,\mathcal{T}_{(\top,\top')}\, y > 0$ for all $x, y \in (0, +\infty]$. □

E.3 Proof of Proposition 5.1

Proof. To make this proof easier to read, $\textcircled{\scriptsize u}_w$ will be simply noted $\textcircled{\scriptsize u}$ and a BBA m^{Ω_s} defined on a frame Ω_s will be simply noted m^s. Accordingly, $m^{\downarrow\Omega_s}$ and $m^{\uparrow\Omega_s}$ will be noted, respectively, $m^{\downarrow s}$ and $m^{\uparrow s}$.

In order to show Proposition 5.1, we merely need to show that if the binary operator $\mathcal{U}$ underlying the conjunctive u-rule $\textcircled{\scriptsize u}$ is such that $\exists x, y \in (0, +\infty)$, $x \ \mathcal{U} \ y \neq xy$, then there exist two nondogmatic BBAs m_1 defined on a frame of discernment Ω_s and m_2 defined on a frame of discernment Ω_t such that

$$(m_1 \textcircled{\scriptsize u} m_2)^{\downarrow z} \neq m_1^{\downarrow z\cap s} \textcircled{\scriptsize u} m_2,$$

for some $z \in D$ such that $t \subseteq z \subseteq t \cup s$.

Let x and y be two arbitrary numbers in $(0, +\infty)$ such that $x \ \mathcal{U} \ y \neq xy$. Let s and t be two sets of variables such that $t \subseteq s$ and let $z = t$. Further, let m_1 be a BBA defined on the frame of discernment Ω_s and m_2 be a BBA defined on the frame of discernment Ω_t. From the fact that the $\textcircled{\scriptsize \ensuremath{\cap}}$ rule satisfies axiom 5, we have:

$$(m_1 \textcircled{\scriptsize \ensuremath{\cap}} m_2)^{\downarrow z} = m_1^{\downarrow z\cap s} \textcircled{\scriptsize \ensuremath{\cap}} m_2,$$

and thus

$$(m_1 \textcircled{\scriptsize \ensuremath{\cap}} m_2)^{\downarrow t} = m_1^{\downarrow t} \textcircled{\scriptsize \ensuremath{\cap}} m_2. \quad \text{(E.5)}$$

The proof consists in choosing the BBAs m_1 and m_2 such that we have $(m_1 \textcircled{\scriptsize u} m_2)^{\downarrow t} = (m_1 \textcircled{\scriptsize \ensuremath{\cap}} m_2)^{\downarrow t}$ and $m_1^{\downarrow t} \textcircled{\scriptsize u} m_2 \neq m_1^{\downarrow t} \textcircled{\scriptsize \ensuremath{\cap}} m_2$. Hence, we have

$$\begin{aligned} (m_1 \textcircled{\scriptsize u} m_2)^{\downarrow t} &= (m_1 \textcircled{\scriptsize \ensuremath{\cap}} m_2)^{\downarrow t} \\ &= m_1^{\downarrow t} \textcircled{\scriptsize \ensuremath{\cap}} m_2 \qquad \text{(from (E.5))} \\ &\neq m_1^{\downarrow t} \textcircled{\scriptsize u} m_2, \end{aligned}$$

and thus

$$(m_1 \textcircled{\scriptsize u} m_2)^{\downarrow z} \neq m_1^{\downarrow z\cap s} \textcircled{\scriptsize u} m_2.$$

We get $m_1^{\downarrow t} \textcircled{\scriptsize u} m_2 \neq m_1^{\downarrow t} \textcircled{\scriptsize \ensuremath{\cap}} m_2$ by choosing the BBAs m_1^s and m_2^t such that:

- $\exists B \in 2^{\Omega_t} \setminus \{\Omega_t\}$ such that $w_1^{s\downarrow t}(B) = x$ and $w_2^t(B) = y$, with $w_1^{s\downarrow t}$ the weight function associated to $m_1^{s\downarrow t}$,
- $\forall A \in 2^{\Omega_t} \setminus \{\Omega_t, B\}$, $w_1^{s\downarrow t}(A) = 1$ or $w_2^t(A) = 1$.

The weight functions $w^t_{1\textcircled{\tiny u}2}$ and $w^t_{1\textcircled{\tiny \ensuremath{\cap}}2}$ associated respectively to $m_1^{\downarrow t} \textcircled{\scriptsize u} m_2$ and $m_1^{\downarrow t} \textcircled{\scriptsize \ensuremath{\cap}} m_2$ are thus as follows:

$$\begin{aligned} w^t_{1\textcircled{\tiny u}2}(B) &\neq w^t_{1\textcircled{\tiny \ensuremath{\cap}}2}(B), \\ w^t_{1\textcircled{\tiny u}2}(A) &= w^t_{1\textcircled{\tiny \ensuremath{\cap}}2}(A), \text{ for } A \neq B. \end{aligned}$$

Consequently

$$m_1^{\downarrow t} \textcircled{\scriptsize u} m_2 \neq m_1^{\downarrow t} \textcircled{\scriptsize \ensuremath{\cap}} m_2.$$

We get $(m_1 \textcircled{\scriptsize u} m_2)^{\downarrow t} = (m_1 \textcircled{\scriptsize \ensuremath{\cap}} m_2)^{\downarrow t}$ by choosing the BBAs m_1^s and m_2^t such that:

- $\forall A \in 2^{\Omega_s} \backslash \{\Omega_s\}, w_1^s(A) = 1$ or $w_2^{t\uparrow s}(A) = 1$, with $w_2^{t\uparrow s}$ the weight function associated to $m_2^{t\uparrow s}$.

The weight functions $w_{1 \circledast 2}^s$ and $w_{1 \bigcirc\!\!\!\!\cap\, 2}^s$ associated respectively to $m_1 \circledast m_2$ and $m_1 \bigcirc\!\!\!\!\cap\, m_2$ are thus as follows:

$$w_{1 \circledast 2}^s(A) = w_{1 \bigcirc\!\!\!\!\cap\, 2}^s(A), \forall A \in 2^{\Omega_s} \backslash \Omega_s.$$

Consequently we have $m_1 \circledast m_2 = m_1 \bigcirc\!\!\!\!\cap\, m_2$ and thus $(m_1 \circledast m_2)^{\downarrow t} = (m_1 \bigcirc\!\!\!\!\cap\, m_2)^{\downarrow t}$.

Let us now provide the BBAs m_1 and m_2, which verify the above scheme.

The operator $\mathcal{U}$ is such that $\exists x, y, x \mathcal{U} y \neq xy$, this implies that $x, y \in (0, +\infty) \backslash \{1\}$ as 1 is the neutral element of $\mathcal{U}$. In the remainder of this proof, we consider thus the cases where:

- Case 1: $x \vee y < 1$,
- Case 2: $x \wedge y > 1$,
- Case 3: $x \vee y > 1$ and $x \wedge y < 1$.

We must thus provide a pair of BBAs m_1 and m_2 verifying the above scheme for each of those three cases. Let us first provide two frames of discernment Ω_s and Ω_t on which we are going to define our three pairs of BBA. Let X and Z be two binary variables whose frames are $\Omega_X = \{x_1, x_2\}$ and $\Omega_Z = \{z_1, z_2\}$, and let Y be a ternary variable whose frame is $\Omega_Y = \{y_1, y_2, y_3\}$. Let t denote the set composed of the variables Y and Z and let s denote the set composed of the variables X, Y and Z. Tables E.1 and E.2 give explicit names to the configurations of the frames Ω_t and Ω_s.

Table E.1: The frame Ω_s

configurations	
s_1	(x_1, y_1, z_1)
s_2	(x_1, y_1, z_2)
s_3	(x_1, y_2, z_1)
s_4	(x_1, y_2, z_2)
s_5	(x_1, y_3, z_1)
s_6	(x_1, y_3, z_2)
s_7	(x_2, y_1, z_1)
s_8	(x_2, y_1, z_2)
s_9	(x_2, y_2, z_1)
s_{10}	(x_2, y_2, z_2)
s_{11}	(x_2, y_3, z_1)
s_{12}	(x_2, y_3, z_2)

Let us now provide the pairs of BBAs m_1 and m_2 satisfying the scheme described at the beginning of the proof, for each of the three possible cases. Case 1 is rather simple; the other cases are more tedious but nonetheless similar.

Table E.2: The frame Ω_t

configurations	
t_1	(y_1, z_1)
t_2	(y_1, z_2)
t_3	(y_2, z_1)
t_4	(y_2, z_2)
t_5	(y_3, z_1)
t_6	(y_3, z_2)

- Case 1:

  Let m_1 be a BBA defined on Ω_s as follows, for $x \in (0,1)$:

$$\begin{aligned} m_1^s(\{s_9, s_{10}\}) &= 1-x, \\ m_1^s(\Omega_s) &= x. \end{aligned}$$

Marginalizing m_1^s on Ω_t yields:

$$\begin{aligned} m_1^{s\downarrow t}(\{t_3, t_4\}) &= 1-x, \\ m_1^{s\downarrow t}(\Omega_t) &= x. \end{aligned}$$

The weight functions associated respectively to m_1^s and $m_1^{s\downarrow t}$ are the following:

$$\begin{aligned} w_1^s(\{s_9, s_{10}\}) &= x, \\ w_1^s(A) &= 1, \end{aligned}$$

for all $A \in 2^{\Omega_s} \setminus \{\Omega_s, \{s_9, s_{10}\}\}$, and

$$\begin{aligned} w_1^{s\downarrow t}(\{t_3, t_4\}) &= x, \\ w_1^{s\downarrow t}(A) &= 1, \end{aligned}$$

for all $A \in 2^{\Omega_t} \setminus \{\Omega_t, \{t_3, t_4\}\}$.

Now, let m_2 be a BBA defined on Ω_t as follows, for $y \in (0,1)$:

$$\begin{aligned} m_2^t(\{t_3, t_4\}) &= 1-y, \\ m_2^t(\Omega_t) &= y. \end{aligned}$$

Vacuously extending m_2^t on Ω_s yields:

$$\begin{aligned} m_2^{t\uparrow s}(\{s_3, s_4, s_9, s_{10}\}) &= 1-y, \\ m_2^{t\uparrow s}(\Omega_s) &= y. \end{aligned}$$

The weight functions associated respectively to m_2^t and $m_2^{t\uparrow s}$ are the following:

$$\begin{aligned} w_2^t(\{t_3, t_4\}) &= y, \\ w_2^t(A) &= 1, \end{aligned}$$

for all $A \in 2^{\Omega_t} \backslash \{\Omega_t, \{t_3, t_4\}\}$, and

$$\begin{aligned} w_2^{t\uparrow s}(\{s_3, s_4, s_9, s_{10}\}) &= y, \\ w_2^{t\uparrow s}(A) = 1, \end{aligned}$$

for all $A \in 2^{\Omega_s} \backslash \{\Omega_s, \{s_3, s_4, s_9, s_{10}\}\}$.

For those two BBAs m_1 and m_2, we thus have:

- $\exists B = \{t_3, t_4\}$ such that $w_1^{s\downarrow t}(B) = x$, and $w_2^t(B) = y$, with $x, y \in (0,1)$;
- $\forall A \in 2^{\Omega_t} \backslash \{\Omega_t, B\}$, $w_1^{s\downarrow t}(A) = 1$ or $w_2^t(A) = 1$;
- $\forall A \in 2^{\Omega_s} \backslash \{\Omega_s\}, w_1^s(A) = 1$ or $w_2^{t\uparrow s}(A) = 1$.

- Case 2: Let m_1 be a BBA defined on Ω_s as follows:

$$\begin{aligned} m_1^s(\{s_1, s_2, s_3\}) &= m_1^s(\{s_1, s_3, s_5\}) \\ &= m_1^s(\{s_2, s_4, s_5\}) \\ &= \alpha, \\ m_1^s(\Omega_s) &= 1 - 3\alpha, \end{aligned}$$

for $\alpha \in (0, 1/3)$.

Marginalizing m_1^s on Ω_t yields:

$$\begin{aligned} m_1^{s\downarrow t}(\{t_1, t_2, t_3\}) &= m_1^{s\downarrow t}(\{t_1, t_3, t_5\}) \\ &= m_1^{s\downarrow t}(\{t_2, t_4, t_5\}) \\ &= \alpha, \\ m_1^{s\downarrow t}(\Omega_t) &= 1 - 3\alpha. \end{aligned}$$

The weight functions associated respectively to m_1^s and $m_1^{s\downarrow t}$ are the following:

$$\begin{aligned} w_1^s(\{s_1, s_2, s_3\}) &= w_1^s(\{s_1, s_3, s_5\}) \\ &= w_1^s(\{s_2, s_4, s_5\}) \\ &= \frac{1-3\alpha}{1-2\alpha}, \\ w_1^s(\emptyset) &= \frac{(1-\alpha)^3(1-3\alpha)}{(1-2\alpha)^3}, \\ w_1^s(\{s_2\}) &= w_1^s(\{s_5\}) \\ &= w_1^s(\{s_1, s_3\}) \\ &= \frac{(1-2\alpha)^2}{(1-\alpha)(1-3\alpha)}, \\ w_1^s(A) &= 1, \end{aligned}$$

for all $A \in 2^{\Omega_s} \backslash \{\Omega_s, \{s_1, s_2, s_3\}, \{s_1, s_3, s_5\}, \{s_2, s_4, s_5\}, \emptyset, \{s_2\}, \{s_5\}, \{s_1, s_3\}\}$,

and

$$\begin{aligned}
w_1^{s\downarrow t}(\{t_1,t_2,t_3\}) &= w_1^{s\downarrow t}(\{t_1,t_3,t_5\}) \\
&= w_1^{s\downarrow t}(\{t_2,t_4,t_5\}) \\
&= \frac{1-3\alpha}{1-2\alpha}, \\
w_1^{s\downarrow t}(\emptyset) &= \frac{(1-\alpha)^3\,(1-3\alpha)}{(1-2\alpha)^3}, \\
w_1^{s\downarrow t}(\{t_2\}) &= w_1^{s\downarrow t}(\{t_5\}) \\
&= w_1^{s\downarrow t}(\{t_1,t_3\}) \\
&= \frac{(1-2\alpha)^2}{(1-\alpha)(1-3\alpha)}, \\
w_1^{s\downarrow t}(A) &= 1,
\end{aligned}$$

for all $A \in 2^{\Omega_t} \setminus \{\Omega_t, \{t_1,t_2,t_3\}, \{t_1,t_3,t_5\}, \{t_2,t_4,t_5\}, \emptyset, \{t_2\}, \{t_5\}, \{t_1,t_3\}\}$.

Let m_2 be a BBA defined on Ω_t as follows:

$$\begin{aligned}
m_2^t(\{t_1,t_2\}) &= m_2^t(\{t_2,t_3\}) = \beta, \\
m_2^t(\Omega_t) &= 1-2\beta,
\end{aligned}$$

for $\beta \in (0, 0.5)$.

Vacuously extending m_2^t on Ω_s yields:

$$\begin{aligned}
m_2^{t\uparrow s}(\{s_1,s_2,s_7,s_8\}) &= m_2^{t\uparrow s}(\{s_2,s_3,s_8,s_9\}) \\
&= \beta, \\
m_2^{t\uparrow s}(\Omega_s) &= 1-2\beta.
\end{aligned}$$

The weight functions associated respectively to m_2^t and $m_2^{t\uparrow s}$ are the following:

$$\begin{aligned}
w_2^t(\{t_1,t_2\}) &= w_2^t(\{t_2,t_3\}) = \frac{1-2\beta}{1-\beta}, \\
w_2^t(\{t_2\}) &= \frac{(1-\beta)^2}{1-2\beta}, \\
w_2^t(A) &= 1,
\end{aligned}$$

for all $A \in 2^{\Omega_t} \setminus \{\Omega_t, \{t_1,t_2\}, \{t_2,t_3\}, \{t_2\}\}$, and

$$\begin{aligned}
w_2^{t\uparrow s}(\{s_1,s_2,s_7,s_8\}) &= w_2^{t\uparrow s}(\{s_2,s_3,s_8,s_9\}) \\
&= \frac{1-2\beta}{1-\beta}, \\
w_2^{t\uparrow s}(\{s_2,s_8\}) &= \frac{(1-\beta)^2}{1-2\beta}, \\
w_2^{t\uparrow s}(A) &= 1,
\end{aligned}$$

for all $A \in 2^{\Omega_s} \setminus \{\Omega_s, \{s_1,s_2,s_7,s_8\}, \{s_2,s_3,s_8,s_9\}, \{s_2,s_8\}\}$.

For those two BBAs m_1 and m_2, we have

– $\exists B = \{t_2\}$ such that $w_1^{s\downarrow t}(B) = x, x \in (1, +\infty)$ as $w_1^{s\downarrow t}(B) = f(\alpha)$ with f a surjective function from $(0, 1/3)$ to $(1, +\infty)$, and $w_2^t(B) = y, y \in (1, +\infty)$ as $w_2^t(B) = g(\beta)$ with g a surjective function from $(0, 0.5)$ to $(1, +\infty)$;

– $\forall A \in 2^{\Omega_t} \backslash \{\Omega_t, B\}$, $w_1^{s\downarrow t}(A) = 1$ or $w_2^t(A) = 1$;

– $\forall A \in 2^{\Omega_s} \backslash \{\Omega_s\}, w_1^s(A) = 1$ or $w_2^{t\uparrow s}(A) = 1$.

- Case 3:

Let m_1 be a BBA defined on Ω_s as follows:

$$\begin{aligned} m_1^s(\{s_1, s_2, s_3\}) &= m_1^s(\{s_1, s_2, s_4\}) \\ &= m_1^s(\{s_1, s_3, s_4\}) \\ &= \alpha, \\ m_1^s(\Omega_s) &= 1 - 3\alpha \end{aligned}$$

for $\alpha \in (0, 1/3)$.

Marginalizing m_1^s on Ω_t yields:

$$\begin{aligned} m_1^{s\downarrow t}(\{t_1, t_2, t_3\}) &= m_1^{s\downarrow t}(\{t_1, t_2, t_4\}) \\ &= m_1^{s\downarrow t}(\{t_1, t_3, t_4\}) \\ &= \alpha, \\ m_1^{s\downarrow t}(\Omega_t) &= 1 - 3\alpha. \end{aligned}$$

The weight functions associated respectively to m_1^s and $m_1^{s\downarrow t}$ are the following:

$$\begin{aligned} w_1^s(\{s_1, s_2, s_3\}) &= w_1^s(\{s_1, s_2, s_4\}) \\ &= w_1^s(\{s_1, s_3, s_4\}) \\ &= \frac{1 - 3\alpha}{1 - 2\alpha}, \\ w_1^s(\{s_1, s_2\}) &= w_1^s(\{s_1, s_3\}) \\ &= w_1^s(\{s_1, s_4\}) \\ &= \frac{(1 - 2\alpha)^2}{(1 - \alpha)(1 - 3\alpha)}, \\ w_1^s(\{s_1\}) &= \frac{(1 - \alpha)^3 (1 - 3\alpha)}{(1 - 2\alpha)^3}, \\ w_1^s(A) &= 1, \end{aligned}$$

for all $A \in 2^{\Omega_s} \backslash \{\Omega_s, \{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}, \{s_1, s_3, s_4\}, \{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_1\}\}$,

and

$$\begin{aligned}
w_1^{s\downarrow t}(\{t_1,t_2,t_3\}) &= w_1^{s\downarrow t}(\{t_1,t_2,t_4\}) \\
&= w_1^{s\downarrow t}(\{t_1,t_3,t_4\}) \\
&= \frac{1-3\alpha}{1-2\alpha}, \\
w_1^{s\downarrow t}(\{t_1,t_2\}) &= w_1^{s\downarrow t}(\{t_1,t_3\}) \\
&= w_1^{s\downarrow t}(\{t_1,t_4\}) \\
&= \frac{(1-2\alpha)^2}{(1-\alpha)(1-3\alpha)}, \\
w_1^{s\downarrow t}(\{t_1\}) &= \frac{(1-\alpha)^3(1-3\alpha)}{(1-2\alpha)^3}, \\
w_1^{s\downarrow t}(A) &= 1,
\end{aligned}$$

for all $A \in 2^{\Omega_t} \setminus \{\Omega_t, \{t_1,t_2,t_3\}, \{t_1,t_2,t_4\}, \{t_1,t_3,t_4\}, \{t_1,t_2\}, \{t_1,t_3\}, \{t_1,t_4\}, \{t_1\}\}$.

Let m_2 be a BBA defined on Ω_t as follows:

$$\begin{aligned}
m_2^t(\{t_1,t_2\}) &= 1-y, \\
m_2^t(\Omega_t) &= y
\end{aligned}$$

for $y \in (0,1)$.

Vacuously extending m_2^t on Ω_s yields:

$$\begin{aligned}
m_2^{t\uparrow s}(\{s_1,s_2,s_7,s_8\}) &= 1-\beta, \\
m_2^{t\uparrow s}(\Omega_s) &= \beta.
\end{aligned}$$

The weight functions associated respectively to m_2^t and $m_2^{t\uparrow s}$ are the following:

$$\begin{aligned}
w_2^t(\{t_1,t_2\}) &= \beta, \\
w_2^t(A) &= 1,
\end{aligned}$$

for all $A \in 2^{\Omega_t} \setminus \{\Omega_t, \{t_1,t_2\}\}$, and

$$\begin{aligned}
w_2^{t\uparrow s}(\{s_1,s_2,s_7,s_8\}) &= \beta, \\
w_2^{t\uparrow s}(A) &= 1,
\end{aligned}$$

for all $A \in 2^{\Omega_s} \setminus \{\Omega_s, \{s_1,s_2,s_7,s_8\}\}$.

For those two BBAs m_1 and m_2, we have

- $\exists B = \{t_1,t_2\}$ such that $w_1^{s\downarrow t}(B) = x, x \in (1,+\infty)$ as $w_1^{s\downarrow t}(B) = f(\alpha)$ with f a surjective function from $(0,1/3)$ to $(1,+\infty)$, and $w_2^t(B) = y, y \in (0,1)$;
- $\forall A \in 2^{\Omega_t} \setminus \{\Omega_t, B\}$, $w_1^{s\downarrow t}(A) = 1$ or $w_2^t(A) = 1$;
- $\forall A \in 2^{\Omega_s} \setminus \{\Omega_s\}, w_1^s(A) = 1$ or $w_2^{t\uparrow s}(A) = 1$.

□

α-Junctions: Proofs

F.1 Proof of Proposition 6.2

In order to show this proposition, five technical lemmas are needed. First, two lemmas related to the BBA $m_{\alpha,\cap}$ appearing in Proposition 6.2, are given. Then, three lemmas related to the α-conditioning operation, are provided.

Let us first show that the BBA $m_{\alpha,\cap}$ of Proposition 6.2 is indeed a BBA.

Lemma F.1. *Let m be a set function defined on Ω such that $m(A) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$, with $\alpha \in [0,1]$ and $\overline{\alpha} = 1-\alpha$. We have*

$$m = \circledcap_{x\in\Omega} \{\Omega \backslash x\}^{\overline{\alpha}},$$

where $\{\Omega\backslash x\}^{\overline{\alpha}}$ denotes a simple BBA m_x such that $m_x(\{\Omega\backslash x\}) = 1-\overline{\alpha}$ and $m_x(\Omega) = \overline{\alpha}$.

Proof. Let m be a set function defined on Ω such that $m(A) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$, with $\alpha \in [0,1]$ and $\overline{\alpha} = 1-\alpha$. Let f be a set function defined on Ω by

$$f(A) = \sum_{B\supseteq A} m(B).$$

Let $A \subseteq \Omega$. The set A has $\binom{|\Omega|-|A|}{k}$ supersets of cardinality $|A|+k$, with $0 \le k \le |\Omega|-|A|$. Hence, for all $A \subseteq \Omega$, we have

$$\begin{aligned}
f(A) &= \sum_{B\supseteq A} m(B)\\
&= \binom{|\Omega|-|A|}{0}\alpha^{|\Omega|-(|A|+0)}\overline{\alpha}^{|A|+0} + \binom{|\Omega|-|A|}{1}\alpha^{|\Omega|-(|A|+1)}\overline{\alpha}^{|A|+1}\\
&\quad +\ldots+ \binom{|\Omega|-|A|}{|\Omega|-|A|}\alpha^{|\Omega|-(|A|+|\Omega|-|A|)}\overline{\alpha}^{|A|+|\Omega|-|A|}\\
&= \sum_{k=0}^{|\Omega|-|A|}\binom{|\Omega|-|A|}{k}\alpha^{|\Omega|-|A|-k}\overline{\alpha}^{|A|+k}\\
&= \overline{\alpha}^{|A|}\sum_{k=0}^{|\Omega|-|A|}\binom{|\Omega|-|A|}{k}\alpha^{|\Omega|-|A|-k}\overline{\alpha}^{k}.
\end{aligned}$$

The binomial theorem shows the following equality:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Hence,

$$\begin{aligned}
\overline{\alpha}^{|A|} \sum_{k=0}^{|\Omega|-|A|} \binom{|\Omega|-|A|}{k} \alpha^{|\Omega|-|A|-k} \overline{\alpha}^{k} &= \overline{\alpha}^{|A|} (\alpha + \overline{\alpha})^{|\Omega|-|A|} \\
&= \overline{\alpha}^{|A|},
\end{aligned}$$

and thus $f(A) = \overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$.

The commonality function q_x associated to a simple BBA $\{\Omega \backslash x\}^{\overline{\alpha}}$, with $x \in \Omega$, is such that

$$q_x(A) = \begin{cases} 1 & \text{if } A \subseteq \Omega \backslash x, \\ \overline{\alpha} & \text{if } A \supseteq x. \end{cases} \tag{F.1}$$

Let m' be a BBA defined by $m' = \circledcap_{x \in \Omega} \{\Omega \backslash x\}^{\overline{\alpha}}$. The commonality function q' associated to m' is such that, for all $A \subseteq \Omega$

$$\begin{aligned}
q'(A) &= \prod_{x \in \Omega} q_x(A) \\
&= \overline{\alpha}^{|A|},
\end{aligned}$$

from (F.1) and the fact that a given set $A \subseteq \Omega$ is the superset of $|A|$ singletons of Ω. Identifying q' to f, we conclude that m is a BBA. □

From Lemma F.1, $m_{\alpha,\cap}$ of Proposition 6.2 is thus indeed a BBA, since it can be obtained by a combination by the TBM conjunctive rule of simple BBAs.

Lemma F.2. *Let m be a BBA such that $m(A) = \alpha^{|\overline{A}|} \overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$, with $\alpha \in [0,1]$ and $\overline{\alpha} = 1 - \alpha$. Let $x \in \Omega$. We have*

$$\sum_{A \subseteq \overline{x}} m(A) = \alpha,$$

and

$$\sum_{A \supseteq x} m(A) = \overline{\alpha}.$$

Proof. Let $x \in \Omega$. Furthermore, let $|\overline{x}| = n$, hence $|\Omega| = n + 1$. Let m be a BBA such that $m(A) = \alpha^{|\overline{A}|} \overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$. We clearly have $m(A) = \alpha^{n+1-|A|} \overline{\alpha}^{|A|}$.

The set $\overline{x}$ has $\left(\begin{array}{c} n \\ k \end{array}\right)$ subsets with cardinality $k \leq n$. Hence, we have

$$\begin{aligned}
\sum_{A\subseteq\overline{x}} m(A) &= \left(\begin{array}{c} n \\ 0 \end{array}\right)\alpha^{n+1-0}\overline{\alpha}^0 + \left(\begin{array}{c} n \\ 1 \end{array}\right)\alpha^{n+1-n}\overline{\alpha}^1 + ... + \left(\begin{array}{c} n \\ 1 \end{array}\right)\alpha^{n+1-n}\overline{\alpha}^n \\
&= \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\alpha^{n+1-k}\overline{\alpha}^k \\
&= \alpha\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\alpha^{n-k}\overline{\alpha}^k \\
&= \alpha,
\end{aligned}$$

by the binomial theorem.

Furthermore, we have

$$\begin{aligned}
1 &= \sum_{A\subseteq\Omega} m(A) \\
&= \sum_{A\supseteq x} m(A) + \sum_{A\not\supseteq x} m(A) \\
&= \sum_{A\supseteq x} m(A) + \sum_{A\subseteq\overline{x}} m(A) \\
&= \sum_{A\supseteq x} m(A) + \alpha,
\end{aligned}$$

and thus $\sum_{A\supseteq x} m(A) = 1 - \alpha$. □

The next lemma is useful in the proof of Lemma F.4, which is itself directly related to the α-conditioning operation.

Lemma F.3. *Let m_1 be a BBA such that $m_1(A) = 0$ for all $A \subset \Omega, |A| < |\Omega| - 1$. We have:*

$$m_1 \circledcap^{\alpha} m_2 = \alpha \cdot m_1 \circledcap m_2 + (1-\alpha) \cdot m_1 \circledcup m_2, \quad \forall m_2. \tag{F.2}$$

Proof. We need to introduce some notations to show this lemma. The matrix $\mathbf{K}_{\{\overline{x}\}}$ (Equation (6.2)) and the quantity $k_{\overline{x}}(A,B)$ (Equation (6.3)) are noted, respectively, $\mathbf{K}^{\alpha}_{\{\overline{x}\}}$ and $k^{\alpha}_{\overline{x}}(A,B)$, to depict the fact that they depend on the value of α. Accordingly, we will write $\mathbf{K}^{1}_{\{\overline{x}\}}$ and $k^{1}_{\overline{x}}(A,B)$ when $\alpha = 1$, and $\mathbf{K}^{0}_{\{\overline{x}\}}$ and $k^{0}_{\overline{x}}(A,B)$ when $\alpha = 0$.

The proof is based on the following remark. Let $\overline{\alpha} = 1 - \alpha$. For all $\alpha \in [0,1]$ and for all $x \in \Omega$, we have:

$$k^{\alpha}_{\overline{x}}(A,B) = \alpha \cdot k^{1}_{\overline{x}}(A,B) + \overline{\alpha} \cdot k^{0}_{\overline{x}}(A,B), \quad \forall A, B \subseteq \Omega.$$

Hence, the following equation clearly holds, for all $\alpha \in [0,1]$ and for all $x \in \Omega$

$$\mathbf{K}^{\alpha}_{\{\overline{x}\}} = \alpha \cdot \mathbf{K}^{1}_{\{\overline{x}\}} + \overline{\alpha} \cdot \mathbf{K}^{0}_{\{\overline{x}\}}. \tag{F.3}$$

For all BBAs m_1 and m_2, we have then

$$\begin{aligned}\mathbf{m}_1 \circledcap^\alpha \mathbf{m}_2 &= \mathbf{K}^\alpha_{m_1} \cdot \mathbf{m}_2 \\ &= (\sum_{X\subseteq\Omega} m_1(X)\cdot \mathbf{K}^\alpha_X)\cdot \mathbf{m}_2 \\ &= (\sum_{X\subset\Omega} m_1(X)\cdot \mathbf{K}^\alpha_X + m_1(\Omega))\cdot \mathbf{m}_2 \\ &= (\sum_{X\subset\Omega} m_1(X)\cdot \prod_{x\notin X} \mathbf{K}^\alpha_{\{\overline{x}\}} + m_1(\Omega))\cdot \mathbf{m}_2,\end{aligned}$$

which from (F.3) is equal to

$$\mathbf{m}_1 \circledcap^\alpha \mathbf{m}_2 = (\sum_{X\subset\Omega} m_1(X)\cdot \prod_{x\notin X}(\alpha\cdot \mathbf{K}^1_{\{\overline{x}\}} + \overline{\alpha}\cdot \mathbf{K}^0_{\{\overline{x}\}}) + m_1(\Omega))\cdot \mathbf{m}_2. \quad \text{(F.4)}$$

Now, suppose m_1 is such that $m_1(A) = 0$ for all $A \subset \Omega, |A| < |\Omega| - 1$. From (F.4), we obtain

$$\begin{aligned}\mathbf{m}_1 \circledcap^\alpha \mathbf{m}_2 &= (\sum_{\substack{X\subset\Omega\\ |X|=|\Omega|-1}} m_1(X)\cdot \prod_{x\notin X}(\alpha\cdot \mathbf{K}^1_{\{\overline{x}\}} + \overline{\alpha}\cdot \mathbf{K}^0_{\{\overline{x}\}}) + m_1(\Omega))\cdot \mathbf{m}_2 \\ &= (\sum_{\substack{X\subset\Omega\\ |X|=|\Omega|-1}} m_1(X)\cdot (\alpha\cdot \mathbf{K}^1_{\{\overline{\overline{X}}\}} + \overline{\alpha}\cdot \mathbf{K}^0_{\{\overline{\overline{X}}\}}) + m_1(\Omega))\cdot \mathbf{m}_2,\end{aligned}$$

since for all $X \subset \Omega$ such that $|X| = |\Omega| - 1$, there is only one element $x \in \Omega$, such that $x \notin X$: this element is $x = \overline{X}$.

Furthermore, we have

$$\begin{aligned}\alpha\cdot \mathbf{m}_1 \circledcap \mathbf{m}_2 + \overline{\alpha}\cdot \mathbf{m}_1 \circledcirc \mathbf{m}_2 &= \alpha\cdot \mathbf{K}^1_{m_1}\cdot \mathbf{m}_2 + \overline{\alpha}\cdot \mathbf{K}^0_{m_1}\cdot \mathbf{m}_2 \\ &= (\alpha\cdot \mathbf{K}^1_{m_1} + \overline{\alpha}\cdot \mathbf{K}^0_{m_1})\cdot \mathbf{m}_2 \\ &= (\alpha\cdot \sum_{X\subseteq\Omega} m_1(X)\cdot \mathbf{K}^1_X + \overline{\alpha}\cdot \sum_{X\subseteq\Omega} m_1(X)\cdot \mathbf{K}^0_X)\cdot \mathbf{m}_2 \\ &= (\sum_{X\subseteq\Omega} m_1(X)\cdot (\alpha\cdot \mathbf{K}^1_X + \overline{\alpha}\cdot \mathbf{K}^0_X))\cdot \mathbf{m}_2 \\ &= (\sum_{X\subseteq\Omega} m_1(X)\cdot (\alpha\cdot \prod_{x\notin X}\mathbf{K}^1_{\{\overline{x}\}} + \overline{\alpha}\cdot \prod_{x\notin X}\mathbf{K}^0_{\{\overline{x}\}}))\cdot \mathbf{m}_2 \\ &= (\sum_{\substack{X\subset\Omega\\ |X|=|\Omega|-1}} m_1(X)\cdot (\alpha\cdot \mathbf{K}^1_{\{\overline{\overline{X}}\}} + \overline{\alpha}\cdot \mathbf{K}^0_{\{\overline{\overline{X}}\}}) + m_1(\Omega))\cdot \mathbf{m}_2,\end{aligned}$$

which completes the proof. □

Lemma F.4. *Let m_B be a categorical BBA focused on $B \subseteq \Omega$, $|B| = |\Omega| - 1$. Let m be a BBA. We have*

$$(m_B \circledcap^\alpha m)(X) = \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X} m(A)\, m_\alpha(C), \quad \forall X \subseteq \Omega, \qquad \text{(F.5)}$$

where m_α is a BBA such that $m_\alpha(A) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$.

Proof. Let $B \subseteq \Omega$, $|B| = |\Omega| - 1$. Let m_B be a categorical BBA focused on B. Let m be a BBA and let m_α be a BBA such that $m_\alpha(A) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$.

Let $X \subseteq \Omega$. We have

$$
\begin{aligned}
& \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X} m(A)\, m_\alpha(C) \\
= & \sum_{\substack{C\supseteq\overline{B} \\ (A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X}} m(A)\, m_\alpha(C) + \sum_{\substack{C\not\supseteq\overline{B} \\ (A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X}} m(A)\, m_\alpha(C) \\
= & \sum_{\substack{C\supseteq\overline{B} \\ (A\cap B)\cup(\overline{A}\cap\overline{B})=X}} m(A)\, m_\alpha(C) + \sum_{\substack{C\not\supseteq\overline{B} \\ (A\cap B)=X}} m(A)\, m_\alpha(C) \\
= & \left(\sum_{C\supseteq\overline{B}} m_\alpha(C)\right) \cdot \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B})=X} m(A) + \left(\sum_{C\subseteq B} m_\alpha(C)\right) \cdot \sum_{(A\cap B)=X} m(A) \\
= & \overline{\alpha} \cdot m \circledcirc m_B(X) + \alpha \cdot m \circledcap m_B(X) \quad \text{(from Lemma F.2 and Equations (6.4) and (1.4))} \\
= & m \circledcap^\alpha m_B(X),
\end{aligned}
$$

using Lemma F.3. □

Lemma F.5. *Let m_B be a categorical BBA focused on $B \subseteq \Omega$. We have*

$$m_B \circledcap^\alpha m = \left(\circledcap^\alpha_{x\notin B} m_{\overline{x}}\right) \circledcap^\alpha m, \quad \forall m, \tag{F.6}$$

where, for all $x \in \Omega$, $m_{\overline{x}}$ denotes a categorical BBA focused on $\overline{x}$.

Proof. The case $B = \Omega$ clearly holds. The case where B is such that $|B| = |\Omega| - 1$ also clearly holds. In order to show that (F.6) holds when B is such that $|B| < |\Omega| - 1$, we need to show first that

$$\mathbf{m}_{\overline{x}} \circledcap^\alpha \mathbf{m} = \mathbf{K}_{\{\overline{x}\}} \cdot \mathbf{m} \tag{F.7}$$

holds, for all $x \in \Omega$. We have, for all $x \in \Omega$

$$
\begin{aligned}
\mathbf{m}_{\overline{x}} \circledcap^\alpha \mathbf{m} &= \mathbf{K}^{\cap,\alpha}_{m_{\overline{x}}} \cdot \mathbf{m} \\
&= \left(\sum_{X\subseteq\Omega} m_{\overline{x}}(X) \cdot \mathbf{K}_X\right) \cdot \mathbf{m} \\
&= m_{\overline{x}}(\overline{x}) \cdot \mathbf{K}_{\overline{x}} \cdot \mathbf{m} \\
&= \mathbf{K}_{\overline{x}} \cdot \mathbf{m} \\
&= \mathbf{K}_{\{\overline{x}\}} \cdot \mathbf{m}.
\end{aligned}
$$

The last line comes from the fact that if we let $X = \overline{x}$ in

$$\mathbf{K}_X = \prod_{x\notin X} \mathbf{K}_{\{\overline{x}\}},$$

we obtain

$$\mathbf{K}_{\overline{x}} = \mathbf{K}_{\{\overline{x}\}},$$

since the only element of Ω that does not belong to $X = \overline{x}$ is x.

Equation (F.7) being proved, we can proceed with the rest of the proof. Let m_B be a categorical BBA focused on $B \subset \Omega$. We have

$$\begin{aligned}
\mathbf{m}_B \circledcap^\alpha \mathbf{m} &= \mathbf{K}^{\cap,\alpha}_{m_B} \cdot \mathbf{m} \\
&= \left(\sum_{X \subseteq \Omega} m_B(X) \cdot \mathbf{K}_X \right) \cdot \mathbf{m} \\
&= m_B(B) \cdot \mathbf{K}_B \cdot \mathbf{m} \\
&= \mathbf{K}_B \cdot \mathbf{m} \\
&= \prod_{x \notin B} \mathbf{K}_{\{\overline{x}\}} \cdot \mathbf{m} \\
&= \prod_{x \notin B \cup y} \mathbf{K}_{\{\overline{x}\}} \cdot \mathbf{K}_{\{\overline{y}\}} \cdot \mathbf{m},
\end{aligned}$$

assuming that there exists $y \in \Omega$ such that $y \notin B$. Using (F.7), the last equation becomes

$$\mathbf{m}_B \circledcap^\alpha \mathbf{m} = \prod_{x \notin B \cup y} \mathbf{K}_{\{\overline{x}\}} \cdot \mathbf{m}_{\overline{y}} \circledcap^\alpha \mathbf{m},$$

where $m_{\overline{y}}$ is a categorical BBA focused on $\overline{y}$. Assuming that there exists also $z \in \Omega$ such that $z \notin B$, we have

$$\begin{aligned}
\mathbf{m}_B \circledcap^\alpha \mathbf{m} &= \prod_{x \notin B \cup y \cup z} \mathbf{K}_{\{\overline{x}\}} \cdot \mathbf{K}_{\{\overline{z}\}} \cdot \mathbf{m}_{\overline{y}} \circledcap^\alpha \mathbf{m} \\
&= \prod_{x \notin B \cup y \cup z} \mathbf{K}_{\{\overline{x}\}} \cdot \mathbf{m}_{\overline{z}} \circledcap^\alpha \mathbf{m}_{\overline{y}} \circledcap^\alpha \mathbf{m}.
\end{aligned}$$

More generally, we have

$$m_B \circledcap^\alpha m = \left(\circledcap^\alpha_{x \notin B} m_{\overline{x}} \right) \circledcap^\alpha m, \quad \forall B \subset \Omega, \quad \forall m.$$

□

Proposition 6.2 may then be showed as follows.

Proof. Let m be a BBA and let m_α be a BBA such that $m_\alpha(A) = \alpha^{|\overline{A}|} \overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$. Lemma F.4 has shown that we have

$$m[B]^\alpha(X) = \sum_{(A \cap B) \cup (\overline{A} \cap \overline{B} \cap C) = X} m(A)\, m_\alpha(C), \quad \forall X \subseteq \Omega, \tag{F.8}$$

for all $B \subseteq \Omega$, $|B| = |\Omega| - 1$. Here, we are first going to show by induction that (F.8) holds for all B such that $|B| = |\Omega| - n$, $1 \leq n \leq \Omega$, i.e., all $B \subset \Omega$. We will show that (F.8) holds for $B = \Omega$ at the end of this proof.

The base case of this proof by induction, i.e., (F.8) holds for all B such that $|B| = |\Omega| - n$ with $n = 1$, has been shown by Lemma F.4. We now show the inductive step, i.e., if (F.8) holds for all B such that $|B| = |\Omega| - n$, then it also holds for all B such that $|B| = |\Omega| - (n+1)$.

Let $B \subseteq \Omega$ such that $|B| = |\Omega| - (n+1)$. Let m_B be a categorical BBA focused on B. Let $y \in \Omega$ such that $y \notin B$. Let $B' \subseteq \Omega$ such that $B' = B \cup y$, thus $|B'| = |\Omega| - n$. Furthermore, let m_A denote a categorical BBA focused on some subset $A \subseteq \Omega$. We have, for all BBA m and all $X \subseteq \Omega$

$$\begin{aligned}
(m_B \circledcap^\alpha m)(X) &= \left(\left(\circledcap^\alpha_{x \notin B} m_{\overline{x}}\right) \circledcap^\alpha m\right)(X) && \text{(from Lemma F.5)}\\
&= \left(m_{\overline{y}} \circledcap^\alpha \left(\circledcap^\alpha_{x \notin B \cup y} m_{\overline{x}}\right) \circledcap^\alpha m\right)(X)\\
&= \left(m_{\overline{y}} \circledcap^\alpha m_{B'} \circledcap^\alpha m\right)(X) && \text{(from Lemma F.5)}\\
&= \left(m_{\overline{y}} \circledcap^\alpha m[B']^\alpha\right)(X)\\
&= \sum_{(A \cap \overline{y}) \cup (\overline{A} \cap y \cap C) = X} m_\alpha(C)\, m[B']^\alpha(A) && \text{(from Lemma F.4)}\\
&= \sum_{(A \cap \overline{y}) \cup (\overline{A} \cap y \cap C) = X} m_\alpha(C) \left(\sum_{(E \cap B') \cup (\overline{E} \cap \overline{B'} \cap D) = A} m(E)\, m_\alpha(D) \right),
\end{aligned} \tag{F.9}$$

assuming that the inductive step holds, i.e., assuming that (F.8) holds if the cardinality of the conditioning set is $|\Omega| - n$. From (F.9), we obtain

$$(m_B \circledcap^\alpha m)(X) = \sum_{\left(\left((E \cap B') \cup (\overline{E} \cap \overline{B'} \cap D)\right) \cap \overline{y}\right) \cup \left(\overline{\left((E \cap B') \cup (\overline{E} \cap \overline{B'} \cap D)\right)} \cap y \cap C\right) = X} m_\alpha(C)\, m(E)\, m_\alpha(D). \tag{F.10}$$

We are now going to modify the expression under the sum sign in the right side of (F.10). To achieve this modification, we will make use of the following simple facts about the sets B and B':

$$\begin{aligned}
B' \cap \overline{y} &= (B \cup y) \cap \overline{y}\\
&= B \cap \overline{y}\\
&= B,
\end{aligned} \tag{F.11}$$

since $y \notin B$ and thus $B \subseteq \overline{y}$.

$$\begin{aligned}
\overline{B'} \cap y &= (\overline{B \cup y}) \cap y\\
&= \overline{B} \cap \overline{y} \cap y\\
&= \emptyset.
\end{aligned} \tag{F.12}$$

$$\begin{aligned}
\overline{B'} \cap \overline{y} &= (\overline{B \cup y}) \cap \overline{y}\\
&= \overline{B} \cap \overline{y} \cap \overline{y}\\
&= \overline{B} \cap \overline{y}.
\end{aligned} \tag{F.13}$$

$$\begin{aligned} B' \cap y &= (B \cup y) \cap y \\ &= y. \end{aligned} \quad \text{(F.14)}$$

$y \notin B$, hence $y \in \overline{B}$, and thus

$$\overline{B} \cup y = \overline{B}. \quad \text{(F.15)}$$

We can now work with the expression under the sum sign in (F.10). We have

$$\begin{aligned}
X &= \Big(\big((E \cap B') \cup (\overline{E} \cap \overline{B'} \cap D)\big) \cap \overline{y}\Big) \cup \Big(\overline{\big((E \cap B') \cup (\overline{E} \cap \overline{B'} \cap D)\big)} \cap y \cap C\Big) \\
&= (E \cap B' \cap \overline{y}) \cup (\overline{E} \cap \overline{B'} \cap D \cap \overline{y}) \cup (\overline{(E \cap B')} \cap \overline{(\overline{E} \cap \overline{B'} \cap D)} \cap y \cap C) \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap D \cap \overline{y}) \cup ((\overline{E} \cup \overline{B'}) \cap (E \cup B' \cup \overline{D}) \cap y \cap C) \\
&\quad \text{(using (F.11) and (F.13))} \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap D \cap \overline{y}) \cup (((\overline{E} \cap y) \cup (\overline{B'} \cap y)) \cap (E \cup B' \cup \overline{D}) \cap C) \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap D \cap \overline{y}) \cup (\overline{E} \cap y \cap (E \cup B' \cup \overline{D}) \cap C) \\
&\quad \text{(using (F.12))} \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap D \cap \overline{y}) \cup (((\overline{E} \cap E) \cup (\overline{E} \cap B') \cup (\overline{E} \cap \overline{D})) \cap y \cap C) \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap D \cap \overline{y}) \cup (((\overline{E} \cap B') \cup (\overline{E} \cap \overline{D})) \cap y \cap C) \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap D \cap \overline{y}) \cup (\overline{E} \cap B' \cap y \cap C) \cup (\overline{E} \cap \overline{D} \cap y \cap C) \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap D \cap \overline{y}) \cup (\overline{E} \cap y \cap C) \cup (\overline{E} \cap y \cap C \cap \overline{D}) \\
&\quad \text{(using (F.14))} \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap D \cap \overline{y}) \cup ((\overline{E} \cap y \cap C) \cap (\Omega \cup \overline{D})) \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap D \cap \overline{y}) \cup (\overline{E} \cap y \cap C) \\
&= (E \cap B) \cup (\overline{E} \cap ((\overline{B} \cap D \cap \overline{y}) \cup (y \cap C))) \\
&= (E \cap B) \cup (\overline{E} \cap ((y \cap C) \cup \overline{B}) \cap ((y \cap C) \cup D) \cap ((y \cap C) \cup \overline{y})) \\
&= (E \cap B) \cup (\overline{E} \cap (y \cup \overline{B}) \cap (C \cup \overline{B}) \cap (y \cup D) \cap (C \cup D) \cap (y \cup \overline{y}) \cap (C \cup \overline{y})) \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap (C \cup \overline{B}) \cap (y \cup D) \cap (C \cup D) \cap (C \cup \overline{y})) \quad \text{(using (F.15))} \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap (y \cup D) \cap (C \cup \overline{y}) \cap (C \cup D)) \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap (((y \cup D) \cap C) \cup ((y \cup D) \cap \overline{y})) \cap (C \cup D)) \\
&= (E \cap B) \cup (\overline{E} \cap \overline{B} \cap ((y \cap C) \cup (D \cap C) \cup (D \cap \overline{y})) \cap (C \cup D)).
\end{aligned} \quad \text{(F.16)}$$

Using (F.16), Equation (F.10) can be rewritten

$$\begin{aligned}
(m_B \circledcap^{\alpha} m)(X) &= \sum_{(E\cap B)\cup(\overline{E}\cap\overline{B}\cap((y\cap C)\cup(D\cap C)\cup(D\cap\overline{y}))\cap(C\cup D))=X} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&= \sum_{\substack{C \supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap((y\cap C)\cup(D\cap C)\cup(D\cap\overline{y}))\cap(C\cup D))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&\quad + \sum_{\substack{C \not\supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap((y\cap C)\cup(D\cap C)\cup(D\cap\overline{y}))\cap(C\cup D))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D).
\end{aligned} \quad \text{(F.17)}$$

Note that for all $C \subseteq \Omega$ such that $C \supseteq y$, we have $y \cap C = y$. Furthermore, for all $C \subseteq \Omega$ such that $C \not\supseteq y$ or, equivalently, such that $C \subseteq \overline{y}$, we have $y \cap C = \emptyset$. Hence, (F.17) reduces to

$$
\begin{aligned}
(m_B \textcircled{\cap}^{\alpha}\, m)(X) &= \sum_{\substack{C \supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap(y\cup(D\cap C)\cup(D\cap\overline{y}))\cap(C\cup D))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&\quad + \sum_{\substack{C \not\supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap((D\cap C)\cup(D\cap\overline{y}))\cap(C\cup D))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&= \sum_{\substack{C \supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap(((y\cup D)\cap(y\cup\overline{y}))\cup(D\cap C))\cap(C\cup D))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&\quad + \sum_{\substack{C \not\supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap(((D\cap C)\cap(C\cup D))\cup((D\cap\overline{y})\cap(C\cup D))))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&= \sum_{\substack{C \supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap(y\cup D\cup(D\cap C))\cap(C\cup D))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&\quad + \sum_{\substack{C \not\supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap((D\cap C)\cup(D\cap\overline{y})))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&= \sum_{\substack{C \supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap(y\cup D)\cap(C\cup D))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&\quad + \sum_{\substack{C \not\supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap D\cap(C\cup\overline{y}))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&= \sum_{\substack{C \supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap(D\cup y))=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&\quad + \sum_{\substack{C \not\supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap D\cap\overline{y})=X}} m_\alpha(C)\, m(E)\, m_\alpha(D) \\
&= \left(\sum_{C \supseteq y} m_\alpha(C)\right) \cdot \sum_{(E\cap B)\cup(\overline{E}\cap\overline{B}\cap(D\cup y))=X} m(E)\, m_\alpha(D) \\
&\quad + \left(\sum_{C \not\supseteq y} m_\alpha(C)\right) \cdot \sum_{(E\cap B)\cup(\overline{E}\cap\overline{B}\cap D\cap\overline{y})=X} m(E)\, m_\alpha(D) \\
&= \overline{\alpha} \cdot \sum_{(E\cap B)\cup(\overline{E}\cap\overline{B}\cap(D\cup y))=X} m(E)\, m_\alpha(D) \\
&\quad + \alpha \cdot \sum_{(E\cap B)\cup(\overline{E}\cap\overline{B}\cap D\cap\overline{y})=X} m(E)\, m_\alpha(D) \qquad \text{(from Lemma F.2)}
\end{aligned}
$$

$$= \sum_{\substack{F\supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap F)=X}} m(E)\, m'(F) + \sum_{\substack{F\not\supseteq y \\ (E\cap B)\cup(\overline{E}\cap\overline{B}\cap F)=X}} m(E)\, m'(F)\,, \tag{F.18}$$

with

$$m'(F) = \begin{cases} \overline{\alpha}(m_\alpha(F\cap\overline{y}) + m_\alpha(F)), & \forall F\supseteq y, \\ \alpha(m_\alpha(F\cup y) + m_\alpha(F)), & \forall F\not\supseteq y, \end{cases}$$

since, for all $F\supseteq y$, we have

$$\begin{aligned}(F\cap\overline{y})\cup y &= F\cup y \\ &= F,\end{aligned}$$

and, for all $F\not\supseteq y$, we have

$$\begin{aligned}(F\cup y)\cap\overline{y} &= F\cap\overline{y} \\ &= F.\end{aligned}$$

From (F.18), we find

$$(m_B \circledcap^\alpha m)(X) = \sum_{(E\cap B)\cup(\overline{E}\cap\overline{B}\cap F)=X} m(E)\, m'(F)\,.$$

Let us now show that $m'(A) = m_\alpha(A)$ or, equivalently, $m'(A) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}$, for all $A\subseteq\Omega$.

If $A\supseteq y$, then

$$\begin{aligned} m'(A) &= \overline{\alpha}\left(\alpha^{|\overline{A\cap\overline{y}}|}\overline{\alpha}^{|A\cap\overline{y}|} + \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}\right) \\ &= \overline{\alpha}\left(\alpha^{|\overline{A}\cup y|}\overline{\alpha}^{|A\cap\overline{y}|} + \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}\right) \\ &= \overline{\alpha}\left(\alpha^{|\overline{A}|+1}\overline{\alpha}^{|A|-1} + \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}\right) \\ &= \alpha^{|\overline{A}|}\left(\alpha\overline{\alpha}^{|A|} + \overline{\alpha}^{|A|+1}\right) \\ &= \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}, \end{aligned}$$

since $\overline{\alpha}^{n+1} + \overline{\alpha}^n\alpha = \overline{\alpha}^n$.

If $A\not\supseteq y$, then

$$\begin{aligned} m'(A) &= \alpha\left(\alpha^{|\overline{A\cup y}|}\overline{\alpha}^{|A\cup y|} + \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}\right) \\ &= \alpha\left(\alpha^{|\overline{A}\cap\overline{y}|}\overline{\alpha}^{|A|+1} + \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}\right) \\ &= \alpha\left(\alpha^{|\overline{A}|-1}\overline{\alpha}^{|A|+1} + \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}\right) \\ &= \alpha^{|\overline{A}|}\left(\overline{\alpha}^{|A|+1} + \alpha\overline{\alpha}^{|A|}\right) \\ &= \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}. \end{aligned}$$

We have thus shown that (F.8) holds for all B such that $|B| = |\Omega| - n$, $1 \leq n \leq \Omega$, i.e., all $B \subset \Omega$. We now show that (F.8) holds for $B = \Omega$. For all BBA m, we have $m[\Omega]^\alpha = m \circledcap^\alpha m_\Omega = m$, and thus $m[\Omega]^\alpha(X) = m(X)$ for all $X \subseteq \Omega$. Furthermore, we have, for all $X \subseteq \Omega$

$$
\begin{aligned}
\sum_{(A\cap\Omega)\cup(\overline{A}\cap\overline{\Omega}\cap C)=X} m(A)\, m_\alpha(C) &= \left(\sum_{C\subseteq\Omega} m_\alpha(C)\right) \cdot \sum_{A=X} m(A) \\
&= m(X).
\end{aligned}
\tag{F.19}
$$

Hence, we have

$$
m[\Omega]^\alpha(X) = \sum_{(A\cap\Omega)\cup(\overline{A}\cap\overline{\Omega}\cap C)=X} m(A)\, m_\alpha(C), \quad \forall X \subseteq \Omega. \tag{F.20}
$$

□

F.2 Proof of Proposition 6.4

Proof. Let m_1 and m_2 be two BBAs. We have, for all $X \subseteq \Omega$,

$$
\begin{aligned}
(m_1 \circledcup^\alpha m_2)(X) &= \overline{(\overline{m_1} \circledcap^\alpha \overline{m_2})}(X) \\
&= (\overline{m_1} \circledcap^\alpha \overline{m_2})(\overline{X}) \\
&= \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=\overline{X}} \overline{m_1}(A)\, \overline{m_2}(B)\, m_{\alpha,\cap}(C),
\end{aligned}
\tag{F.21}
$$

using Proposition 6.3, where $m_{\alpha,\cap}(A) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$. From (F.21), we obtain

$$
\begin{aligned}
(m_1 \circledcup^\alpha m_2)(X) &= \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=\overline{X}} m_1\left(\overline{A}\right) m_2\left(\overline{B}\right) m_{\alpha,\cap}(C) \\
&= \sum_{(\overline{A}\cap\overline{B})\cup(A\cap B\cap C)=\overline{X}} m_1(A)\, m_2(B)\, m_{\alpha,\cap}(C) \\
&= \sum_{\overline{(\overline{A}\cap\overline{B})\cup(A\cap B\cap C)}=X} m_1(A)\, m_2(B)\, m_{\alpha,\cap}(C) \\
&= \sum_{(A\cup B)\cap(\overline{A}\cup\overline{B}\cup\overline{C})=X} m_1(A)\, m_2(B)\, m_{\alpha,\cap}(C) \\
&= \sum_{(A\cup B)\cap(\overline{A}\cup\overline{B}\cup C)=X} m_1(A)\, m_2(B)\, m_{\alpha,\cap}\left(\overline{C}\right) \\
&= \sum_{(A\cup B)\cap(\overline{A}\cup\overline{B}\cup C)=X} m_1(A)\, m_2(B)\, \overline{m_{\alpha,\cap}}(C) \\
&= \sum_{(A\cup B)\cap(\overline{A}\cup\overline{B}\cup C)=X} m_1(A)\, m_2(B)\, m_{\alpha,\cup}(C),
\end{aligned}
\tag{F.22}
$$

with, for all $A \subseteq \Omega$,

$$\begin{aligned} m_{\alpha,\cup}(A) &= \overline{m_{\alpha,\cap}}(A) \\ &= m_{\alpha,\cap}\left(\overline{A}\right) \\ &= \alpha^{|\overline{\overline{A}}|}\overline{\alpha}^{|\overline{A}|} \\ &= \alpha^{|A|}\overline{\alpha}^{|\overline{A}|}. \end{aligned}$$

Let us now work with the expression under the sum sign in (F.22). We have

$$\begin{aligned} X &= (A \cup B) \cap (\overline{A} \cup \overline{B} \cup C) \\ &= (A \cap (\overline{A} \cup \overline{B} \cup C)) \cup (B \cap (\overline{A} \cup \overline{B} \cup C)) \\ &= (A \cap \overline{B}) \cup (A \cap C) \cup (\overline{A} \cap B) \cup (B \cap C) \\ &= (A \cap \overline{B}) \cup (A \cap C \cap (B \cup \overline{B})) \cup (\overline{A} \cap B) \cup (B \cap C \cap (A \cup \overline{A})) \\ &= (A \cap \overline{B}) \cup (A \cap C \cap B) \cup (A \cap C \cap \overline{B}) \cup (\overline{A} \cap B) \cup (B \cap C \cap A) \cup (B \cap C \cap \overline{A}) \\ &= (A \cap B \cap C) \cup (A \cap \overline{B} \cap \Omega) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap \Omega) \cup (\overline{A} \cap B \cap C) \\ &= (A \cap B \cap C) \cup (A \cap \overline{B} \cap (\Omega \cup C)) \cup (\overline{A} \cap B \cap (\Omega \cup C)) \\ &= (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (A \cap B \cap C). \end{aligned}$$

Hence, we have, for all $X \subseteq \Omega$

$$(m_1 \circledcup^{\alpha} m_2)(X) = \sum_{(A\cap\overline{B})\cup(\overline{A}\cap B)\cup(A\cap B\cap C)=X} m_1(A)\, m_2(B)\, m_{\alpha,\cup}(C).$$

□

F.3 Proof of Theorem 6.1

In order to show Theorem 6.1, the following technical lemma is needed.

Lemma F.6. *Let $T1 = \{t1, f1\}$ and $T2 = \{t2, f2\}$. Let Ω be a frame of discernment. Let m_{xand} be a BBA on $T1 \times T2$ defined by*

$$m_{xand}^{T1\times T2}(\{(t1,t2),(f1,f2)\}) = 1. \tag{F.23}$$

Let $m^{T1\times T2}[x]$ denote a BBA such that

$$m^{T1\times T2}[x] = \{(t1,t2),(f1,t2),(t1,f2)\}^{\overline{\alpha}},$$

for some $x \in \Omega$.

We have

$$m_{xand}^{T1\times T2\uparrow\Omega\times T1\times T2} \circledcap (\circledcap_{x\in\Omega} m^{T1\times T2}[x]^{\Uparrow\Omega\times T1\times T2}) = m,$$

with m a BBA defined on $\Omega \times T1 \times T2$ by

$$m((A \times (f1,f2)) \cup (\Omega \times (t1,t2))) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}, \quad \forall A \subseteq \Omega.$$

Proof. We have

$$m_{xand}^{T1\times T2\uparrow\Omega\times T1\times T2}((\Omega\times(f1,f2))\cup(\Omega\times(t1,t2))) \quad = \quad 1$$

and

$$\begin{aligned} & m^{T1\times T2}[x]^{\Uparrow\Omega\times T1\times T2} \\ = \quad & \{(\{\Omega\backslash x\}\times(f1,f2))\cup(\Omega\times(t1,t2))\cup(\Omega\times(f1,t2))\cup(\Omega\times(t1,f2))\}^{\overline{\alpha}}. \end{aligned}$$

From Lemma F.1, it is clear that the BBA m' defined on $\Omega\times T1\times T2$ by

$$m' = \circledcap_{x\in\Omega} m^{T1\times T2}[x]^{\Uparrow\Omega\times T1\times T2},$$

is such that

$$m'(\{(A\times(f1,f2))\cup(\Omega\times(t1,t2))\cup(\Omega\times(f1,t2))\cup(\Omega\times(t1,f2))\}) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}, \quad \forall A\subseteq\Omega.$$

Let $m = m_{xand}^{T1\times T2\uparrow\Omega\times T1\times T2} \circledcap m'^{\Omega\times T1\times T2}$. It is direct to show that

$$m((A\times(f1,f2))\cup(\Omega\times(t1,t2))) \quad = \quad \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}, \quad \forall A\subseteq\Omega$$

holds. □

We may now show Theorem 6.1 as follows.

Proof. From Lemma F.6, Equation (6.17) may be rewritten

$$m_1 \circledcap^{\alpha} m_2 \quad = \quad (m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \circledcap m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \circledcap m)^{\downarrow\Omega},$$

with m a BBA defined on $\Omega\times T1\times T2$ by

$$m((A\times(f1,f2))\cup(\Omega\times(t1,t2))) \quad = \quad \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}, \quad \forall A\subseteq\Omega.$$

We have for all $A\subseteq\Omega$,

$$m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2}(D) \quad = \quad m_1^{\Omega}(A),$$

where

$$D \quad = \quad \left\{(A\times(t1,t2))\cup(A\times(t1,f2))\cup(\overline{A}\times(f1,t2))\cup(\overline{A}\times(f1,f2))\right\}$$

and for all $B\subseteq\Omega$,

$$m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2}(E) \quad = \quad m_2^{\Omega}(B),$$

where

$$E \quad = \quad \left\{(B\times(t1,t2))\cup(\overline{B}\times(t1,f2))\cup(B\times(f1,t2))\cup(\overline{B}\times(f1,f2))\right\}.$$

Let

$$D \quad = \quad \left\{(A\times(t1,t2))\cup(A\times(t1,f2))\cup(\overline{A}\times(f1,t2))\cup(\overline{A}\times(f1,f2))\right\},$$

for some $A \subseteq \Omega$. Let

$$E = \left\{(B \times (t1,t2)) \cup (\overline{B} \times (t1,f2)) \cup (B \times (f1,t2)) \cup (\overline{B} \times (f1,f2))\right\},$$

for some $B \subseteq \Omega$. Let

$$F = \{(C \times (f1,f2)) \cup (\Omega \times (t1,t2))\},$$

for some $C \subseteq \Omega$. We have

$$(D \cap E \cap F) \downarrow \Omega = (A \cap B) \cup (\overline{A} \cap \overline{B} \cap C).$$

Furthermore, we have

$$\begin{aligned}
& (m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \circledcap m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \circledcap m)^{\downarrow\Omega}(X) \\
= & \sum_{\substack{D,E,F\subseteq\Omega\times T1\times T2 \\ (D\cap E\cap F)\downarrow\Omega=X}} m_{1'}(D)m_{2'}(E)m(F) \\
= & \sum_{\substack{A,B,C\subseteq\Omega, \\ D=\left\{(A\times(t1,t2))\cup(A\times(t1,f2))\cup(\overline{A}\times(f1,t2))\cup(\overline{A}\times(f1,f2))\right\}, \\ E=\left\{(B\times(t1,t2))\cup(\overline{B}\times(t1,f2))\cup(B\times(f1,t2))\cup(\overline{B}\times(f1,f2))\right\}, \\ F=\{(C\times(f1,f2))\cup(\Omega\times(t1,t2))\}, \\ (A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X}} m_{1'}(D)m_{2'}(E)m(F) \\
& \text{(since } m_{1'}, m_{2'} \text{ and } m \text{ are non null only for the sets } D, E \text{ and } F \\
& \text{ described under the sum sign)} \\
= & \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X} m_1(A)m_2(B)m_\alpha(C) \quad \text{(from the definitions of } m_{1'}, m_{2'} \text{ and } m\text{),}
\end{aligned} \tag{F.24}$$

with m_α a BBA defined on Ω such that $m_\alpha(A) = \alpha^{|\overline{A}|}\overline{\alpha}^{|A|}$, for all $A \subseteq \Omega$. The theorem is then proved with Proposition 6.3. □

F.4 Proof of Theorem 6.3

Proof.

Notations

This proof requires that we introduce a few notations. Let Ω^n denote the set Ω with n elements, i.e., $\Omega^n = \{x_1, ..., x_n\}$. Further, let Ω^{n+1} denote the set defined by $\Omega^{n+1} = \{x_1, ..., x_n, x_{n+1}\}$. Let $\mathbf{G}^n$ and $\mathbf{V}_X^n$ denote, respectively, the $2^n \times 2^n$ matrices $\mathbf{G}^{\cap,\alpha}$ and $\mathbf{V}_X^{\cap,\alpha}$ when the frame is Ω^n. Let $\mathbf{G}^n(\cdot, X)$ denote the X column of $\mathbf{G}^n$ (note that $\mathbf{G}^n(\cdot, X)$ is a column vector). Let $\mathbf{M}^n$ denote the $2^n \times 2^n$ matrix defined, for all $n \geq 1$, by:

$$\mathbf{M}^n = \mathbf{Kron}\left(\begin{bmatrix} 1 & 1 \\ \alpha - 1 & 1 \end{bmatrix}, \mathbf{M}^{n-1}\right), \quad \mathbf{M}^0 = 1. \tag{F.25}$$

Let $\mathbf{x}$ be a column vector of size n and let ${}^{i}\mathbf{x}$ denote a column vector of size $2^i \cdot n$ (i is a nonnegative integer), whose elements are $\mathbf{x}$, e.g., let $\mathbf{x} = [2\,3]'$, ${}^{2}\mathbf{x}$ denotes then a column vector of size $2^2 \cdot 2$, whose elements are $\mathbf{x}$, i.e., we have

$$\begin{aligned} {}^{2}\mathbf{x} &= \begin{bmatrix} \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \end{bmatrix} \\ &= [2\,3\,2\,3\,2\,3\,2\,3]'. \end{aligned}$$

Proof by induction

The aim of this proof is to show that $\mathbf{M}^n = \mathbf{G}^n$ for all $n \geq 2$, or, equivalently that the X column of $\mathbf{M}^n$, noted $\mathbf{M}^n(\cdot, X)$, is equal to $\mathbf{V}_X^n \cdot {}^{n}1$ (since the X column of the matrix $\mathbf{G}^n$ is such that $\mathbf{G}^n(\cdot, X) = \mathbf{V}_X^n \cdot {}^{n}1$) for all $n \geq 2$ and all $X \subseteq \Omega^n$. In other words, we must prove that the following equation holds, for all $X \subseteq \Omega^n$ and all $n \geq 2$:

$$\mathbf{M}^n(\cdot, X) = \mathbf{V}_X^n \cdot {}^{n}1. \tag{F.26}$$

We show (F.26) by induction.

First, we must show the *base case*, i.e., that (F.26) holds for $n = 2$. This can be done by comparing the matrix $\mathbf{G}^{\cap,\alpha}$ when $|\Omega| = 2$ (i.e., "$\mathbf{G}^2$") with $\mathbf{M}^2$. Using the definition of $\mathbf{G}^{\cap,\alpha}$ given in Section 6.4.1, we find:

$$\mathbf{G}^2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha - 1 & 1 & \alpha - 1 & 1 \\ \alpha - 1 & \alpha - 1 & 1 & 1 \\ (\alpha - 1)^2 & \alpha - 1 & \alpha - 1 & 1 \end{bmatrix}.$$

From the definition (F.25) of $\mathbf{M}^n$, we have

$$\begin{aligned} \mathbf{M}^2 &= \begin{bmatrix} \mathbf{M}^1 & \mathbf{M}^1 \\ (\alpha - 1)\mathbf{M}^1 & \mathbf{M}^1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha - 1 & 1 & \alpha - 1 & 1 \\ \alpha - 1 & \alpha - 1 & 1 & 1 \\ (\alpha - 1)^2 & \alpha - 1 & \alpha - 1 & 1 \end{bmatrix}, \end{aligned}$$

hence $\mathbf{G}^2 = \mathbf{M}^2$.

Now, we must show the *inductive step*, i.e., if $\mathbf{M}^n(\cdot, X) = \mathbf{V}_X^n \cdot {}^{n}1$ holds for all $X \subseteq \Omega^n$, then

$$\mathbf{M}^{n+1}(\cdot, X) = \mathbf{V}_X^{n+1} \cdot {}^{n+1}1 \tag{F.27}$$

holds for all $X \subseteq \Omega^{n+1}$.

Let us first work with the left side of (F.27). From (F.25), we have:

$$\mathbf{M}^{n+1} = \begin{bmatrix} \mathbf{M}^n & \mathbf{M}^n \\ (\alpha - 1)\mathbf{M}^n & \mathbf{M}^n \end{bmatrix}. \tag{F.28}$$

As a consequence of the matrix notation and (F.28), it may be shown that we have:

$$\mathbf{M}^{n+1}(\cdot, X) = \begin{cases} \begin{bmatrix} \mathbf{M}^n(\cdot, X) \\ (\alpha-1)\mathbf{M}^n(\cdot, X) \end{bmatrix} & \text{if } X \subseteq \Omega^n, \\ \begin{bmatrix} \mathbf{M}^n(\cdot, X \backslash x_{n+1}) \\ \mathbf{M}^n(\cdot, X \backslash x_{n+1}) \end{bmatrix} & \text{otherwise,} \end{cases} \tag{F.29}$$

where $\backslash$ denotes set difference, e.g., if $X = \{x_1, x_3\}$ and $n = 2$ (hence $\Omega^n = \Omega^2 = \{x_1, x_2\}$ and $\Omega^{n+1} = \Omega^3 = \{x_1, x_2, x_3\}$), then $X \backslash x_{n+1} = \{x_1\}$.

If $\mathbf{M}^n(\cdot, X) = \mathbf{V}^n_X \cdot {}^n1$ holds for all $X \subseteq \Omega^n$, then we have from (F.29):

$$\mathbf{M}^{n+1}(\cdot, X) = \begin{cases} \begin{bmatrix} \mathbf{V}^n_X \cdot {}^n1 \\ (\alpha-1)\,(\mathbf{V}^n_X \cdot {}^n1) \end{bmatrix} & \text{if } X \subseteq \Omega^n, \\ \begin{bmatrix} \mathbf{V}^n_{X \backslash x_{n+1}} \cdot {}^n1 \\ \mathbf{V}^n_{X \backslash x_{n+1}} \cdot {}^n1 \end{bmatrix} & \text{otherwise.} \end{cases} \tag{F.30}$$

Let us now work with the right side of (F.27).

First, we can remark that $\mathbf{V}^{n+1}_X$ is necessarily a diagonal matrix since it is a product of matrices $\mathbf{V}^{n+1}_{\{\overline{x}\}}$ that are themselves diagonal (see the definition (6.21) of $\mathbf{V}^{\cap,\alpha}_{\{\overline{x}\}}$ in Section 6.4.1). We can further note that those matrices $\mathbf{V}^{n+1}_{\{\overline{x}\}}$ have some kind of structure, which we present below.

Considering a frame Ω^1 that has one element denoted by x_1, we have

$$\mathbf{V}^1_{\{\overline{x_1}\}} = \mathbf{Diag}(\mathbf{v}_{\{\overline{x_1}\}}),$$

where

$$\mathbf{v}_{\{\overline{x_1}\}} = \begin{bmatrix} {}^01 \\ {}^0(\alpha-1) \end{bmatrix} = \begin{bmatrix} 1 \\ (\alpha-1) \end{bmatrix}.$$

Now, if we consider a frame Ω^2 with two elements x_1 and x_2, we have

$$\mathbf{V}^2_{\{\overline{x_1}\}} = \mathbf{Diag}(\begin{bmatrix} \mathbf{v}_{\{\overline{x_1}\}} \\ \mathbf{v}_{\{\overline{x_1}\}} \end{bmatrix}),$$

and

$$\mathbf{V}^2_{\{\overline{x_2}\}} = \mathbf{Diag}(\mathbf{v}_{\{\overline{x_2}\}}),$$

with

$$\mathbf{v}_{\{\overline{x_2}\}} = \begin{bmatrix} {}^11 \\ {}^1(\alpha-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ (\alpha-1) \\ (\alpha-1) \end{bmatrix}.$$

Now, if we consider a frame Ω^3 with three elements x_1, x_2 and x_3, we have

$$\mathbf{V}^3_{\{\overline{x_1}\}} = \mathbf{Diag}(\begin{bmatrix} \mathbf{v}_{\{\overline{x_1}\}} \\ \mathbf{v}_{\{\overline{x_1}\}} \\ \mathbf{v}_{\{\overline{x_1}\}} \\ \mathbf{v}_{\{\overline{x_1}\}} \end{bmatrix}),$$

and

$$\mathbf{V}^3_{\{\overline{x_2}\}} = \mathbf{Diag}(\begin{bmatrix} \mathbf{v}_{\{\overline{x_2}\}} \\ \mathbf{v}_{\{\overline{x_2}\}} \end{bmatrix}),$$

and

$$\mathbf{V}^3_{\{\overline{x_3}\}} = \mathbf{Diag}(\mathbf{v}_{\{\overline{x_3}\}}),$$

with

$$\mathbf{v}_{\{\overline{x_3}\}} = \begin{bmatrix} {}^2 1 \\ {}^2(\alpha - 1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ (\alpha - 1) \\ (\alpha - 1) \\ (\alpha - 1) \\ (\alpha - 1) \end{bmatrix}.$$

More generally, it may be shown that we have, for a frame Ω^i with i elements $x_1, ..., x_i$ and with $j \leq i$:

$$\mathbf{V}^i_{\{\overline{x_j}\}} = \mathbf{Diag}({}^{i-j}\mathbf{v}_{\{\overline{x_j}\}}),$$

(remember that ${}^{i-j}\mathbf{v}_{\{\overline{x_j}\}}$ designates the column vector whose elements are the column vector $\mathbf{v}_{\{\overline{x_j}\}}$, and that the size of ${}^{i-j}\mathbf{v}_{\{\overline{x_j}\}}$ is equal to 2^{i-j} times the size of the vector $\mathbf{v}_{\{\overline{x_j}\}}$) or, equivalently:

$$\mathbf{V}^i_{\{\overline{x_j}\}} \cdot {}^i 1 = {}^{i-j}\mathbf{v}_{\{\overline{x_j}\}}, \tag{F.31}$$

with

$$\mathbf{v}_{\{\overline{x_j}\}} = \begin{bmatrix} {}^{j-1}1 \\ {}^{j-1}(\alpha - 1) \end{bmatrix}.$$

Let us now consider two cases.

1. For $j < n + 1$, we have from (F.31):

$$\begin{aligned} \mathbf{V}^{n+1}_{\{\overline{x_j}\}} \cdot {}^{n+1}1 &= {}^{n+1-j}\mathbf{v}_{\{\overline{x_j}\}} \\ &= \begin{bmatrix} {}^{n-j}\mathbf{v}_{\{\overline{x_j}\}} \\ {}^{n-j}\mathbf{v}_{\{\overline{x_j}\}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V}^n_{\{\overline{x_j}\}} \cdot {}^n 1 \\ \mathbf{V}^n_{\{\overline{x_j}\}} \cdot {}^n 1 \end{bmatrix}. \end{aligned} \tag{F.32}$$

2. If $n + 1 = j$, then we have

$$\begin{aligned} \mathbf{V}^{n+1}_{\{\overline{x_j}\}} \cdot {}^{n+1}1 &= {}^{0}\mathbf{v}_{\{\overline{x_j}\}} \\ &= \begin{bmatrix} {}^{(n+1)-1}1 \\ {}^{(n+1)-1}(\alpha - 1) \end{bmatrix} \\ &= \begin{bmatrix} {}^{n}1 \\ {}^{n}(\alpha - 1) \end{bmatrix}. \end{aligned} \tag{F.33}$$

Now, for all $X \subseteq \Omega^{n+1}$, we have by definition:

$$\mathbf{V}_X^{n+1} = \prod_{x \notin X} \mathbf{V}_{\{\overline{x}\}}^{n+1}$$

Since the matrices $\mathbf{V}_{\{\overline{x}\}}^{n+1}$ are diagonal, we have:

$$\begin{aligned} \mathbf{V}_X^{n+1} \cdot {}^{n+1}1 &= \left(\prod_{x \notin X} \mathbf{V}_{\{\overline{x}\}}^{n+1}\right) \cdot {}^{n+1}1 \\ &= \otimes_{x \notin X} \left(\mathbf{V}_{\{\overline{x}\}}^{n+1} \cdot {}^{n+1}1\right), \end{aligned} \quad \text{(F.34)}$$

where $\otimes$ denotes the pointwise product of column vectors.

We may now consider two cases.

1. Let $X \subseteq \Omega^n$ (thus $x_{n+1} \notin X$). We have

$$\begin{aligned} \mathbf{V}_X^{n+1} \cdot {}^{n+1}1 &= \otimes_{x \notin X} \left(\mathbf{V}_{\{\overline{x}\}}^{n+1} \cdot {}^{n+1}1\right) \quad \text{(from (F.34))} \\ &= \left(\mathbf{V}_{\{\overline{x_{n+1}}\}}^{n+1} \cdot {}^{n+1}1\right) \otimes \left(\otimes_{\substack{x \notin X, \\ x \neq x_{n+1}}} \left(\mathbf{V}_{\{\overline{x}\}}^{n+1} \cdot {}^{n+1}1\right)\right) \quad \text{(since } x_{n+1} \notin X) \\ &= \begin{bmatrix} {}^{n}1 \\ {}^{n}(\alpha - 1) \end{bmatrix} \otimes \left(\otimes_{\substack{x \notin X, \\ x \neq x_{n+1}}} \left(\mathbf{V}_{\{\overline{x}\}}^{n+1} \cdot {}^{n+1}1\right)\right) \quad \text{(using (F.33))} \\ &= \begin{bmatrix} {}^{n}1 \\ {}^{n}(\alpha - 1) \end{bmatrix} \otimes \left(\otimes_{\substack{x \notin X, \\ x \neq x_{n+1}}} \left(\begin{bmatrix} \mathbf{V}_{\{\overline{x}\}}^{n} \cdot {}^{n}1 \\ \mathbf{V}_{\{\overline{x}\}}^{n} \cdot {}^{n}1 \end{bmatrix}\right)\right) \quad \text{(using (F.32))} \\ &= \begin{bmatrix} {}^{n}1 \\ {}^{n}(\alpha - 1) \end{bmatrix} \otimes \begin{bmatrix} \otimes_{\substack{x \notin X, \\ x \neq x_{n+1}}} \left(\mathbf{V}_{\{\overline{x}\}}^{n} \cdot {}^{n}1\right) \\ \otimes_{\substack{x \notin X, \\ x \neq x_{n+1}}} \left(\mathbf{V}_{\{\overline{x}\}}^{n} \cdot {}^{n}1\right) \end{bmatrix} \\ &= \begin{bmatrix} {}^{n}1 \\ {}^{n}(\alpha - 1) \end{bmatrix} \otimes \begin{bmatrix} \mathbf{V}_X^{n} \cdot {}^{n}1 \\ \mathbf{V}_X^{n} \cdot {}^{n}1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V}_X^{n} \cdot {}^{n}1 \\ (\alpha - 1)\left(\mathbf{V}_X^{n} \cdot {}^{n}1\right) \end{bmatrix} \\ &= \mathbf{M}^{n+1}(\cdot, X). \quad \text{(using (F.30))} \end{aligned}$$

2. Let $X \not\subseteq \Omega^n$ and $X \subseteq \Omega^{n+1}$ (thus $x_{n+1} \in X$). Thus $x \notin X$ is equivalent to $x \in \Omega^n \backslash (X \backslash x_{n+1})$. From (F.34) and (F.32), we have

$$\begin{aligned} \mathbf{V}_X^{n+1} \cdot {}^{n+1}1 &= \otimes_{x \in \Omega^n \backslash (X \backslash x_{n+1})} \left(\mathbf{V}_{\{\overline{x}\}}^{n+1} \cdot {}^{n+1}1\right) \\ &= \otimes_{x \in \Omega^n \backslash (X \backslash x_{n+1})} \left(\begin{bmatrix} \mathbf{V}_{\{\overline{x}\}}^{n} \cdot {}^{n}1 \\ \mathbf{V}_{\{\overline{x}\}}^{n} \cdot {}^{n}1 \end{bmatrix}\right). \end{aligned} \quad \text{(F.35)}$$

For all $X \nsubseteq \Omega^n$ and $X \subseteq \Omega^{n+1}$ we also have:

$$\begin{aligned} \mathbf{V}^n_{X \backslash x_{n+1}} &= \prod_{x \notin X \backslash x_{n+1}} \mathbf{V}^n_{\{\overline{x}\}} \\ &= \prod_{x \in \Omega^n \backslash (X \backslash x_{n+1})} \mathbf{V}^n_{\{\overline{x}\}}. \end{aligned} \tag{F.36}$$

Hence, using (F.35) and (F.36), we obtain

$$\begin{aligned} \mathbf{V}^{n+1}_X \cdot {}^{n+1}1 &= \begin{bmatrix} \mathbf{V}^n_{X \backslash x_{n+1}} \cdot {}^n1 \\ \mathbf{V}^n_{X \backslash x_{n+1}} \cdot {}^n1 \end{bmatrix} \\ &= \mathbf{M}^{n+1}(\cdot, X). \qquad \text{(using (F.30))} \end{aligned}$$

□

F.5 Proof of Proposition 6.6

This proof requires the following technical lemma.

Lemma F.7. *Let* $\mathbf{M}$ *be a* $2^n \times 2^n$ *matrix defined by*

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ *are* $2^{n-1} \times 2^{n-1}$ *matrices. Let* $\mathbf{J}^n$ *and* $\mathbf{J}^{n-1}$ *denote, respectively, the* $2^n \times 2^n$ *and* $2^{n-1} \times 2^{n-1}$ *matrices which elements are zeros except those on the secondary diagonal. We have*

$$\begin{aligned} \mathbf{J}^n \cdot \mathbf{M} &= \mathbf{J}^n \cdot \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{J}^{n-1} \cdot \mathbf{C} & \mathbf{J}^{n-1} \cdot \mathbf{D} \\ \mathbf{J}^{n-1} \cdot \mathbf{A} & \mathbf{J}^{n-1} \cdot \mathbf{B} \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{M} \cdot \mathbf{J}^n &= \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \cdot \mathbf{J}^n \\ &= \begin{bmatrix} \mathbf{B} \cdot \mathbf{J}^{n-1} & \mathbf{A} \cdot \mathbf{J}^{n-1} \\ \mathbf{D} \cdot \mathbf{J}^{n-1} & \mathbf{C} \cdot \mathbf{J}^{n-1} \end{bmatrix}, \end{aligned}$$

Proof. The proof is direct when one remarks that if $\mathbf{J}$ is placed before a matrix, it inverses the rows of this matrix, and if $\mathbf{J}$ is placed behind a matrix, it inverses the columns of this matrix. □

We may then show Proposition 6.6 as follows.

Proof. Let us first show that $\mathbf{G}^{\cup,\alpha}_{new}$ is a matrix of left eigenvectors of $\mathbf{K}^{\cup,\alpha}_{m}$: this follows from the fact that premultiplying a matrix (here the matrix $\mathbf{G}^{\cup,\alpha}_{smets}$) by the matrix $\mathbf{J}$ performs a permutation of the rows of this matrix.

Let us now show that $\mathbf{G}^{\cup,\alpha}_{new}$ generalizes $\mathbf{B}$. For that we merely need to show that the matrix $\mathbf{G}^{\cup,\alpha}_{new}$ is based on a building block that generalizes the building block of the matrix $\mathbf{B}$.

In order to demonstrate this, let us first remark that the matrix $\mathbf{G}^{\cup,\alpha}_{Smets}$ is based on the following building block (this can be shown in a similar manner as Theorem 6.3 was shown):

$$\begin{bmatrix} 1 & 1 \\ 1 & \alpha-1 \end{bmatrix} \tag{F.37}$$

In the remainder of this proof, let $\mathbf{G}^{n}_{smets}$ and $\mathbf{G}^{n}_{new}$ denote, respectively, the $2^n \times 2^n$ matrices $\mathbf{G}^{\cup,\alpha}_{Smets}$ and $\mathbf{G}^{\cup,\alpha}_{new}$ when the frame Ω has cardinality n. From the fact that the matrix $\mathbf{G}^{\cup,\alpha}_{Smets}$ is based on the building block given by (F.37), we may obtain

$$\mathbf{G}^{n+1}_{Smets} = \begin{bmatrix} \mathbf{G}^{n}_{Smets} & \mathbf{G}^{n}_{Smets} \\ \mathbf{G}^{n}_{Smets} & (\alpha-1)\mathbf{G}^{n}_{Smets} \end{bmatrix}.$$

Hence, we have:

$$\begin{aligned}
\mathbf{G}^{n+1}_{new} &= \mathbf{J}^{n+1} \cdot \mathbf{G}^{n+1}_{Smets} \\
&= \mathbf{J}^{n+1} \cdot \begin{bmatrix} \mathbf{G}^{n}_{Smets} & \mathbf{G}^{n}_{Smets} \\ \mathbf{G}^{n}_{Smets} & (\alpha-1)\mathbf{G}^{n}_{Smets} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{J}^{n} \cdot \mathbf{G}^{n}_{Smets} & (\alpha-1)\mathbf{J}^{n} \cdot \mathbf{G}^{n}_{Smets} \\ \mathbf{J}^{n} \cdot \mathbf{G}^{n}_{Smets} & \mathbf{J}^{n} \cdot \mathbf{G}^{n}_{Smets} \end{bmatrix} \quad \text{(By Lemma F.7)} \\
&= \mathbf{Kron}\left(\begin{bmatrix} 1 & \alpha-1 \\ 1 & 1 \end{bmatrix}, \mathbf{J}^{n} \cdot \mathbf{G}^{n}_{Smets} \right) \\
&= \mathbf{Kron}\left(\begin{bmatrix} 1 & \alpha-1 \\ 1 & 1 \end{bmatrix}, \mathbf{G}^{n}_{new} \right)
\end{aligned} \tag{F.38}$$

The matrix $\mathbf{G}^{\cup,\alpha}_{new}$ is thus a matrix based on a building block that generalizes the building block of the matrix $\mathbf{B}$. □

F.6 Proof of Theorem 7.1

The proof of Theorem 7.1 requires the following technical lemma.

Lemma F.8. *(Inspired from [77, Lemma 5.2])*

Suppose Ω *is a finite set and* $sq : 2^\Omega \to (-\infty,+\infty)\setminus\{0\}$ *and* $sw : 2^\Omega\setminus\{\Omega\} \to (-\infty,+\infty)\setminus\{0\}$ *are two functions such that* $sq(\emptyset) = 1$. *Then*

$$sq(A) = \prod_{B \not\supseteq A, B\neq\Omega} sw(B) \tag{F.39}$$

for all $A \subseteq \Omega$ *if and only if*

$$sw(A) = \prod_{B\supseteq A} sq(B)^{(-1)^{|B|-|A|+1}} \tag{F.40}$$

for all $A \subset \Omega$.

Proof.

1. Suppose (F.40) holds. Then, for a given $A \subseteq \Omega$, we have (the following equation is inspired by an equation in the proof of [39, Theorem 1]):

$$\begin{aligned}\prod_{B \not\supseteq A, B \neq \Omega} sw(B) &= \prod_{B \not\supseteq A, B \neq \Omega} sw(B) \cdot \left(\prod_{B \supseteq A, B \neq \Omega} sw(B)\right) / \left(\prod_{B \supseteq A, B \neq \Omega} sw(B)\right) \\ &= \left(\prod_{B \subset \Omega} sw(B)\right) / \left(\prod_{B \supseteq A, B \neq \Omega} sw(B)\right) \qquad \text{(F.41)}\end{aligned}$$

Let us study the term $\prod_{B \subset \Omega} sw(B)$ of (F.41). We have, using (F.40):

$$\prod_{B \subset \Omega} sw(B) = \prod_{B \subset \Omega} \left(\prod_{C \supseteq B} sq(C)^{(-1)^{|C|-|B|+1}}\right) \qquad \text{(F.42)}$$

A given $C \in 2^{\Omega} \setminus \{\emptyset\}$ with cardinality $|C| = n$ has $\begin{pmatrix} n \\ 0 \end{pmatrix}$ subsets with cardinality 0, $\begin{pmatrix} n \\ 1 \end{pmatrix}$ subsets with cardinality 1, and more generally $\begin{pmatrix} n \\ k \end{pmatrix}$ subsets with cardinality $k \leq n$. The binomial theorem shows the following equality:

$$(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^k b^{n-k}. \qquad \text{(F.43)}$$

Let $b = 1$ and $a = -1$. From (F.43), we have:

$$\begin{aligned}(1-1)^n &= \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \\ &= \begin{pmatrix} n \\ 0 \end{pmatrix} - \begin{pmatrix} n \\ 1 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix} - \begin{pmatrix} n \\ 3 \end{pmatrix} + \dots + (-1)^n \begin{pmatrix} n \\ n \end{pmatrix} \\ &= 0\end{aligned}$$

It follows from this last equation that a given set $C \in 2^{\Omega} \setminus \{\emptyset\}$ has as many subsets of even cardinality as subsets of odd cardinality. Let us remark that it may also be shown that a given set $C \in 2^{\Omega} \setminus \{\Omega\}$ has as many supersets of even cardinality as supersets of odd cardinality.

Let B and C be two subsets of Ω such that $C \supseteq B$, and x be a non null real number.

   (i) If C has odd cardinality and B has even cardinality, or if C has even cardinality and B has odd cardinality, then $x^{(-1)^{|C|-|B|+1}} = x$;

   (ii) If C has odd cardinality and B has odd cardinality, or if C has even cardinality and B has even cardinality, then $x^{(-1)^{|C|-|B|+1}} = 1/x$

Now, let us consider the right term of (F.42). For a given set $C \in 2^{\Omega} \setminus \{\Omega, \emptyset\}$, the term $sq(C)^{(-1)^{|C|-|B|+1}}$ appears in (F.42) as many times as there are sets B such that $B \subseteq C$. Suppose the cardinality of C is even. Since there are as many $B \subseteq C$ of even cardinality and of odd cardinality and because of (i) and (ii), the term $sq(C)$ appears as many times as the term $1/sq(C)$ in (F.42). The same can be said if the cardinality of C is odd. Hence the term $sq(C)^{(-1)^{|C|-|B|+1}}$ vanishes from (F.42) for all $C \in 2^{\Omega} \setminus \{\Omega, \emptyset\}$.

When $C = \emptyset$, there is a unique set B such that $B \subseteq C$, i.e., $B = \emptyset$. Thus the term $sq(\emptyset)^{(-1)^{|\emptyset|-|\emptyset|+1}} = 1/sq(\emptyset)$ appears once in (F.42).

When $C = \Omega$, the term $sq(\Omega)^{(-1)^{|\Omega|-|B|+1}}$ appears in (F.42) as many times as there are sets B such that $B \subset \Omega$. Since $B \neq \Omega$, there will be one missing $sq(\Omega)^{(-1)^{|\Omega|-|\Omega|+1}} = 1/sq(\Omega)$ to cancel a term $sq(\Omega)$. This implies that the term $sq(\Omega)$ appears once in (F.42).

We thus have

$$\begin{aligned}
\prod_{B\subset\Omega} sw(B) &= \prod_{B\subset\Omega}\left(\prod_{C\supseteq B} sq(C)^{(-1)^{|C|-|B|+1}}\right) \\
&= \frac{sq(\Omega)}{sq(\emptyset)} \\
&= sq(\Omega),
\end{aligned} \tag{F.44}$$

since $sq(\emptyset) = 1$.

Using a similar reasoning, it may be shown that the term $\prod_{B\supseteq A, B\neq\Omega} sw(B)$ of (F.41) is equal to $\frac{sq(\Omega)}{sq(A)}$. This is easily seen when one rewrites $\prod_{B\subset\Omega} sw(B)$ into $\prod_{B\supseteq\emptyset, B\neq\Omega} sw(B)$ and compares this last term with $\prod_{B\supseteq A, B\neq\Omega} sw(B)$.

Consequently, we have

$$\begin{aligned}
\prod_{B\not\supseteq A, B\neq\Omega} sw(B) &= sq(\Omega)/\left(sq(\Omega)/sq(A)\right) \\
&= sq(A)
\end{aligned}$$

2. Suppose (F.39) holds. Then, for a given $A \subset \Omega$, we have

$$\begin{aligned}
&\prod_{B \supseteq A} sq(B)^{(-1)^{|B|-|A|+1}} \\
&= \prod_{B \supseteq A} \left(\prod_{C \not\supseteq B, C \neq \Omega} sw(C) \right)^{(-1)^{|B|-|A|+1}} \\
&= \prod_{B \supseteq A} \left(\prod_{C \not\supseteq B, C \neq \Omega} sw(C) \cdot \frac{\prod_{C \supseteq B, C \neq \Omega} sw(C)}{\prod_{C \supseteq B, C \neq \Omega} sw(C)} \right)^{(-1)^{|B|-|A|+1}} \\
&= \prod_{B \supseteq A} \left(\frac{\prod_{C \subset \Omega} sw(C)}{\prod_{C \supseteq B, C \neq \Omega} sw(C)} \right)^{(-1)^{|B|-|A|+1}} \\
&= \prod_{B \supseteq A} \left(\left(\prod_{C \subset \Omega} sw(C) \right)^{(-1)^{|B|-|A|+1}} \cdot \frac{1}{\left(\prod_{C \supseteq B, C \neq \Omega} sw(C) \right)^{(-1)^{|B|-|A|+1}}} \right) \\
&= \prod_{B \supseteq A} \left(\prod_{C \subset \Omega} sw(C) \right)^{(-1)^{|B|-|A|+1}} \cdot \prod_{B \supseteq A} \left(\frac{1}{\left(\prod_{C \supseteq B, C \neq \Omega} sw(C) \right)^{(-1)^{|B|-|A|+1}}} \right) \\
&= \prod_{B \supseteq A} \left(\prod_{C \subset \Omega} sw(C) \right)^{(-1)^{|B|-|A|+1}} \cdot \prod_{B \supseteq A} \left(\frac{1}{\prod_{C \supseteq B, C \neq \Omega} sw(C)^{(-1)^{|B|-|A|+1}}} \right)
\end{aligned} \tag{F.45}$$

Let us study the term $\prod_{B \supseteq A} \left(\prod_{C \subset \Omega} sw(C) \right)^{(-1)^{|B|-|A|+1}}$ of (F.45). It can be equivalently written $\prod_{B \supseteq A} x^{(-1)^{|B|-|A|+1}}$, with $x = \prod_{C \subset \Omega} sw(C)$. Since $A \subset \Omega$ has as many supersets with even cardinality as supersets with odd cardinality and because of (i) and (ii), it is clear that the term $\prod_{B \supseteq A} x^{(-1)^{|B|-|A|+1}}$ vanishes. We thus have

$$\prod_{B \supseteq A} sq(B)^{(-1)^{|B|-|A|+1}} = \prod_{B \supseteq A} \left(\frac{1}{\prod_{C \supseteq B, C \neq \Omega} sw(C)^{(-1)^{|B|-|A|+1}}} \right) \tag{F.46}$$

For a given set $C \neq \Omega$ and $C \supseteq A$, the term $\frac{1}{sw(C)^{(-1)^{|B|-|A|+1}}}$ appears as many times in (F.46) as there are sets B such that $A \subseteq B$ and $B \subseteq C$.

Suppose the cardinality of A is even and $C \supset A$. Since there are as many sets B, $A \subseteq B$ and $B \subseteq C$, that are of even cardinality and of odd cardinality[1] and because of (i) and (ii), there are as many terms $sw(C)$ as terms $1/sw(C)$ in

[1] The conditions $A \subseteq B$ and $B \subseteq C$ that B must respect can be transformed into a condition $E \subseteq D$ to be respected by a set E with $E = B - A$ and $D = C - A$, because $A - A = \emptyset \subseteq B - A = E$ and $B - A = E \subseteq C - A = D$. From the first part of this proof, we know that D, which is $\neq \emptyset$ (because $C \supset A$), has as many subsets E of even cardinality as subsets E of odd cardinality.

(F.46). The same can be said if the cardinality of A is odd. Hence, the term $\frac{1}{sw(C)^{(-1)^{|B|-|A|+1}}}$ vanishes from (F.46), for all $C \neq \Omega$ and $C \supset A$.

When $C = A$, there is only one set B such that $A \subseteq B \subseteq C$, i.e. $B = A$. Hence

$$\begin{aligned}\prod_{B \supseteq A} \left(\frac{1}{\prod_{C \supseteq B, C \neq \Omega} sw(C)^{(-1)^{|B|-|A|+1}}} \right) &= \frac{1}{\frac{1}{sw(A)}} \\ &= sw(A).\end{aligned}$$

□

Theorem 7.1 can then be proved as follows.

Proof. (Inspired from the proof of [85, Theorem 1])

Since sm is sq-invertible, sq satisfies $sq(A) \neq 0$ for all $A \subseteq \Omega$. Besides, it also verifies $sq(\emptyset) = 1$, since sm is regular. From Lemma F.8, there exists a function $sw : 2^{\Omega} \setminus \{\Omega\} \rightarrow (-\infty, +\infty) \setminus \{0\}$ defined by (F.40) such that

$$\begin{aligned}sq(A) &= \prod_{B \not\supseteq A} sw(B) \\ &= \prod_{B \subset \Omega} sq_B(A), \qquad \text{(F.47)}\end{aligned}$$

where sq_B is the signed commonality function of a simple BSMA $B^{sw(B)}$, i.e., such that:

$$sq_B(A) = \begin{cases} 1 & \text{if } A \subseteq B, \\ sw(B) & \text{otherwise.} \end{cases} \qquad \text{(F.48)}$$

From (F.47), one obtains (7.5). □

F.7 Proof of Proposition D.1

Proof. We have

$$\left(\left(m^{\Omega}[r1]^{\Uparrow R1 \times \Omega} \right)^{\uparrow R1 \times R2 \times \Omega} \right)(D) = m_1^{\Omega}(A), \quad \forall A \subseteq \Omega,$$

where

$$D = \left\{ (A \times (r1, r2)) \cup (A \times (r1, \overline{r2})) \cup (\Omega \times (\overline{r1}, r2)) \cup (\Omega \times (\overline{r1}, \overline{r2})) \right\},$$

and

$$\left(\left(m^{\Omega}[r2]^{\Uparrow R2 \times \Omega} \right)^{\uparrow R1 \times R2 \times \Omega} \right)(E) = m_1^{\Omega}(B), \quad \forall B \subseteq \Omega,$$

where

$$E = \left\{ (B \times (r1, r2)) \cup (B \times (\overline{r1}, r2)) \cup (\Omega \times (r1, \overline{r2})) \cup (\Omega \times (\overline{r1}, \overline{r2})) \right\},$$

and

$$(m_{xor}^{R1\times R2\uparrow R1\times R2\times\Omega})(F) \;=\; 1$$

where

$$F \;=\; \left\{(\Omega\times(\overline{r1},r2))\cup(\Omega\times(r1,\overline{r2}))\right\},$$

and

$$(m_{or}^{R1\times R2\uparrow R1\times R2\times\Omega})(F) \;=\; 1$$

where

$$G \;=\; \left\{(\Omega\times(\overline{r1},r2))\cup(\Omega\times(r1,\overline{r2}))\cup(\Omega\times(r1,r2))\right\}.$$

Let

$$D \;=\; \left\{(A\times(r1,r2))\cup(A\times(r1,\overline{r2}))\cup(\Omega\times(\overline{r1},r2))\cup(\Omega\times(\overline{r1},\overline{r2}))\right\},$$

for some $A\subseteq\Omega$.

Let

$$E \;=\; \left\{(B\times(r1,r2))\cup(B\times(\overline{r1},r2))\cup(\Omega\times(r1,\overline{r2}))\cup(\Omega\times(\overline{r1},\overline{r2}))\right\},$$

for some $B\subseteq\Omega$.

Let

$$F \;=\; \left\{(\Omega\times(\overline{r1},r2))\cup(\Omega\times(r1,\overline{r2}))\right\}.$$

Let

$$G \;=\; \left\{(\Omega\times(\overline{r1},r2))\cup(\Omega\times(r1,\overline{r2}))\cup(\Omega\times(r1,r2))\right\}.$$

We have

$$\begin{aligned}(D\cap E\cap F)\downarrow\Omega &= (A\cap\Omega)\cup(\Omega\cap B)\\ &= A\cup B,\end{aligned}$$

and

$$\begin{aligned}(D\cap E\cap G)\downarrow\Omega &= (A\cap\Omega)\cup(\Omega\cap B)\cup(A\cap B)\\ &= A\cup B.\end{aligned}$$

Following a similar reasoning to the one at the end of the proof of Theorem 6.1 (see Appendix F.3), we obtain, for all $X\subseteq\Omega$

$$\begin{aligned}&\left(\left(m^{\Omega}[r1]^{\Uparrow R1\times\Omega}\right)^{\uparrow R1\times R2\times\Omega} \textcircled{\cap} \left(m^{\Omega}[r2]^{\Uparrow R2\times\Omega}\right)^{\uparrow R1\times R2\times\Omega} \textcircled{\cap} m_{xor}^{R1\times R2\uparrow R1\times R2\times\Omega}\right)^{\downarrow\Omega}(X)\\ &= \sum_{A\cup B=X} m_1(A)m_2(B)\\ &= (m_1 \textcircled{\cup} m_2)(X)\end{aligned}$$

and

$$\begin{aligned}
& \left(\left(m^{\Omega}[r1]^{\Uparrow R1\times\Omega}\right)^{\uparrow R1\times R2\times\Omega} \cap \left(m^{\Omega}[r2]^{\Uparrow R2\times\Omega}\right)^{\uparrow R1\times R2\times\Omega} \cap m_{or}^{R1\times R2\uparrow R1\times R2\times\Omega}\right)^{\downarrow\Omega}(X) \\
= & \sum_{A\cup B=X} m_1(A)m_2(B) \\
= & (m_1 \cup m_2)(X).
\end{aligned}$$

□

F.8 Proof of Proposition D.2

Proof. This proposition can be proved rapidly using two degenerate cases of Theorem 6.2. However, it is perhaps more instructive to demonstrate it in the style of the proof of Proposition D.1, in order to enhance the differences between the approach based on the reliability of the sources and the approach based on the truthfulness of the sources.

We have for all $A \subseteq \Omega$,

$$m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2}(D) = m_1^{\Omega}(A),$$

where

$$D = \left\{(A\times(t1,t2)) \cup (A\times(t1,f2)) \cup (\overline{A}\times(f1,t2)) \cup (\overline{A}\times(f1,f2))\right\}$$

and for all $B \subseteq \Omega$,

$$m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2}(E) = m_2^{\Omega}(B),$$

where

$$E = \left\{(B\times(t1,t2)) \cup (\overline{B}\times(t1,f2)) \cup (B\times(f1,t2)) \cup (\overline{B}\times(f1,f2))\right\},$$

and

$$(m_{xor}^{T1\times T2\uparrow T1\times T2\times\Omega})(F) = 1$$

where

$$F = \{(\Omega\times(f1,t2)) \cup (\Omega\times(t1,f2))\},$$

and

$$(m_{or}^{T1\times T2\uparrow T1\times T2\times\Omega})(G) = 1$$

where

$$G = \{(\Omega\times(f1,t2)) \cup (\Omega\times(t1,f2)) \cup (\Omega\times(t1,t2))\}.$$

Let

$$D = \left\{(A \times (t1, t2)) \cup (A \times (t1, f2)) \cup (\overline{A} \times (f1, t2)) \cup (\overline{A} \times (f1, f2))\right\},$$

for some $A \subseteq \Omega$. Let

$$E = \left\{(B \times (t1, t2)) \cup (\overline{B} \times (t1, f2)) \cup (B \times (f1, t2)) \cup (\overline{B} \times (f1, f2))\right\},$$

for some $B \subseteq \Omega$. Let

$$F = \{(\Omega \times (f1, t2)) \cup (\Omega \times (t1, f2))\}.$$

Let

$$G = \{(\Omega \times (f1, t2)) \cup (\Omega \times (t1, f2)) \cup (\Omega \times (t1, t2))\}.$$

We have

$$(D \cap E \cap F) \downarrow \Omega = (A \cap \overline{B}) \cup (\overline{A} \cap B)$$

and

$$\begin{aligned}(D \cap E \cap G) \downarrow \Omega &= (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (A \cap B) \\ &= A \cup B. \qquad \text{(from Remark 6.2)}\end{aligned}$$

Following a similar reasoning to the one at the end of the proof of Theorem 6.1 (see Appendix F.3), we obtain, for all $X \subseteq \Omega$

$$\begin{aligned}&(m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \circledcap m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \circledcap m_{xor}^{T1\times T2\uparrow\Omega\times T1\times T2})^{\downarrow\Omega}(X) \\ = &\sum_{(A\cap\overline{B})\cup(\overline{A}\cap B)=X} m_1(A)m_2(B) \\ = &(m_1 \underline{\circledcup} m_2)(X)\end{aligned}$$

and

$$\begin{aligned}&(m_{1'}^{\Omega\times T1\uparrow\Omega\times T1\times T2} \circledcap m_{2'}^{\Omega\times T2\uparrow\Omega\times T1\times T2} \circledcap m_{or}^{T1\times T2\uparrow\Omega\times T1\times T2})^{\downarrow\Omega}(X) \\ = &\sum_{A\cup B=X} m_1(A)m_2(B) \\ = &(m_1 \circledcup m_2)(X).\end{aligned}$$

□

Bibliography

[1] Aregui, A.: Détection de nouveauté dans le cadre de la théorie des fonctions de croyance. Application à la surveillance d'un système d'incinération de déchets. English title: Novelty detection in the belief function framework, with application to the monitoring of a waste incineration process. PhD thesis (in English), Université de Technologie de Compiègne, France (2007)

[2] Ayoun, A., Smets, Ph.: Data association in multi-target detection using the transferable belief model. Int. J. Intell. Syst. 10(16), 1167–1182 (2001)

[3] Baudrit, C., Couso, I., Dubois, D.: Joint propagation of probability and possibility in risk analysis: Towards a formal framework. Int. J. Approx. Reasoning 45, 82–105 (2007)

[4] Benavoli, A., Chisci, L., Ristic, B., Farina, A., Graziano, A.: Reasoning under uncertainty: from Bayes to Valuation Based Systems. Application to target classification and threat evaluation. Selex Sistemi Integrati, Università degli Studi di Firenze, Italy (2007)

[5] Ben Yaghlane, A., Denœux, T., Mellouli, K.: Constructing Belief Functions from Qualitative Expert Opinions. In: Proceeding of ICTTA'06, Damascus, Syria (2006)

[6] Bloch, I.: Incertitude, imprécision et additivité en fusion de données: point de vue historique. Revue Traitement du Signal 13(4), 267–288 (1996)

[7] Cattaneo, M. E. G. V.: Combining belief functions issued from dependent sources. In: Bernard, J. M., Seidenfeld, T., Zaffalon, M. (eds.) Proceedings of the Third International Symposium on Imprecise Probabilities and Their Applications (ISIPTA'03), Lugano, Switzerland, 2003, pp 133–147. Carleton Scientific (2003)

[8] Choquet, G.: Theory of capacities. Annales de l'Institut Fourier 5, 131–296. Université de Grenoble (1953)

[9] Cobb, B. R., Shenoy, P. P.: On the plausibility transformation method for translating belief function models to probability models. Int. J. Approximate Reasoning 41, 314–330 (2006)

[10] Delgado, M., Moral, S.: A definition of inclusion for evidence. Fuzzy Mathematics 7, 81–87 (1987)

[11] Dempster, A. P.: Upper and lower probabilities induced by a multivalued mapping. Ann. Math. Statistics 38, 325–339 (1967)

[12] Denœux, T.: A k-nearest neighbor classification rule based on Dempster-Shafer theory. IEEE Transactions on Systems, Man and Cybernetics 25(05), 804–813 (1995)

[13] Denœux, T.: Application du modèle des Croyances Transférables en Reconnaissance de Formes. Traitement du Signal 14(5), 443–451 (1998)

[14] Denœux, T.: A neural network classifier based on Dempster-Shafer theory. IEEE transactions on Systems, Man and Cybernetics A 30(2), 131–150 (2000)

[15] Denœux, T.: Modeling vague beliefs using fuzzy-valued belief structures. Fuzzy Sets and Systems 116(2), 167–199 (2000)

[16] Denœux, T.: The cautious rule of combination for belief functions and some extensions. In: Proceedings of the 9th International Conference on Information Fusion (FUSION'2006), Florence, Italy, July (2006)

[17] Denœux, T.: Constructing Belief Functions from Sample Data Using Multinomial Confidence Regions. Int. J. Approximate Reasoning 42(3), 228–252 (2006)

[18] Denœux, T.: Conjunctive and Disjunctive Combination of Belief Functions Induced by Non Distinct Bodies of Evidence. Artificial Intelligence 172, 234–264 (2008)

[19] Denœux, T.: Théorie des fonctions de croyance: Fusion d'informations. Lecture notes SIT 53: Fusion d'informations, Université de Technologie de Compiègne, accessed online on 23 October 2008 at http://www.hds.utc.fr/sit53/

[20] Denœux, T., Smets, Ph.: Classification using Belief Functions: the Relationship between the Case-based and Model-based Approaches. IEEE Transactions on Systems, Man and Cybernetics B 36(6), 1395–1406 (2006)

[21] Destercke, S., Dubois, D., Chojnacki, E.: Cautious Conjunctive Merging of Belief Functions. In: Mellouli, K. (ed.) Proceedings of the 9th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'07), Hammamet, Tunisia, 2007. Lecture Notes in Computer Science. Lecture Notes in Artificial Intelligence, vol. 4724, pp 332–343. Springer–Verlag (2007)

[22] Dietterich, T.: Approximate statistical tests for comparing supervised classification algorithms. Neural Computation 10(7), 1895–1923 (1998)

[23] Dubois, D., Grabisch, M., Prade, H., Smets, Ph.: Using the transferable belief model and a qualitative possibility theory approach on an illustrative example: the assessment of the value of a candidate. Int. J. of Intelligent Systems 16(11), 1245–1272 (2001)

[24] Dubois, D., Prade, H.: On several representations of an uncertain body of evidence. In: Gupta, M. M., Sanchez, E. (eds.) Fuzzy Information and Decision Processes, pp. 167–181. North-Holland (1982)

[25] Dubois, D., Prade, H.: On the Unicity of Dempster Rule of Combination. Int. J. Intell. Syst. 1, 133–142 (1986)

[26] Dubois, D., Prade, H.: A set-theoretic view of belief functions: logical operations and approximations by fuzzy sets. Int. J. Gen Syst 12(3), 193–226 (1986)

[27] Dubois, D., Prade, H.: Possibility Theory: An approach to computerized processing of uncertainty. Plenum Press, New-York (1988)

[28] Dubois, D., Prade, H.: Representation and combination of uncertainty with belief functions and possibility measures. Computational Intelligence 4, 244–264 (1988)

[29] Dubois, D., Prade, H.: La fusion d'informations imprécises. Traitement du Signal, 11(6), 447–458 (1994)

[30] Dubois, D., Prade, H.: Possibility theory in information fusion. In: Della Riccia, G., Lenz, H., Kruse, R. (eds.) Data Fusion and Perception. CISM Courses and Lectures, Vol. 431, pp. 53–76. Springer, Berlin (2001)

[31] Dubois, D., Prade, H.: On the use of aggregation operations in information fusion processes. Fuzzy Sets and Systems, 142(1), 143–161 (2004)

[32] Dubois, D., Prade, H., Smets, Ph.: New semantics for quantitative possibility theory. In: Benferhat, S., Besnard, Ph. (eds.) Proceedings of the 6th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'01), Toulouse, France, 2001. Lecture Notes in Computer Science, vol. 2143, pp 410–421. Springer–Verlag (2001)

[33] Dubois, D., Prade, H., Smets, Ph.: "Not Impossible" vs. "Guaranteed Possible" in Fusion and Revision. In: Benferhat, S., Besnard, Ph. (eds.) Proceedings of the 6th European Conference on Symbolic and Quantitative Approaches to

Reasoning with Uncertainty (ECSQARU'01), Toulouse, France, 2001. Lecture Notes in Computer Science, vol. 2143, pp 522–531. Springer–Verlag (2001)

[34] Dubois, D., Prade, H., Smets, Ph.: A Definition of Subjective Possibility. Int. J. Approximate Reasoning 48(2), 352–364 (2008)

[35] Duda, R. O., Hart, P. E., Stork, D. G.: Pattern Classification. John Wiley and Sons, New-York (2001)

[36] Elouedi, Z., Mellouli, K.: Pooling dependent expert opinions using the theory of evidence. In: Proceedings of the Seventh Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'98), Paris, France, 1998. Congres Internationaux, vol. 1, pp 32–39. E.D.K., France (1998)

[37] Faux, F., Luthon, F.: Étude de différentes règles de fusion d'information couleur appliquées à la détection d'un visage en temps réel. In: Recontres Francophones sur la Logique Floue et ses Applications (LFA '07), Nîmes, France, pp 9–16. Cépaduès-Editions (2007)

[38] Ginsberg, M. L.: Non-Monotonic Reasoning Using Dempster's Rule. In: Proceedings of the National Conference on Artificial Intelligence (AAAI'84), Austin, Texas, USA, pp 126–129 (1984)

[39] Grabisch, M.: Belief functions on lattices. Int. J. of Intelligent Systems, to appear.

[40] Ha-Duong, M.: Hierarchical fusion of expert opinion in the Transferable Belief Model, application to climate sensitivity. Int. J. Approximate Reasoning 49(3), 555–574 (2008)

[41] Haenni, R.: Uncover Dempster's Rule Where It Is Hidden. In: Proceedings of the 9th International Conference on Information Fusion (FUSION'2006), Florence, Italy, July (2006)

[42] Hájek, P.: Deriving Dempster's rule. In: Valverde, L., Yager, R. R. (eds.) Uncertainty in Intelligent Systems, pp 75–83. North-Holland, Amsterdam (1993)

[43] Hájek, P., Havránek, T., Jirousek, R.: Uncertain Information Processing in Expert Systems. CRC Press, Boca Raton, Florida (1992)

[44] Hájek, P., Valdes, J. V.: A generalized algebraic approach to uncertainty processing in rule-based expert systems (dempsteroids). Computers and Artificial Intelligence 10(1), 29–42 (1991)

[45] Joy, K. I.: On-Line Geometric Modeling Notes. Eigenvalues and eigenvectors. Computer Science Department, University of California, Davis (2000). Accessed online on 11 September 2008 at http://graphics.cs.ucdavis.edu/education/CAGDNotes/Eigenvalues.pdf.

[46] Kennes, R., Smets, Ph.: Computational aspects of the Möbius transformation. In: Bonissone, P.P., Henrion, M., Kanal, L. N., Lemmer, J. F. (eds.) Proceedings of the 6th Conference on Uncertainty in Artificial Intelligence (UAI'90), pp 401–416. Elsevier Science, New York (1990)

[47] Klawonn, F., Schwecke, E.: On the axiomatic justification of Dempster's rule of combination. Int. J. Intell. Syst. 7(5), 469–478 (1992)

[48] Klawonn, F., Smets, Ph.: The dynamic of belief in the transferable belief model and specialization-generalization matrices. In: Dubois, D., Wellman, M.P., D'Ambrosio, B., Smets, Ph. (eds.) Proceedings of the 8th conference on Uncertainty in Artificial Intelligence (UAI'92), pp 130–137. Morgan Kaufmann, San Meteo, CA (1992)

[49] Klement, E. P., Mesiar, R., Pap, E.: Triangular norms. Kluwer, Dordrecht (2000)

[50] Klir, G. J.: Uncertainty and Information: Foundations of Generalized Information Theory. John Wiley, Hoboken, NJ (2006)

[51] Kohlas, J., Shenoy, P. P.: Computation in Valuation Algebras. In: Gabbay, D. M., Smets, Ph. (eds.) Handbook of Defeasible Reasoning and Uncertainty Management Systems: Algorithms for Uncertainty and Defeasible Reasoning, vol. 5, pp 5–39. Kluwer, Dordrecht (2000)

[52] Kramosil, I.: Measure-theoretic approach to the inversion problem for belief functions. Fuzzy Sets and Systems 102(3), 363–369 (1999)

[53] Kramosil, I.: Probabilistic Analysis of Belief Functions. IFSR Int. Series on Systems Science and Engineering, vol. 16. Kluwer, New York (2001)

[54] Labreuche, Ch., Grabisch, M.: Modeling positive and negative pieces of evidence in uncertainty. In: Nielsen, T. D., Zhang, N. L. (eds.) Proceedings of the 7th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'03), Aalborg, Denmark, 2003. Lecture Notes in Computer Science, vol. 2711, pp 279–290. Springer–Verlag (2003)

[55] Lauritzen, S. L., Spiegelhalter, D. J.: Local Computations with Probabilities on Graphical Structures and their Application to Expert Systems. J. of Royal Stat. Soc. 50(2), 157–224 (1988)

[56] Ling, X. N., Rudd, W. G.: Combining opinions from several experts. Applied Artificial Intelligence 3, 439–452 (1989)

[57] Mercier, D., Cron, G., Denœux, T., Masson, M.-H.: Fusion of multi-level decision systems using the transferable belief model. In: Proceedings of the 8th Int. Conf. on Information Fusion (FUSION'2005), Philadelphia, USA, paper C8-2, July 25-29 (2005)

[58] Mercier, D., Cron, G., Denœux, T., Masson, M.-H.: Decision fusion for postal address recognition using belief functions. Expert Systems with Applications 36(3), 5643–5653 (2009)

[59] Mercier, D., Quost, B., Denœux, T.: Refined modeling of sensor reliability in the belief function framework using contextual discounting. Information Fusion 9, 246–258 (2008)

[60] Miranda, E., Couso, I., Gil, P.: A random set characterisation of possibility measures. Information Sciences 168, 51–75 (2004)

[61] Miranda, E., De Cooman, G., Couso, I.: Lower previsions induced by multi-valued mappings. J. Statist. Plann. Inference 133, 173–197 (2005)

[62] Moral, S.: Información difusa, relaciones entre probabilidad y posibilidad. PhD thesis (in Spanish), University of Granada, Spain (1985)

[63] Pichon, F., Denœux, T.: On Latent Belief Structures. In: Mellouli, K. (ed.) Proceedings of the 9th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'07), Hammamet, Tunisia, 2007. Lecture Notes in Computer Science. Lecture Notes in Artificial Intelligence, vol. 4724, pp 368–380. Springer–Verlag (2007)

[64] Pichon, F., Denœux, T.: A new justification of the unnormalized Dempster's rule of combination from the Least Commitment Principle. In: Proceedings of the 21th Florida Artificial Intelligence Research Society Int. Conference (FLAIRS'08), Special Track on Uncertain Reasoning, Coconut Grove, Florida, USA, pp 666–671, 15-17 May (2008)

[65] Pichon, F., Denœux, T.: T-norm and Uninorm-Based Combination of Belief Functions. In: Proceedings of the North American Fuzzy Information Processing Society Int. Conference (NAFIPS'08), New York, USA, 19-22 May (2008). Best Student Paper Award.

[66] Pichon, F., Denœux, T.: A new singular property of the unnormalized Dempster's rule among uninorm-based combination rules. In: Magdalena, L., Ojeda-Aciego, M., Verdegay, J.L. (eds) Proceedings of the 12th Int. Conf. on

Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'08), pp 314–321, Torremolinos (Malaga), Spain, 22-27 June (2008)

[67] Pichon, F., Josset, F.-X., Denœux, T.: Structures de croyance latente: Règles de combinaison conjonctive et prise de décision. In: Rencontres Francophones sur la Logique Floue et ses Applications (LFA'07), pp 91–98, Nîmes, France. Cépaduès-Editions (2007)

[68] Pouly, M.: A generic Architecture for Local Computation. PhD thesis, Department of Informatics, University of Fribourg, Switzerland (2008)

[69] Quost, B., Denœux, T., Masson, M.-H.: Pairwise classifier combination using belief functions. Pattern Recognition Letters 28(5), 644–653 (2007)

[70] Quost, B., Denœux, T., Masson, M.-H.: Adapting a combination rule to non-independent information sources. In: Magdalena, L., Ojeda-Aciego, M., Verdegay, J.L. (eds) Proceedings of the 12th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'08), pp 448–455, Torremolinos (Malaga), Spain, 22-27 June (2008)

[71] Quost, B., Denœux, T., Masson, M.-H.: Refined classifier combination using belief functions. In: Proceedings of the 11th Int. Conf. on Information Fusion (FUSION'08), pp 776–782, Cologne, Germany, 30 June - 03 July (2008)

[72] Ristic, B., Smets, Ph.: Target identification using belief functions and implication rules. IEEE Transactions on Aerospace and Electronic Systems 41(3), 1097–1103 (2005)

[73] Rota, G. C.: On the foundations of combinatorial theory I. Theory of Möbius functions. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 2, 340–368 (1964)

[74] Savage, L. J.: Foundations of statistics. Wiley, New York (1954).

[75] Schubert, J.: Clustering decomposed belief functions using generalized weights of conflict. Int. J. Approximate Reasoning 48(2), 466–480 (2008)

[76] Sentz, K., Ferson, S.: Combination of evidence in Dempster-Shafer Theory. Technical report, SANDIA Tech. Report, SAND2002-0835 (2002)

[77] Shafer, G.: A mathematical theory of evidence. Princeton University Press, Princeton, N.J. (1976)

[78] Shenoy, P. P.: Conditional independence in valuation-based systems. Int. J. Approximate Reasoning 10, 203–234 (1994)

[79] Shenoy, P. P., Shafer, G.: Axioms for probability and belief-function propagation. In: Shafer, G., Pearl, J. (eds) Readings in uncertain reasoning, pp 575–610. Morgan Kaufmann, San Mateo, CA (1990)

[80] Smets, Ph.: Un modèle mathématico-statistique simulant le processus du diagnostic médical. PhD thesis (in French), Université Libre de Bruxelles, Belgium (1978)

[81] Smets, Ph.: Information Content of an Evidence. Int. J. Man Machine Studies 19, 33–43 (1983)

[82] Smets, Ph.: The Combination of Evidence in the Transferable Belief Model. IEEE Transactions on Pattern Analysis and Machine Intelligence 12(5), 447–458, (1990)

[83] Smets, Ph.: Belief functions: the disjunctive rule of combination and the generalized Bayesian theorem. Int. J. Approximate Reasoning 9, 1–35 (1993)

[84] Smets, Ph.: What is Dempster-Shafer's model ? In: Yager, R. R., Kacprzyk, J., Fedrizzi, M. (eds.) Advances in the Dempster-Shafer theory of evidence, pp 5–34. Wiley, New-York (1994)

[85] Smets, Ph.: The canonical decomposition of a weighted belief. In: Proceedings of the 14th Int. Joint Conf. on Artificial Intelligence (IJCAI'95), San Mateo, California, USA, 1995, pp 1896–1901. Morgan Kaufmann (1995)

[86] Smets, Ph.: Imperfect Information: Imprecision and Uncertainty. In: Motro, A., Smets, Ph. (eds.) Uncertainty Management in Information Systems: From Needs to Solution, pp 225–254. Kluwer Academic Publishers, Boston (1996)

[87] Smets, Ph.: The α-junctions: combination operators applicable to belief functions. In: Gabbay, D.M., Kruse, R., Nonnengart, A., Ohlbach, H.J. (eds.), Proceedings of the First International Joint Conference on Qualitative and Quantitative Practical Reasoning (ECSQARU-FAPR'97), Bad Honnef, Germany, 1997. Lecture Notes in Computer Science, vol. 1244, pp 131–153. Springer (1997)

[88] Smets, Ph.: The Transferable Belief Model for quantified belief representation. In: Gabbay, D. M., Smets, Ph. (eds.), Handbook of Defeasible Reasoning and Uncertainty Management Systems, vol. 1, pp 267–301. Kluwer, Dordrecht (1998)

[89] Smets, Ph.: Quantified possibility theory seen as an hypercautious transferable belief model. Rencontres Francophones sur les Logiques Floues et ses Applications (LFA 2000, La Rochelle, France), pp 343–353. Cepadues-Editions, Toulouse, France (2000)

[90] Smets, Ph.: Belief functions and the Transferable Belief Model. Documentation Section on the website of the Society for Imprecise Probability Theory and Applications (SIPTA) http://www.sipta.org/documentation/(2000)

[91] Smets, Ph.: The application of the matrix calculus to belief functions. Int. J. Approximate Reasoning 31(1–2), 1–30 (2002)

[92] Smets, Ph.: Belief functions on real numbers. Int. J. Approximate Reasoning 40, 181–223 (2005)

[93] Smets, Ph.: Decision Making in the TBM: the Necessity of the Pignistic Transformation. Int. J. Approximate Reasoning 38, 133–147 (2005)

[94] Smets, Ph.: Managing deceitful reports with the transferable belief model. In: Proceedings of the Eighth International Conference on Information Fusion, IEEE, pp. C8–3/1–7, Piscataway, NJ (2005)

[95] Smets, Ph.: Analyzing the combination of conflicting belief functions. Information Fusion 8(4), 387–412 (2007)

[96] Smets, Ph., Kennes, R.: The Transferable Belief Model. Artificial Intelligence 66, 191–243 (1994)

[97] Smets, Ph., Kruse, R.: The Transferable Belief Model for Belief Representation. In: Motro, A., Smets, Ph. (eds.) Uncertainty Management in Information Systems: From Needs to Solution, pp 343–368. Kluwer Academic Publishers, Boston (1996)

[98] Smets, Ph., Magrez, P.: Additive structure of the measure of information content. In: Gupta, M., Kandel, A., Bandler, W., Kiszkaed, J. (eds.) Approximate reasoning in expert systems, pp 195–197 (1985)

[99] Voorbraak, F.: A computationally efficient approximation of Dempster-Shafer theory. Int. Journal of Man-Machine Studies 30, 525–536 (1989)

[100] Walley, P.: Statistical Reasoning with Imprecise Probabilities. Chapman and Hall, London (1991)

[101] Weisstein, E. W.: Eigen Decomposition Theorem. Wolfram MathWorld, accessed online on 11 September 2008 at http://mathworld.wolfram.com/EigenDecompositionTheorem.html

[102] Weisstein, E. W.: Eigenvector. Wolfram MathWorld, accessed online on 11 September 2008 at http://mathworld.wolfram.com/Eigenvector.html.

[103] Weisstein, E. W.: Eigen Decomposition. Wolfram MathWorld, accessed online on 11 September 2008 at http://mathworld.wolfram.com/EigenDecomposition.html.

[104] Xu, H., Smets, Ph.: Generating Explanations for Evidential Reasoning. In: Besnard, Ph., Hanks, S. (eds.) Proceedings of the 11th Conference on Uncertainty in Artificial Intelligence (UAI'95), pp 574–581. Morgan Kaufmann, San Francisco, CA (1995)

[105] Yager, R. R.: An Introduction to Applications of Possibility Theory. Human Systems Management 3, 246–269 (1983)

[106] Yager, R. R.: The entailment principle for Dempster-Shafer granules. Int. J. Intell. Syst. 1, 247–262 (1986)

[107] Yager, R. R.: On the Dempster-Shafer framework and new combination rules. Information Sciences 41, 93–137 (1987)

[108] Yager, R. R., Rybalov, A.: Uninorm aggregation operators. Fuzzy Sets and Systems 80, 111–120 (1996)

[109] Zadeh, L. A.: Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems 1, 3–28 (1978)

[110] Zouhal, L. M., Denœux, T.: An evidence-theoretic k-NN rule with parameter optimization. IEEE Transactions on Systems, Man and Cybernetics C 28(2), 263–271 (1998)

Titre :
Fonctions de Croyance: Décompositions Canoniques et Règles de Combinaison

Résumé :
Comparé à la théorie des possibilités, le Modèle des Croyances Transférables (MCT) – une interprétation non probabiliste de la théorie de Dempster-Shafer – dispose d'assez peu de choix en terme d'opérateurs d'agrégation pour la fusion d'informations. Dans cette thèse, ce problème de manque de flexibilité pour la combinaison des fonctions de croyance – l'outil mathématique permettant la représentation de l'information dans le MCT – est abordé. Notre première contribution est la mise à jour de familles infinies de règles de combinaison conjonctives et disjonctives, rejoignant ainsi la situation en théorie des possibilités en ce qui concerne les opérateurs de fusion conjonctive et disjonctive. Notre deuxième contribution est un ensemble de résultats rendant intéressante, d'un point de vue applicatif, une famille infinie de règles de combinaison, appelée les α-jonctions et introduite initialement de manière purement formelle. Tout d'abord, nous montrons que ces règles correspondent à une connaissance particulière quant à la véracité des sources d'information. Ensuite, nous donnons plusieurs nouveaux moyens simples de calculer la combinaison par une α-jonction.

Mots-clés :
Modèle des Croyances Transférables, Théorie de Dempster-Shafer, Théorie de l'Évidence, Fonctions de Croyance, Décomposition Canonique, Fusion d'Informations, Raisonnement Incertain, Règles de Combinaison.

Title :
Belief Functions: Canonical Decompositions and Combination Rules

Abstract:
In comparison to possibility theory, the Transferable Belief Model (TBM) – a nonprobabilistic interpretation of Dempster-Shafer theory – enjoys little choice in terms of aggregation operators for the fusion of information. In this thesis, the problem of introducing some flexibility for the combination of belief functions – the mathematical tool allowing the representation of information in the TBM – is tackled. Our first contribution is to introduce infinite families of conjunctive and disjunctive combination rules for belief functions, mimicking thus the situation in possibility theory for what concerns conjunctive and disjunctive operators. Our second contribution is a set of results making an infinite family of combination rules, introduced initially in a pure formal manner and called the α-junctions, of practical interest. First, we show that these rules correspond to a particular form of knowledge about the truthfulness of the sources of information. Second, we provide several new simple means for the computation of the combination by an α-junction.

Keywords :
Transferable Belief Model, Dempster-Shafer Theory, Evidence Theory, Belief Functions, Canonical Decomposition, Information Fusion, Uncertain Reasoning, Combination Rules.